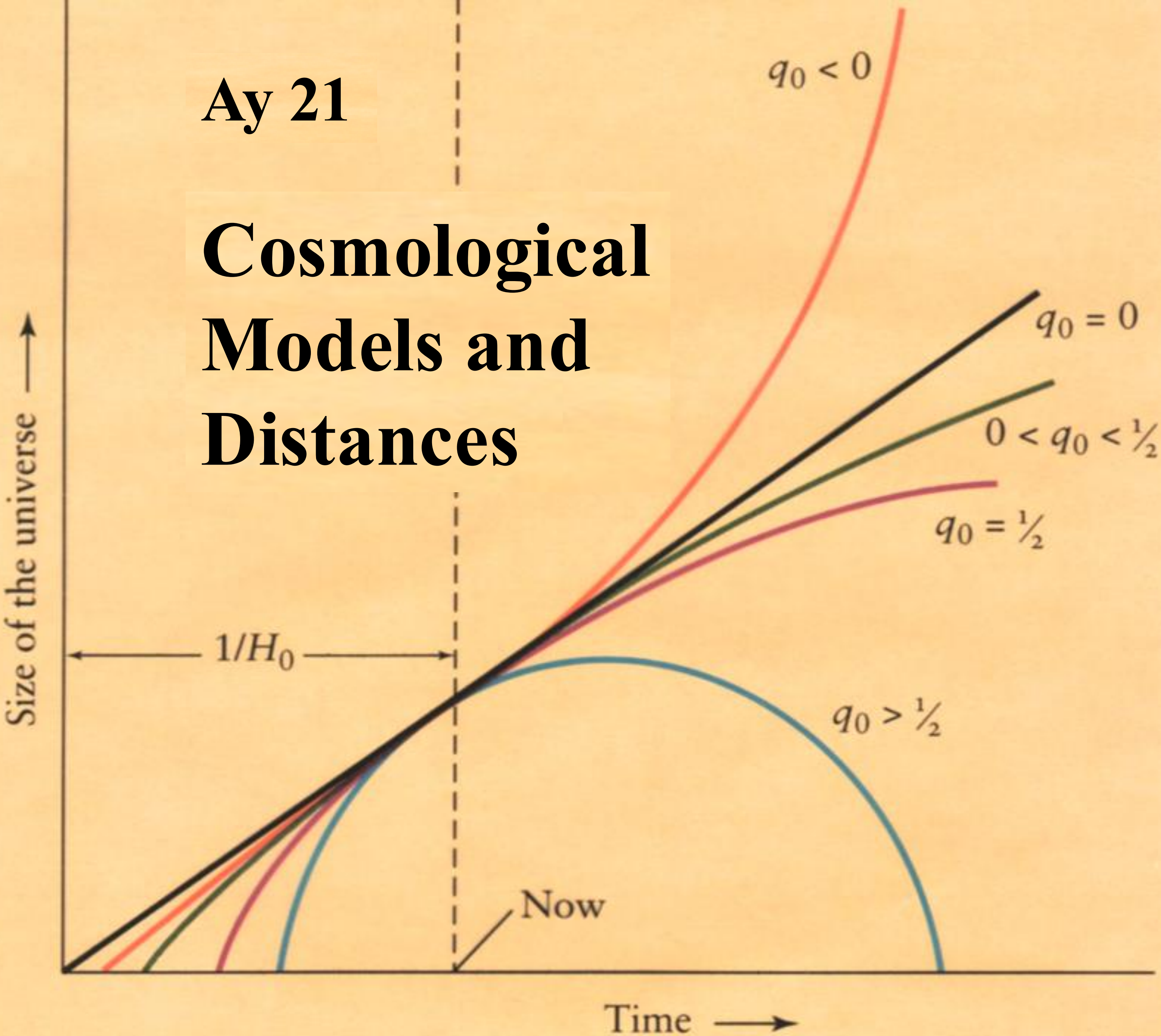
Ay 21

Cosmological Models and Distances



Solving the Friedmann Equation

Expansion rate

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}$$

Density measures

In order to solve it, we also need to define the behavior of the mass/energy density *ρ (a)* of any given mass/energy component. Recall the basic GR paradigm:

Density determines the expansion

Expansion changes the density

Each component will lead to a different evolution in redshift

We already saw that:

$$\rho_{\mathrm{m}}(t) = \rho_{\mathrm{m},0}\, a^{-3}(t)$$
$$\rho_{\mathrm{r}}(t) = \rho_{\mathrm{r},0}\, a^{-4}(t)$$
$$\rho_{\mathrm{v}}(t) = \rho_{\mathrm{v}} = \mathrm{const.}$$

