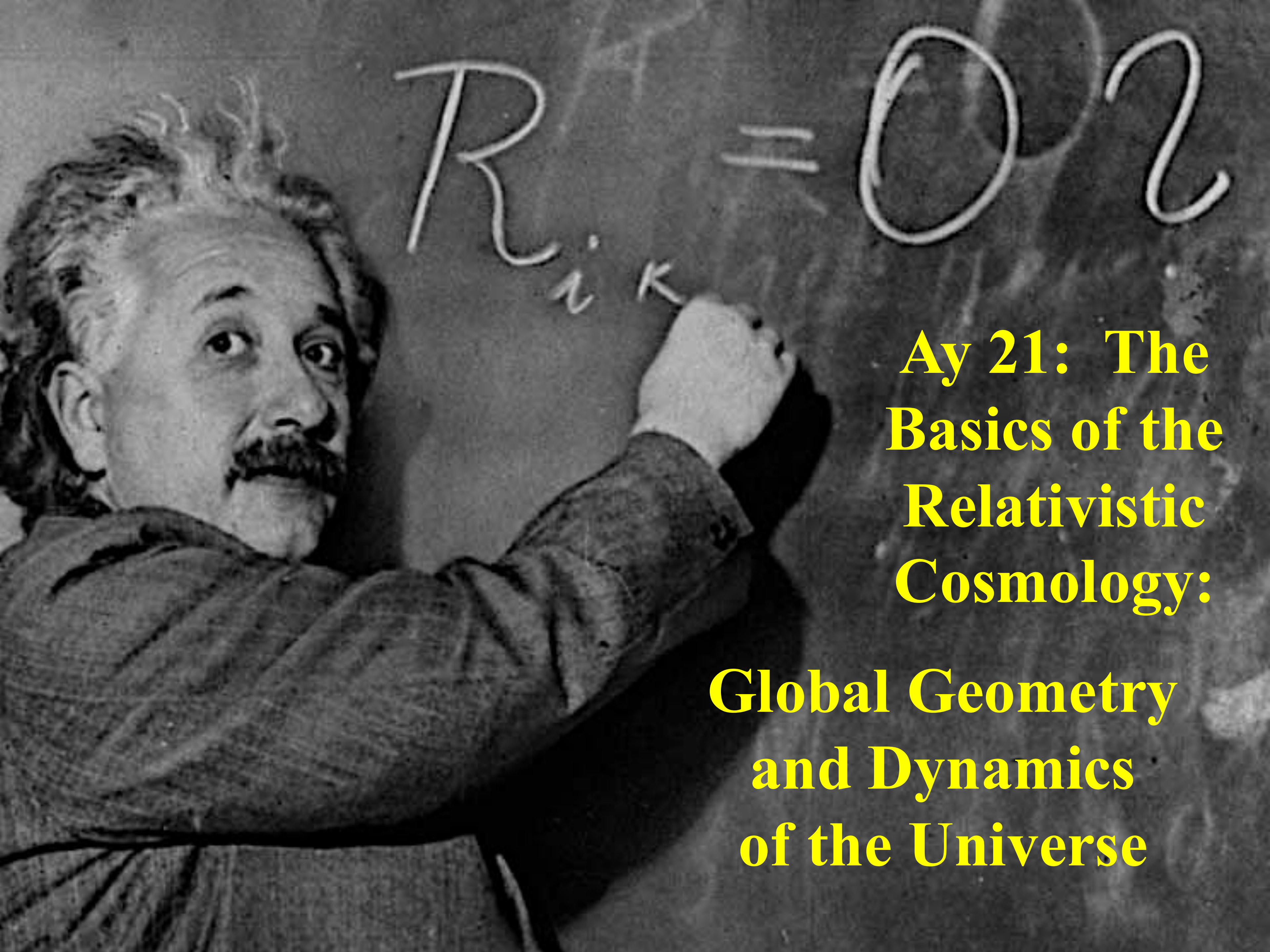


Ay 21: The Basics of the Relativistic Cosmology:

Global Geometry and Dynamics of the Universe

Special Relativity (1905)

- A fundamental change in viewing the physical space and time, now unified in spacetime
- Postulates equivalence among all *unaccelerated* frames of reference (inertial observers)
- Reconciles classical electrodynamics and coordinate and velocity transformations
- Novel effects:
  - c is the maximum velocity
  - Lorentz contraction
  - Time dilation
  - Equivalence of mass and energy
  - Explains the anomalous precession of Mercury's orbit

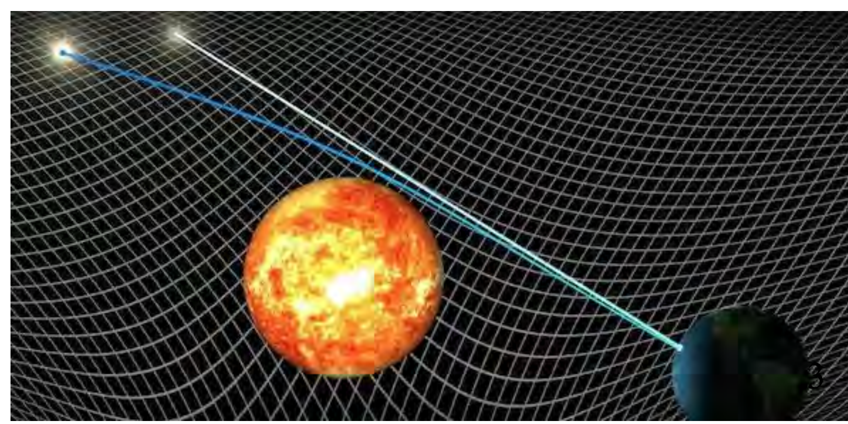
General Relativity (1915)

- An even more fundamental change regarding the space and time, and matter/energy, connecting them
- Postulates equivalence among ***all*** frames of reference (including accelerated ones), and is thus the ***theory of gravity***

Presence of mass/energy determines the geometry of space
Geometry of space determines the motion of mass/energy

- Introduces curvature of space, predicting a number of new effects:
  - Light deflection by masses
  - Gravitational redshift

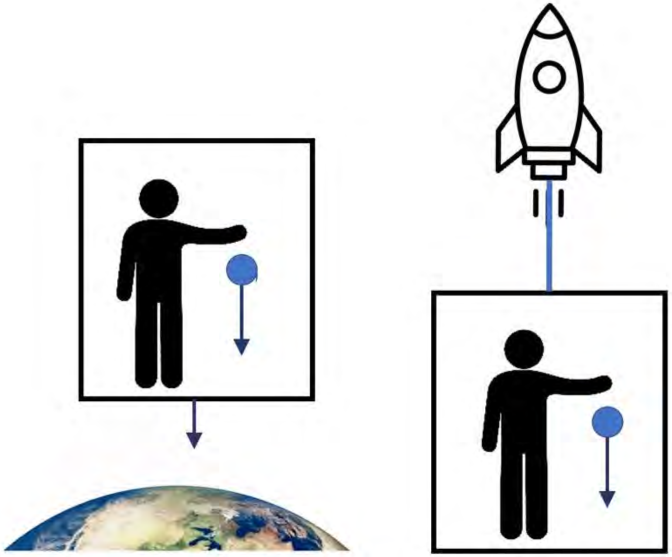
  etc. etc.



The Equivalence Principle

Mach's Principle: The gravitational interaction of mass in the universe causes all inertial forces. In an empty universe, there would be no inertia.

Einstein argued that gravity is an **inertial force**: a frame linearly accelerating relative to an inertial frame in special relativity is locally identical to a frame at rest in a gravitational field.



Thus, **inertial mass = gravitational mass**, which is the ***equivalence principle*** (In the Newtonian physics they need not be the same). This results in two effects:

- Light should be blue/redshifted in a gravitational field
- Light paths in a gravitational field should be curved

Einstein already predicted the gravitational deflection of light in 1911. Here is his letter to G. E. Hale in 1913, asking him about the possibility of observing the light deflection around the Sun.

Zürich 14. X. 13.

Aus

Hoch geehrter Herr Kollege!

Eine einfache theoretische Überlegung macht die Annahme plausibel, dass Lichtstrahlen in einem Gravitationsfelde eine Deviation erfahren.

Grav. Feld

Lichtstrahl

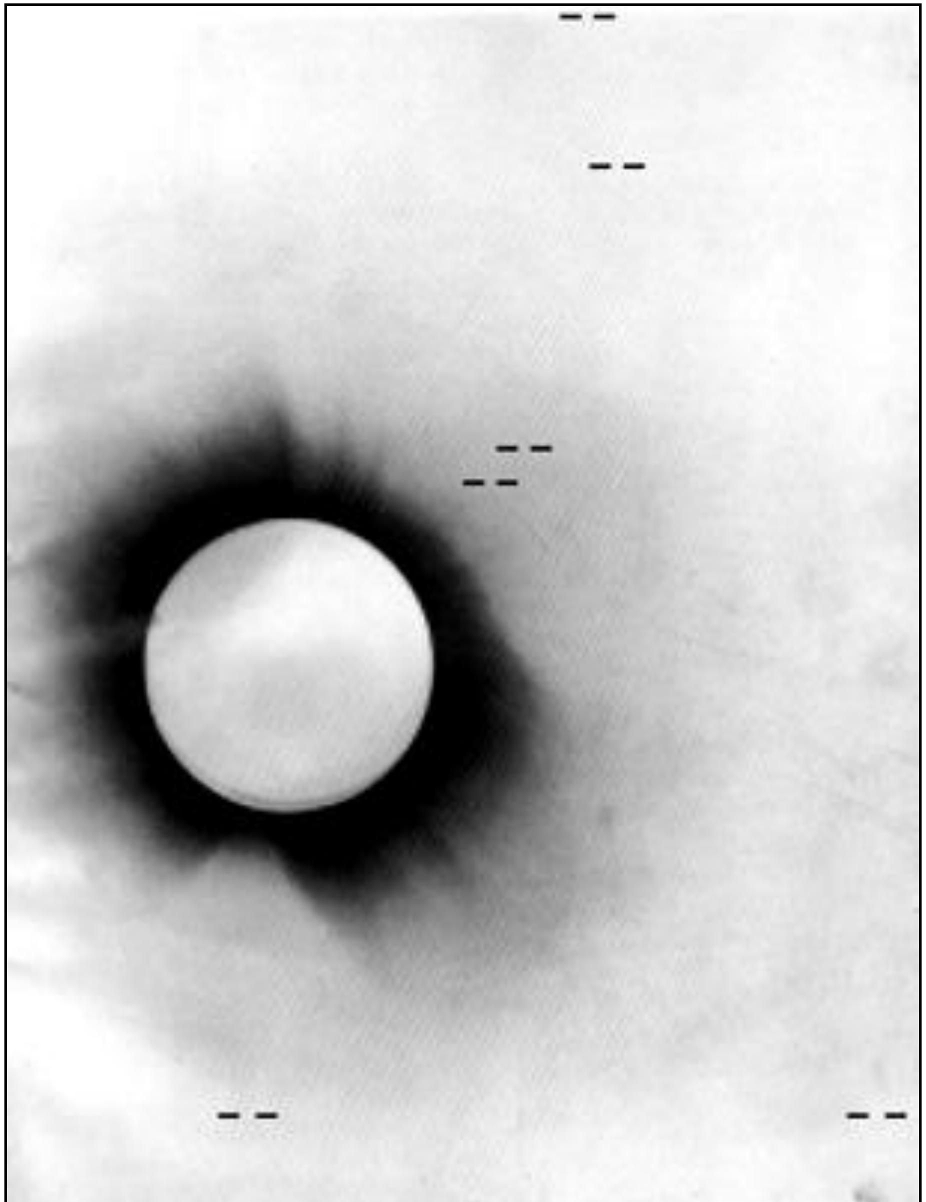
Am Sonnenrande müsste diese Ablenkung 0,84" betragen und wie $\frac{1}{R}$ abnehmen (R = ~~Sonnenradius~~ Entfernung vom Sonnen-mittelpunkt).

0,84"

Sonne

Es wäre deshalb von grösstem Interesse, bis zu wie grosser Sonnennähe ~~grosse~~ helle Fixsterne bei Anwendung der stärksten Vergrösserungen bei Tage (ohne Sonnenfinsternis) gesehen werden können.

Confirmation of the GR



Eddington's 1919 eclipse observations "confirmed" Einstein's relativistic prediction of α = 1.78 arcsec

Later observations have provided more accurate evidence of light deflection due to the influence of GR

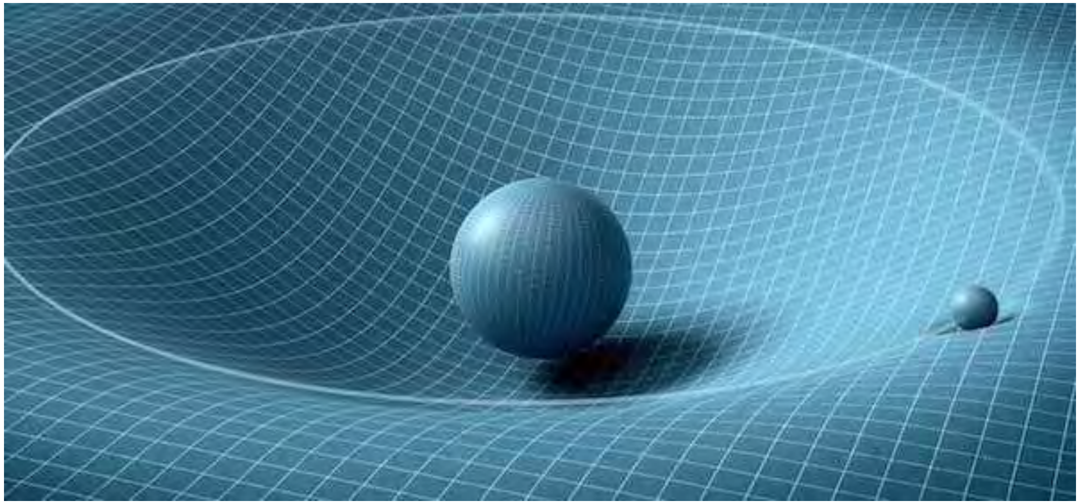
GR has passed *all* experimental tests to date

General Relativity

Remember the key notion:

Presence of mass/energy determines the geometry of space
Geometry of space determines the motion of mass/energy

Thus, the distribution of the matter and energy in space must be consistent with its spatial geometry



Mathematical expression of that statement is given by the **Einstein equation(s)**