The Equation of State

- Defines the dependence of the density vs. volume for a given matter/energy component, to enter in the Friedman eq.
- Usually written as $p = w \rho$
- This is not necessarily the best way to describe the matter / energy density; it implies a fluid of some kind… This may be OK for the matter and radiation we know, but maybe it is not an optimal description for the dark energy
- Special values:

  $w = 0$ means $p = 0$, e.g., non-relativistic matter

  $w = 1/3$ is radiation or relativistic matter

  $w = -1$ looks just like a cosmological constant

  … but it can have in principle any value, and it can be changing in redshift

Evolution of the Density

Generally, $\rho \sim a^{-3(w+1)}$

- Matter dominated ($w = 0$): $\rho \sim a^{-3}$
- Radiation dominated ($w = 1/3$): $\rho \sim a^{-4}$
- Cosmological constant ($w = -1$): $\rho = constant$
- Dark energy with $w < -1$ e.g., $w = -2$: $\rho \sim a^{+3}$
  - Energy density *increases* as is stretched out!
  - Eventually would dominate over even the energies holding atoms together! ("Big Rip")

In a mixed universe, different components will dominate the global dynamics at different times

Note also that in principle, w could be a function of time, density, etc.

What is Dominant When?

Matter dominated ($w = 0$): $\rho \sim a^{-3}$

Radiation dominated ($w = 1/3$): $\rho \sim a^{-4}$

Dark energy ($w \sim -1$): $\rho \sim constant$

- Radiation density decreases the fastest with time
  - Must increase fastest on going back in time
  - Radiation must dominate early in the Universe
- Dark energy with $w \sim -1$ dominates last; it is the dominant component now, and in the (infinite?) future

Radiation domination → Matter domination → Dark energy domination

Some Simple Models

The empty universe:

$\rho = 0, \Lambda = 0, K = -1$

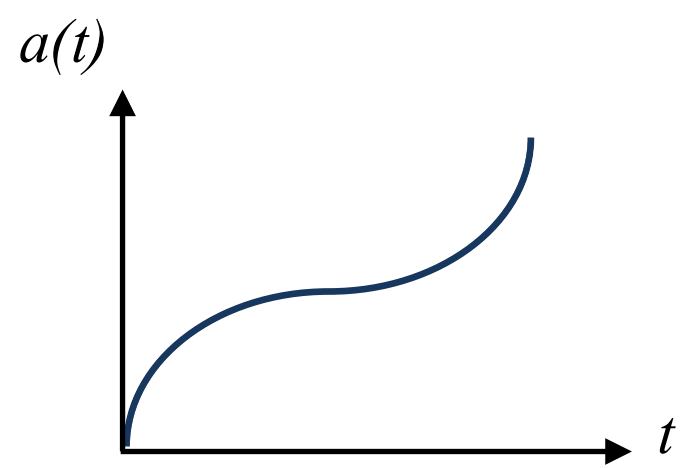
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2} + \frac{\Lambda}{3}$$

$\dot{a}^2 = c^2 \qquad da = c\,dt \qquad a = c\,t \qquad \rightarrow$ Linear expansion

Einstein's static universe:

Cosmological constant is fine-tuned to balance the self-gravity of the matter, so that both $\dot{a} = 0$ and $\ddot{a} = 0$. This requires $K = +1$ and $\Lambda_E = 4\pi G\rho/c^2$

However, this model is *unstable*, and even a slightest perturbation leads to a resumed expansion. This is Lemaitre's **loitering universe:**



$K = 0$, Matter Dominated Einstein – de Sitter (EdS) Model

Friedman Equation with $\Lambda = 0$, $K = 0$: $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_0 a^{-3}$$

$$\Rightarrow \dot{a}^2 = \frac{8}{3}\pi G\rho_0 a^{-1}$$

$$\Rightarrow \frac{\partial a}{\partial t} = \pm\sqrt{\frac{8}{3}\pi G\rho_0}\, a^{-1/2}$$

$$\Rightarrow \int a^{1/2}\, da = \pm\sqrt{\frac{8}{3}\pi G\rho_0} \int dt$$

$$\Rightarrow a^{3/2} = \pm(3/2)\sqrt{\frac{8}{3}\pi G\rho_0}\quad t$$

$$\boxed{\Rightarrow a \propto t^{2/3}}$$

Note that since matter dominates over Λ early on, this is a good approximation for the early universe