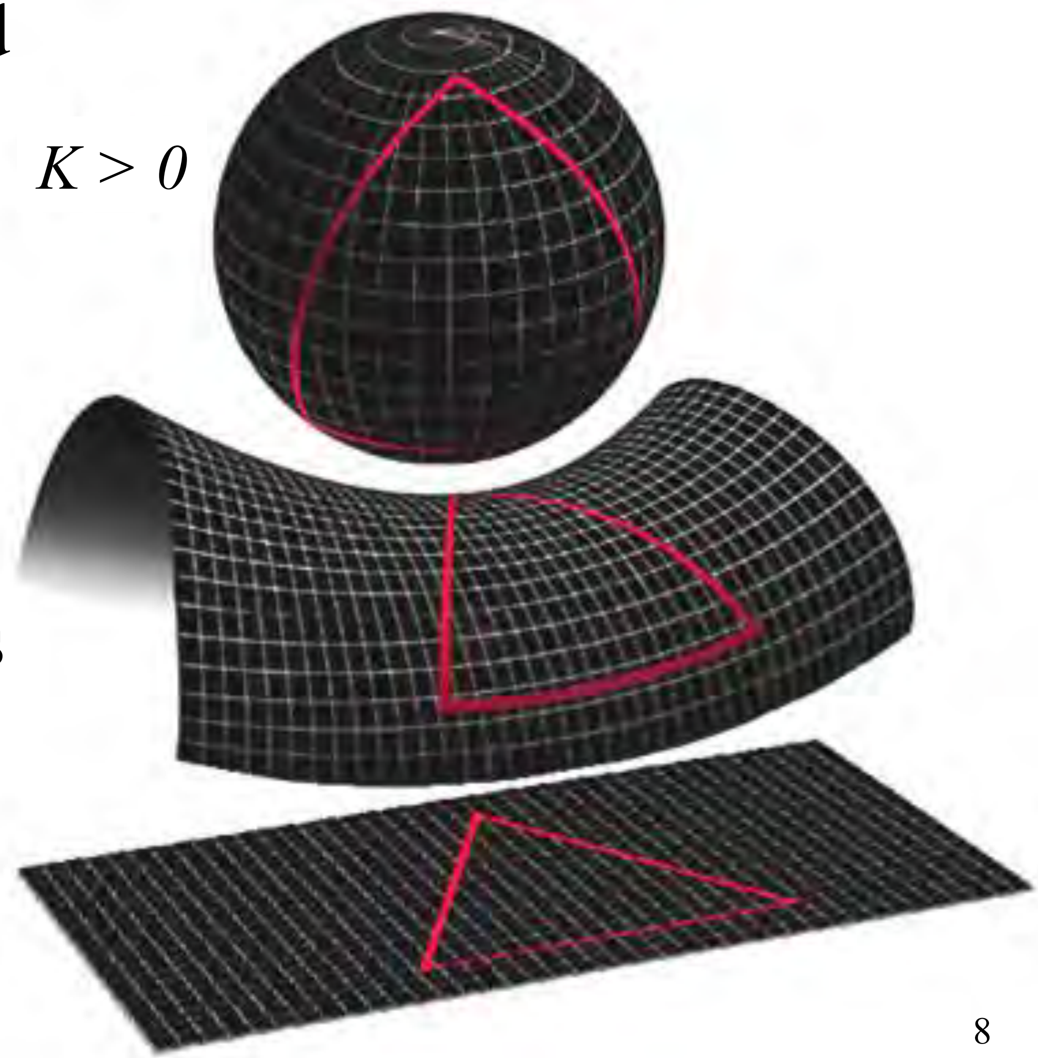
Their derivation is well beyond the scope of this class, but here is just a little flavor…

What is the Global Geometry of the Universe?

Generally, it can be curved

The curvature will depend on its matter-energy content …

… and it will determine its global dynamics and evolution



Metric and Spacetime

Geometry of space can be generally defined through the **metric**, enabling one to compute the distance between any two points:

$$ds^2 = \sum_a \sum_b g_{ab} dx^a dx^b$$

where g_{ab} is the **metric tensor**. Indices $\{a,b\}$ run 0 to 3, for the spacetime (0 is the time dimension, 1,2,3 are the spatial ones, i.e., xyz)

In a simple Euclidean geometry, it is a diagonal unit tensor (matrix): $g_{aa} = 1$, $g_{a\neq b} = 0$, where $\{a,b\} = \{1,2,3\}$

The metric coefficients g_{ab} are generally functions of the spacetime position, and a proper theory of spacetime has to specify these functions

Metric: Quantifying the Geometry

- The geometry of spacetime is completely specified by a **metric**, $g_{\mu\nu}$
- A special relativistic, Euclidean case is the **Minkowski metric:**

$$ds^2=(c\ dt)^2 - (dx^2 + dy^2 + dz^2)$$

- A general case for a GR, *homogeneous and isotropic* universe is the **Robertson-Walker metric:**

$$ds^2 = (c\ dt)^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

where $k = -1, 0, +1$ for a (negative,flat,positive) curvature

The Einstein Equation(s)

A tensor equation - a shorthand for 16 partial differential eqs., connecting the geometry and mass/energy density:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Spacetime geometry

Matter distribution

where:

$G_{\mu\nu}$ = The Einstein tensor

$R_{\mu\nu}$ = The Ricci tensor

$g_{\mu\nu}$ = The metric tensor

R = The Ricci scalar

$T_{\mu\nu}$ = The stress-energy tensor

Homogeneity and isotropy requirements reduce this set of 16 eqs. to only 1, $G_{00} = T_{00}$, which becomes the **Friedmann Equation**

Introducing the Cosmological Constant

Gravitation is an attractive force, so what is to prevent all matter and energy falling to one gigantic lump?

Einstein introduced a negative potential term to balance the attractive gravity:

$$\nabla^2\phi - \lambda\phi = 4\pi G\rho$$

λ could be thought of as an integration constant, or a new constant of nature, or a new aspect of gravity

The Einstein Equations now become:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

$$\Lambda = 1/L^2,\ \rho = \Lambda/4\pi G$$

This is the **cosmological constant**. Note that the theory does not specify its value, or even the sign!

The Cosmological Principle

Relativistic cosmology uses some symmetry assumptions or principles in order to make the problem of "solving the universe" viable. The **Cosmological Principle** states that

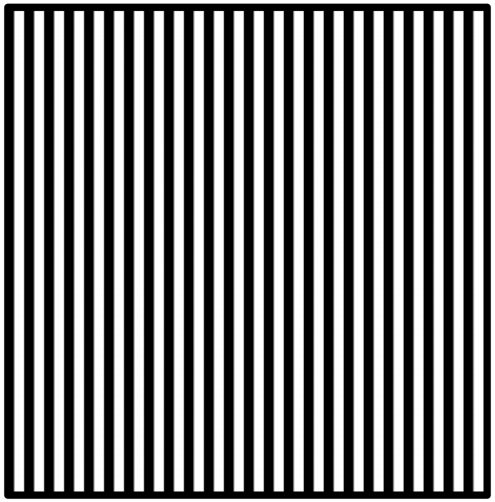
At each epoch, the universe is the same at all locations and in all directions, except for local irregularities

Therefore, globally the Universe is assumed to be **homogeneous** and **isotropic** at any given time; and its dynamics should be the same everywhere

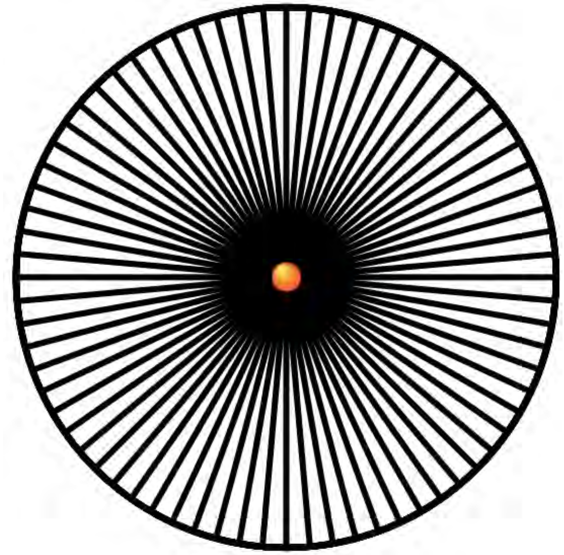
Note: the **Perfect Cosmological Principle** states that the Universe appears the same at all times and it is unchanging - it is also homogeneous in time - this is the basis of the "Steady State" model

Homogeneity and Isotropy

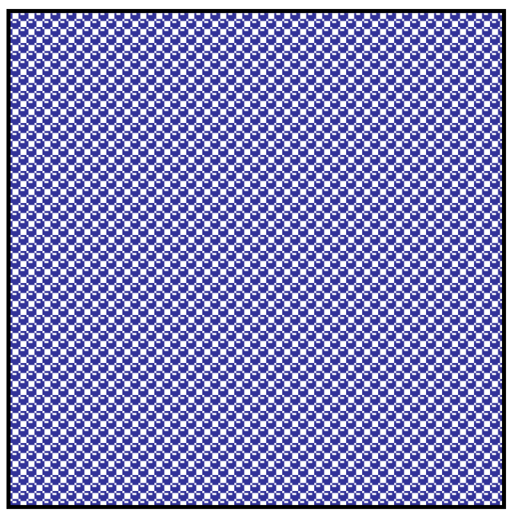
Homogeneous but not isotropic



Isotropic at ● but not homogeneous



Homogeneous *and* Isotropic

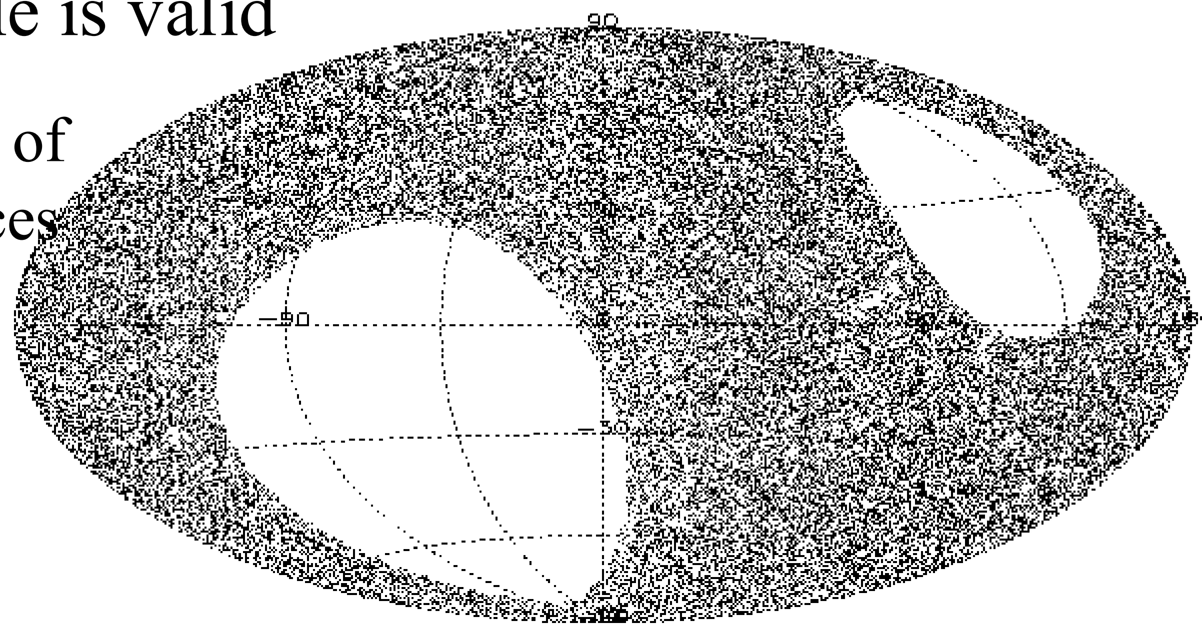


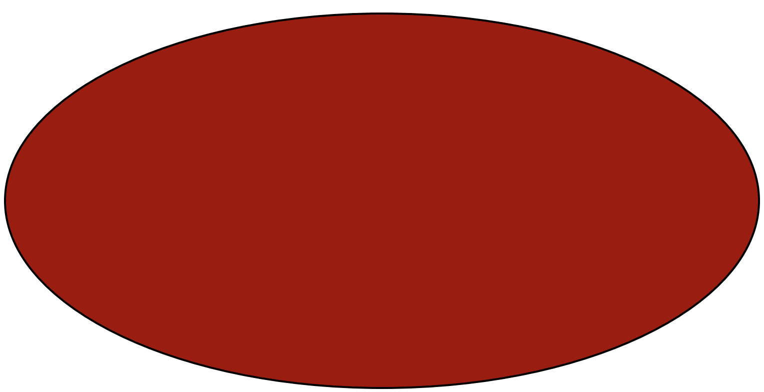
In a homogeneous and isotropic space, only the radial coordinate matters, which greatly simplifies things: 16 Einstein equations reduce to only 1

So, is the Universe Really Homogeneous and Isotropic?

Globally, on scales larger than ~ 100 Mpc, say, it is - so the cosmological principle is valid

Distribution on the sky ⍰ of 65000 distant radio sources from the Texas survey, a cosmological population



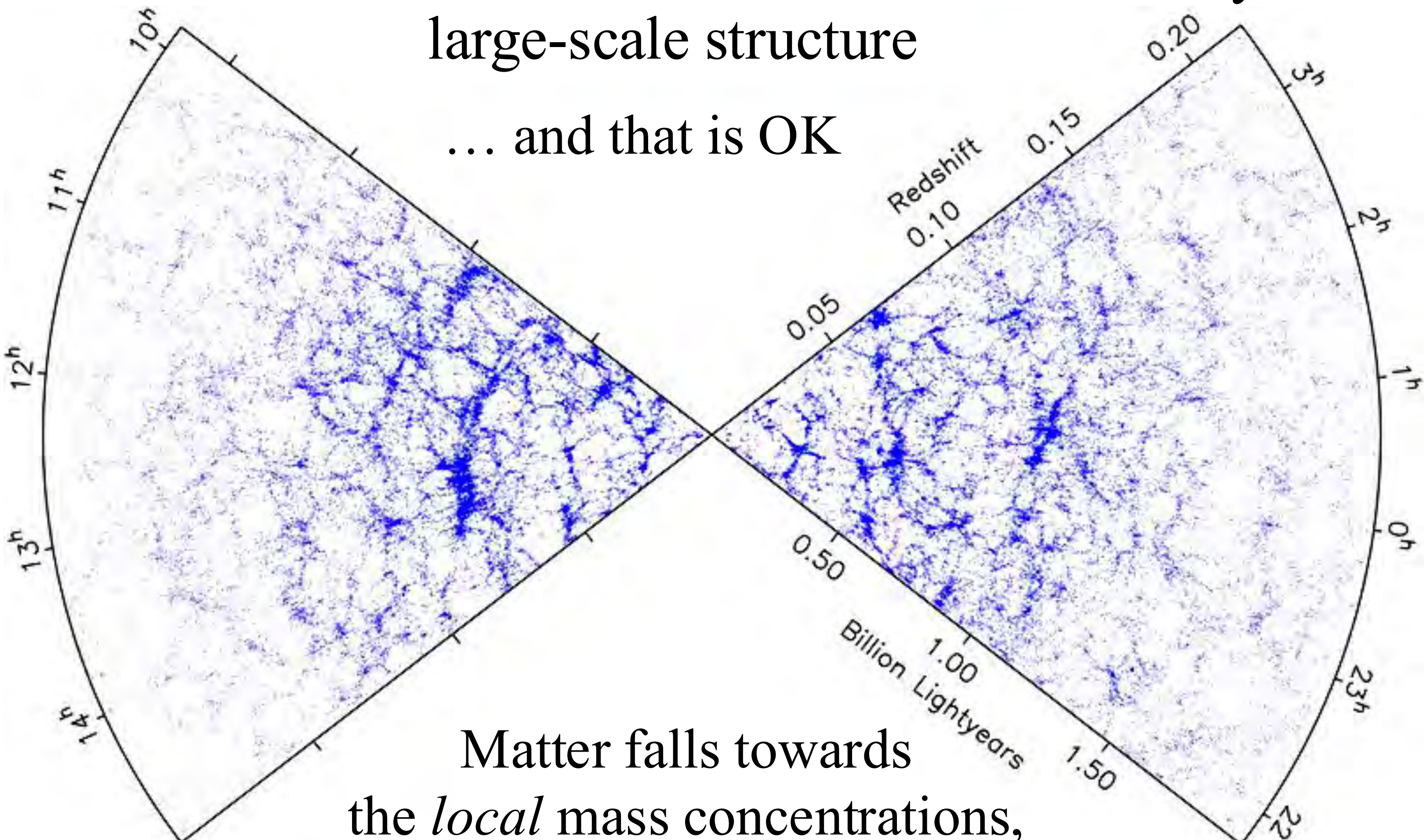


… and of course the CMBR sky, uniform to better than $\Delta T/T < 10^{-5}$, after taking the dipole out

So, is the Universe Really Homogeneous and Isotropic?

Not so much on scales up to ~ 100 Mpc, as shown by the large-scale structure

… and that is OK



Matter falls towards the *local* mass concentrations, which can counter the *global* expansion

Expansion Relative to What? Comoving and Proper Coordinates

There are fundamentally two kinds of coordinates in a GR cosmology:

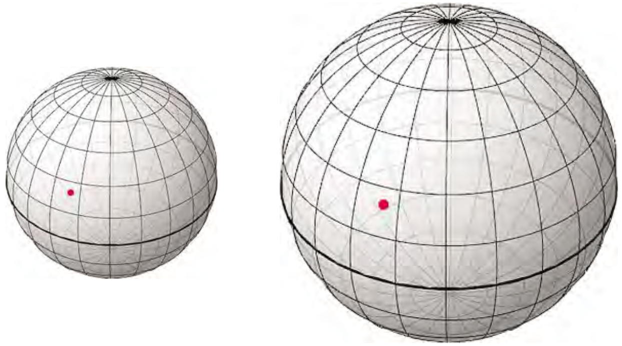
- ***Comoving coordinates*** = expand with the universe

  Examples:
  - Unbound systems, e.g., any two distant galaxies
  - Wavelengths of massless quanta, e.g., photons
- ***Proper coordinates*** = stay fixed, space expands relative to them. Examples:
  - Sizes of atoms, molecules, solid bodies
  - Gravitationally bound systems, e.g., Solar system, stars, galaxies …

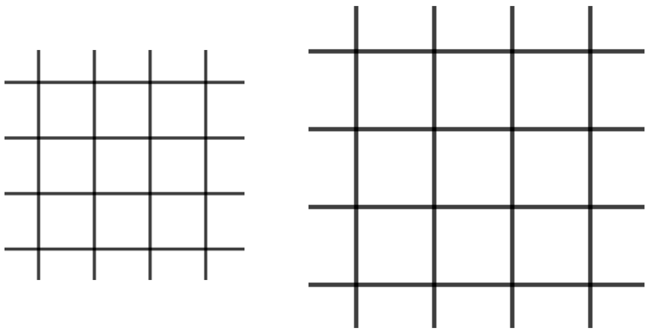
Expansion into What?

Into itself. There is nothing "outside" the universe
(Let's ignore the multiverse hypothesis for now)

A positive curvature universe is like the surface of a sphere, but in one extra dimension. Its volume is finite, but changes with the expansion of space.



A flat or a negative curvature universe is infinite in all directions; the comoving coordinate grid stretches relative to the proper coordinates



In either case, there is no "edge", and there is no center
(homogeneity and isotropy*)*

Expansion and the Hubble's Law

Consider a point at a comoving distance x. At some time t it will be at a radial distance $r(t) = a(t)\, x$, where $a(t)$ is the expansion factor. We will designate values for "here and now" with a subscript 0, t_0 = now, and $a_0 = a(t_0) = 1$. The recession velocity is:

$$\boldsymbol{v}(\boldsymbol{r},t) = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}(t) = \frac{\mathrm{d}a}{\mathrm{d}t}\boldsymbol{x} \equiv \dot{a}\,\boldsymbol{x} = \frac{\dot{a}}{a}\boldsymbol{r} \equiv H(t)\,\boldsymbol{r}$$

Where $H(t) := \dfrac{\dot{a}}{a}$ is the normalized *expansion rate*

Thus:

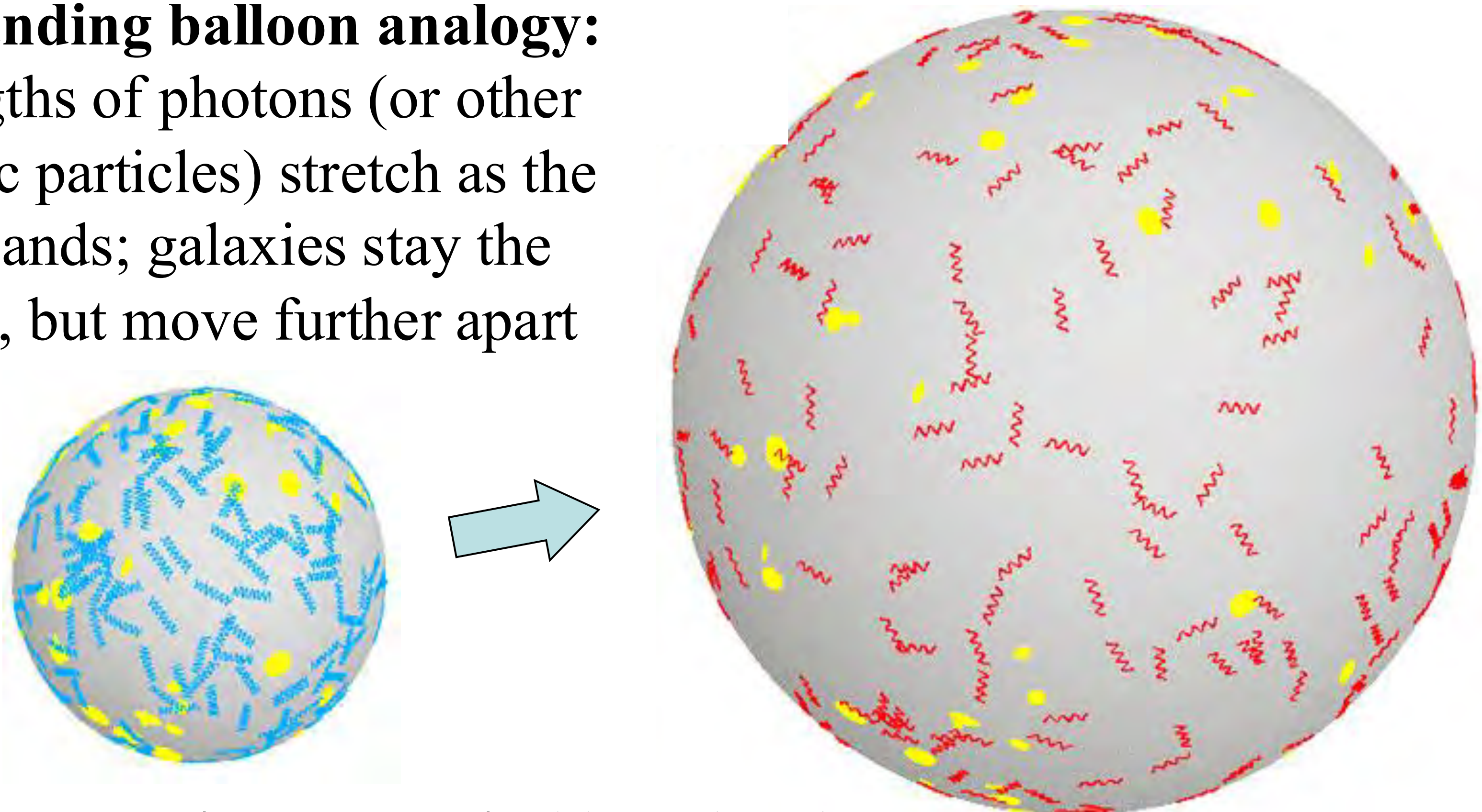
$$\Delta\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}+\Delta\boldsymbol{r},t) - \boldsymbol{v}(\boldsymbol{r},t) = H(t)\,\Delta\boldsymbol{r}$$

Which is the same as the Hubble's law: $v = H_0\, D$

H_0 is the value of the expansion rate here and now. Note that it is not a constant, but it depends on $a(t)$.

The Cosmological Redshift

The expanding balloon analogy: Wavelengths of photons (or other relativistic particles) stretch as the space expands; galaxies stay the same size, but move further apart



$\lambda_{obs} / \lambda_{em} = a_{now} / a_{then} = a_0 / a(z) = (1+z)$

Expansion factor $a(z) = 1 / (1+z)$

But thanks to the Hubble law, *the cosmological redshift turns out to be equivalent to the Doppler shift*

Redshift as Doppler Shift

We define **doppler redshift** to be the shift in spectral lines due to motion:

(special relativity formula)

$$z = \frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1+v/c}{1-v/c}} - 1$$

which, in the case of **v<<c** reduces to the familiar

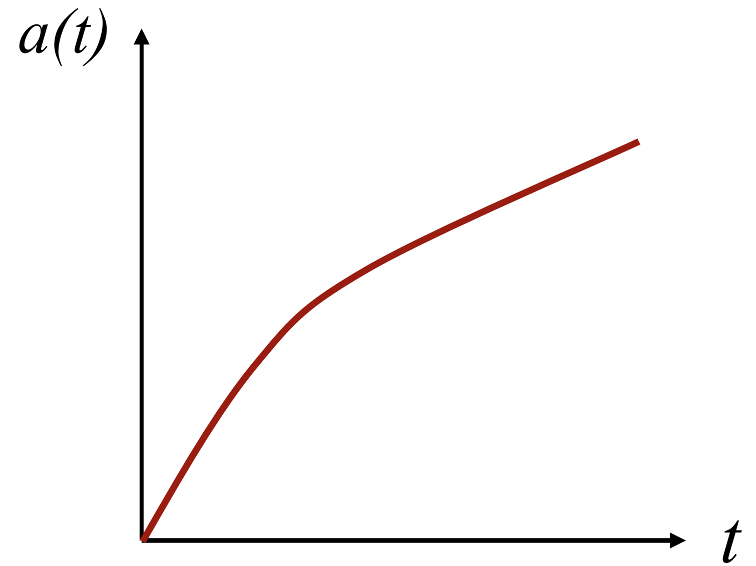
(nonrelativistic formula)

$$z = \frac{v}{c}$$

The **cosmological redshift** is something different, although we are often sloppy and refer to it in the same terms of the doppler redshift. The cosmological redshift is actually due to the **expansion of space** itself.

Quantifying the Kinematics of the Universe

We introduce a **scale factor**, commonly denoted as ***R(t)*** or ***a(t)*:** *a spatial distance between any two unaccelerated frames which move with their comoving coordinates*

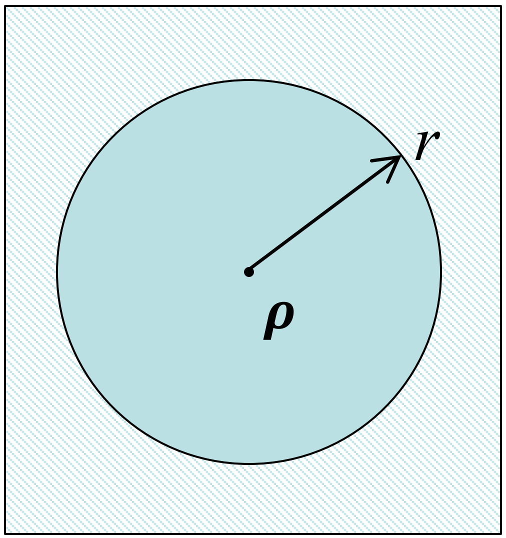


This fully describes the evolution of a homogeneous, isotropic universe

Computing ***a(t)*** and various derived quantities defines the **cosmological models**. This is accomplished by solving the **Friedmann Equation**

Deriving the Friedmann Equation

… in a Newtonian approximation. It works, because locally (small distances, weak gravitational fields) Newtonian approximation is OK.



Consider a test particle on the surface of a sphere with the radius r, and the mean density ρ (in a homogeneous universe, any point and any sphere would work). The enclosed mass is:

$$M(x) = \frac{4\pi}{3}\rho_0\, x^3 = \frac{4\pi}{3}\rho(t)\, r^3(t)$$

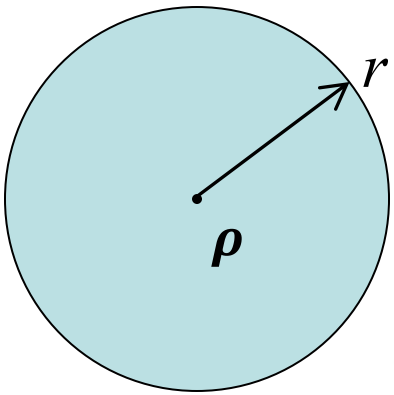
$$= \frac{4\pi}{3}\rho(t)\, a^3(t)\, x^3 \;,$$

$$\rho(t) = \rho_0\, a^{-3}(t)$$

According to Birkhoff' s theorem, for a spherically symmetric system, the force due to gravity at radius r is determined only by the mass *interior* to that radius.

Deriving the Friedmann Equation

The equation of motion of our test particle is then:



$$\ddot{r}(t) \equiv \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = -\frac{G\,M(x)}{r^2} = -\frac{4\pi G}{3}\,\frac{\rho_0\,x^3}{r^2}$$

Or, since $r(t) = a(t)\,x$:

$$\ddot{a}(t) = \frac{\ddot{r}(t)}{x} = -\frac{4\pi G}{3}\,\frac{\rho_0}{a^2(t)} = -\frac{4\pi G}{3}\,\rho(t)\,a(t)$$

Multiply the equation by $2\dot{a}$ and integrate in time using a substitution $\mathrm{d}(\dot{a}^2)/\mathrm{d}t = 2\dot{a}\ddot{a}$, and $\mathrm{d}(-1/a)/\mathrm{d}t = \dot{a}/a^2$:

$$\dot{a}^2 = \frac{8\pi G}{3}\,\rho_0\,\frac{1}{a} - Kc^2 = \frac{8\pi G}{3}\,\rho(t)\,a^2(t) - Kc^2$$

Where Kc^2 is an integration constant; K is the **curvature constant**, which can be set to 0, +1, or –1

Deriving the Friedmann Equation

This is equivalent to the conservation of energy (which is always locally valid), after multiplying by $x^2 / 2$:

$$\frac{v^2(t)}{2} - \frac{G\,M}{r(t)} = -Kc^2\,\frac{x^2}{2}$$

Consider the equation:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho_0\,\frac{1}{a} - Kc^2$$

- **If $K = 0$** (flat universe): $\dot{a}^2 > 0$, universe expands for ever, but as $a \longrightarrow \infty$, $\dot{a} \longrightarrow 0$
- **If $K < 0$** (open universe): $\dot{a}^2 > 0$, universe expands for ever, but $\dot{a} \longrightarrow c$
- **If $K > 0$** (closed universe): the expansion peaks when $\dot{a}^2 = 0$, $a_{\max} = (8\pi G\rho_0)/(3Kc^2)$ and since $\ddot{a}(t)$ is always < 0 (in the absence of the cosmological constant), the expansion reverses, and the universe collapses ("the big crunch")

The Friedmann-Lemaitre Equations

A more complete derivation, including the cosmological constant term Λ, gives:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{\Lambda}{3}$$

where

P = pressure

$\boldsymbol{\rho}$ = total matter+radiation density

$$\underbrace{\rho = \rho_{\mathrm{m}}}_{\text{matter}} + \underbrace{\rho_{\mathrm{r}}}_{\text{radiation}} + \underbrace{\rho_{\mathrm{v}}}_{\text{vacuum }(\Lambda)} = \rho_{\mathrm{m+r}} + \rho_{\mathrm{v}}\ , \qquad P = P_{\mathrm{r}} + P_{\mathrm{v}}$$

$$P_{\mathrm{r}} = \frac{1}{3}\rho_{\mathrm{r}}\, c^2 \qquad P_{\mathrm{v}} = -\rho_{\mathrm{v}}\, c^2$$

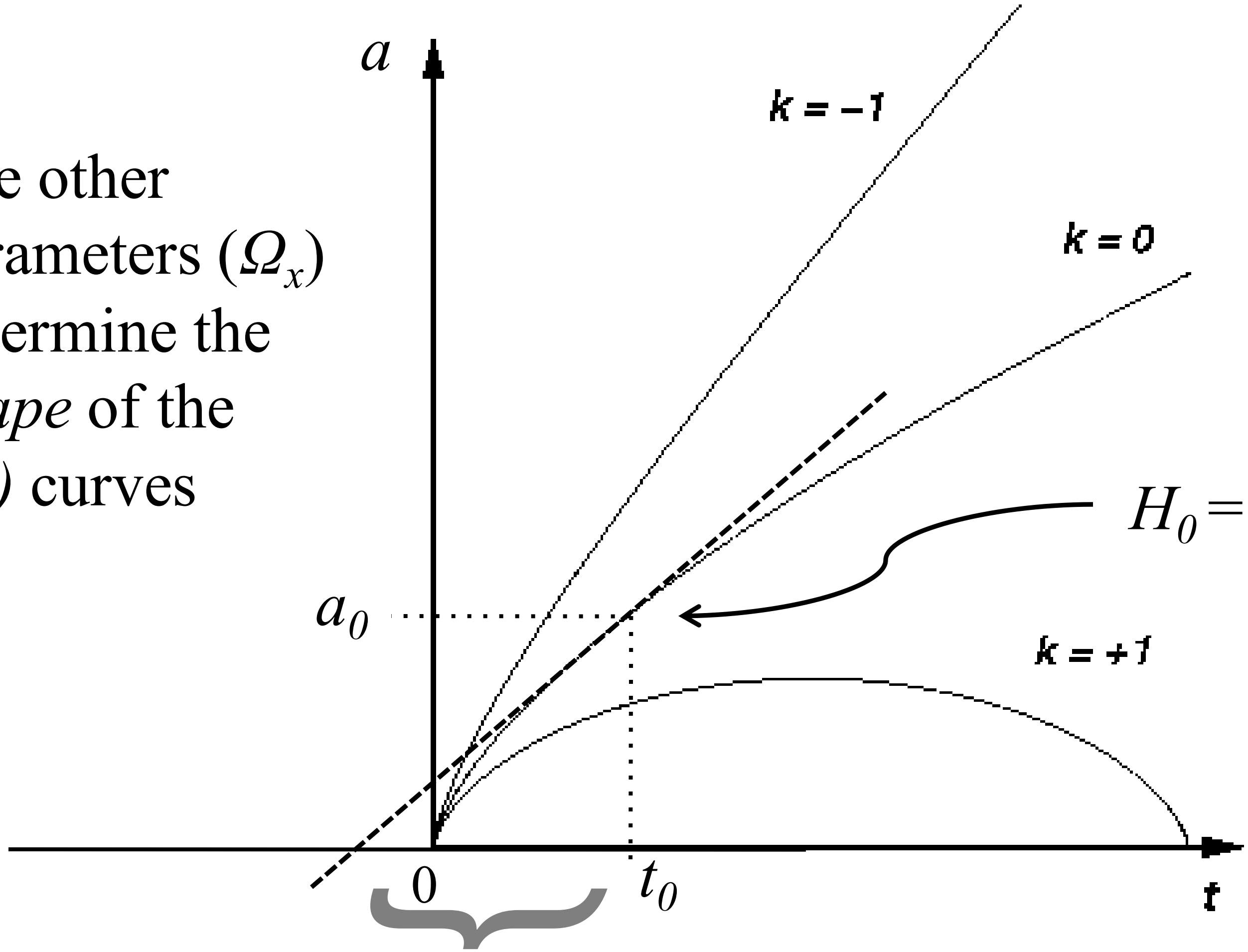
These are effectively *the **equations of motion** for a relativistic, homogeneous, isotropic universe.*

The presence of Λ also leads to a richer dynamical behavior.

Cosmological Parameters

H_0 defines the spatial and temporal *scale* of the universe

The other parameters (Ω_x) determine the *shape* of the $a(t)$ curves



$1 / H_0$ = Hubble time $\quad$ c / H_0 = Hubble length

Cosmological Parameters

The Matter Density Parameter

Rewriting the Friedmann Eqn. using the Hubble parameter, and for now set $\Lambda = 0$:

$$H^2 - \frac{8}{3}\pi G\rho = -\frac{kc^2}{a}$$

The Universe is flat if k=0, or if it has a critical density of

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

We define the **matter density parameter** as

$$\boxed{\Omega_M = \frac{\rho}{\rho_{crit}}}$$

Cosmological Parameters

The "dark energy" density parameter

We can express a similar density parameter for lambda again by using the Friedmann equation and setting $\rho = 0$. We then get

$$\boxed{\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}}$$

The total density parameter is then

$$\boxed{\Omega = \Omega_M + \Omega_\Lambda}$$

The deceleration parameter

$$\boxed{q = -\ddot{a}\, a / \dot{a} = \frac{\Omega_M}{2} - \Omega_\Lambda}$$

Density Components

Each matter/energy component of the universe is characterized by its energy density, and they add up:

$$\rho = \underset{\text{matter}}{\underset{\downarrow}{\rho_{\mathrm{m}}}} + \underset{\text{radiation}}{\underset{\downarrow}{\rho_{\mathrm{r}}}} + \underset{\text{vacuum }(\Lambda)}{\underset{\downarrow}{\rho_{\mathrm{v}}}}$$

vacuum (Λ) = $\rho_{\mathrm{v}} = \dfrac{\Lambda}{8\pi G}$

They are diluted by the universal expansion in different ways:

$$\rho_{\mathrm{m}}(t) = \rho_{\mathrm{m},0}\, a^{-3}(t)$$
$$\rho_{\mathrm{r}}(t) = \rho_{\mathrm{r},0}\, a^{-4}(t)$$
$$\rho_{\mathrm{v}}(t) = \rho_{\mathrm{v}} = \text{const.}$$

Thus they can dominate the expansion dynamics at different times

For each of them we can define the corresponding density parameter:

$$\Omega_{\mathrm{m}} = \frac{\rho_{\mathrm{m},0}}{\rho_{\mathrm{cr}}}\ ; \quad \Omega_{\mathrm{r}} = \frac{\rho_{\mathrm{r},0}}{\rho_{\mathrm{cr}}}\ ; \quad \Omega_{\Lambda} = \frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{cr}}} = \frac{\Lambda}{3H_0^2}$$

Cosmological Parameters

The Hubble *parameter* is usually called the Hubble *constant* (even though it changes in time!) and it is often written as:

$\boldsymbol{h} = \mathrm{H}_0 \,/\, (100\ \mathrm{km\ s^{-1}\ Mpc^{-1}})$, or $\boldsymbol{h_{70}} = \mathrm{H}_0 \,/\, (70\ \mathrm{km\ s^{-1}\ Mpc^{-1}})$

Define the ***critical density***:

$$\rho_{\mathrm{cr}} := \frac{3H_0^2}{8\pi G}$$

The current physical value of the critical density is

$$\rho_{0,\mathrm{crit}} = 0.921 \times 10^{-29}\, h_{70}^{\;2}\ \mathrm{g\ cm^{-3}}$$

Define the ***density parameter*:**

$$\Omega_0 := \frac{\rho_0}{\rho_{\mathrm{cr}}}$$

The total density parameter is a sum of different components:

$$\Omega_0 = \Omega_{\mathrm{m}} + \Omega_{\mathrm{r}} + \Omega_{\Lambda}$$

Friedmann Equation Redux

We can now rewrite the Friedmann Eqn. as:

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 &= H^2(t)\\ &= H_0^2\left[\frac{\Omega_\mathrm{r}}{a^4(t)}+\frac{\Omega_\mathrm{m}}{a^3(t)}+\frac{(1-\Omega_\mathrm{m}-\Omega_\Lambda)}{a^2(t)}+\Omega_\Lambda\right]\\ &\equiv H_0^2 E^2(t)\end{aligned}$$

where $E(t) = H(t)/H_0$

or

$$H^2(t) = H_0^2\left[\frac{\Omega_\mathrm{r}}{a^4(t)}+\frac{\Omega_\mathrm{m}}{a^3(t)}-\frac{Kc^2}{H_0^2\,a^2(t)}+\Omega_\Lambda\right]$$

Define also the *deceleration parameter*:

$$q_0 := -\ddot{a}\,a/\dot{a}^2 \qquad q_0 = \Omega_\mathrm{m}/2 - \Omega_\Lambda$$

Is Energy (or Momentum) Conserved in an Expanding (or Contracting) Universe?

No

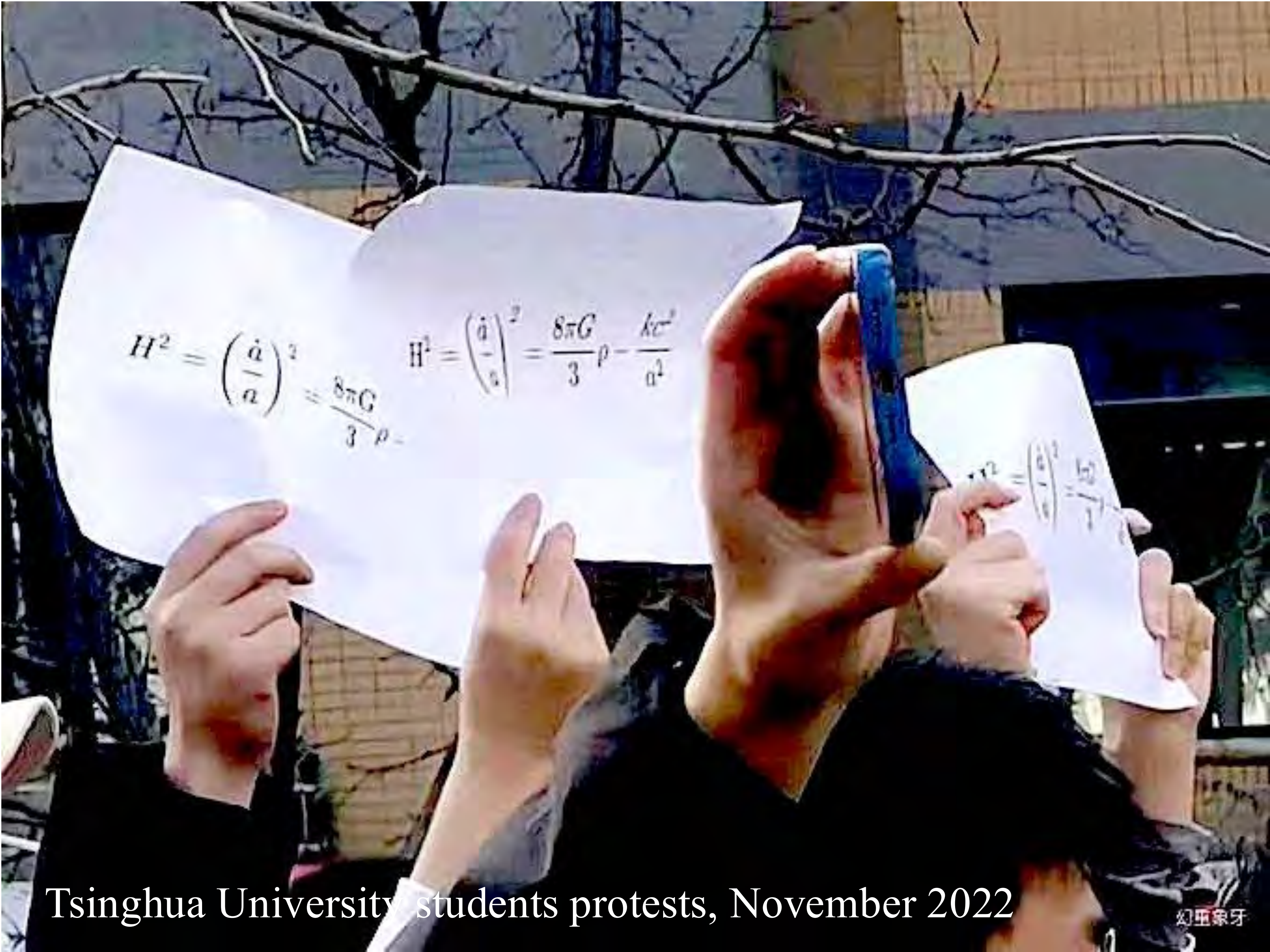
- Consider energies of photons
- Consider potential energies of unbound systems
- Consider dark energy

But "locally", at scales << cosmological, it is still an excellent approximation

Ultimately this is related to the Noether Theorem, which connects conservation laws with the symmetries of space and time:

Homogeneity of time → conservation of energy

Homogeneity of space → conservation of momentum



Tsinghua University students protests, November 2022