Using Matter Dominated EdS Model

Critical density: $$\rho_{cr} := \frac{3H_0^2}{8\pi G}$$

Hubble time: $t_H = 1/H_0$

For $H_0 = 70$ km/s/Mpc, $t_H = 14$ Gyr

Thus: $8\pi G\rho_{cr} / 3 = 1/t_H^2$

$$\dot{a} = (8\pi G\rho_{cr} / 3)^{1/2}\, a^{-1/2} = a^{-1/2}/t_H$$

$$a^{1/2}\, da = dt/t_H$$

Big Bang: $t = 0$, $a = 0$

Now: $a = 1$, evaluate t_{now}:

$$\int_0^1 a^{1/2}\, da = \int_0^{\text{now}} dt/t_H$$

$$\left. 2/3\; a^{3/2} \right]_0^1 = 2/3 = t_{\text{now}}/t_H$$

$$t_{\text{now}} = 2/3\; t_H$$

You can also evaluate the age of the universe at any redshift z by setting the upper limit of the integral to $a = 1 / (1+z)$

$K = 0$, Radiation Dominated Model

Similar, but with a steeper dependence of the energy density on the expansion factor

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho$$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_0 a^{-4}$$

$$\Rightarrow \dot{a} \propto a^{-1}$$

$$\Rightarrow \int a\,da \propto t$$

$$\Rightarrow a^2 \propto t$$

$$\boxed{\Rightarrow a \propto t^{1/2}}$$

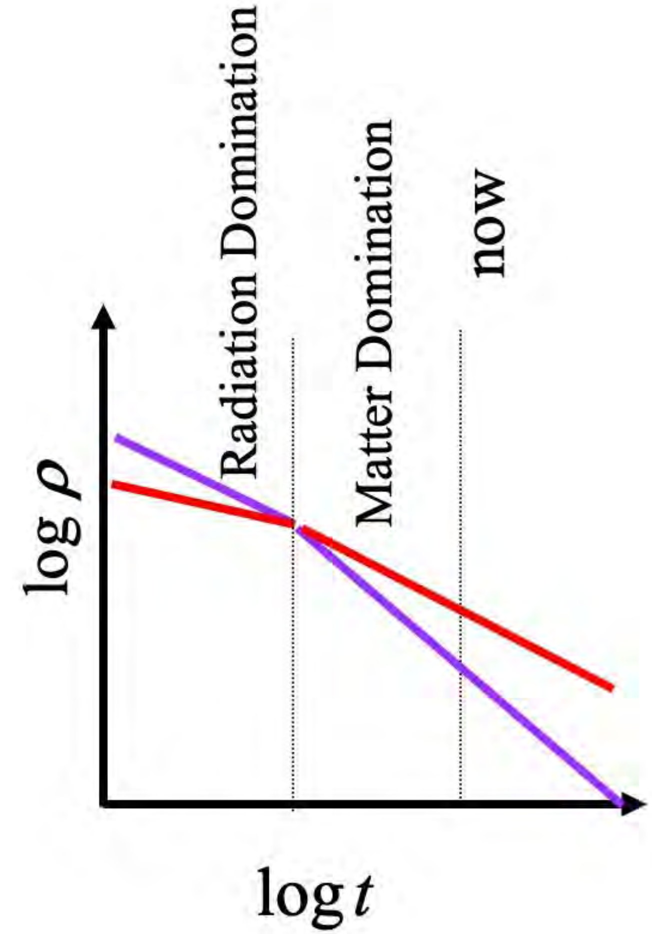
This is an excellent approximation for the very early (radiation dominated) universe

Models With Both Matter & Radiation

Harder to solve for ***ρ (t)***

However, to a good approximation, we can assume that $K = 0$ and either radiation or matter dominate

	γ-dom	m-dom
$a(t)$	$\propto t^{1/2}$	$\propto t^{2/3}$
$\rho_m \propto a^{-3}$	$\propto t^{-3/2}$	$\propto t^{-2}$
$\rho_\gamma \propto a^{-4}$	$\propto t^{-2}$	$\propto t^{-8/3}$



Generally, $$\frac{8\pi G\rho}{3} = H_0^{\,2}\left(\Omega_{\Lambda,0} + \Omega_{m,0}a^{-3} + \Omega_{\gamma,0}a^{-4}\right)$$

Positive Curvature Model: $k = +1$

Friedman Equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{kc^2}{a^2} \quad \Rightarrow \quad \dot{a}^2 = \frac{8}{3}\pi G\rho a^2 - c^2$$

if $\rho \propto a^{-3}$ or $\rho \propto a^{-4}$ then $\rho a^2 \propto a^{-n}$ $(n = 1,2)$ decreases

at some point $\dot{a}^2 = 0$

The acceleration equation is: $$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\left(\rho + 3\frac{P}{c^2}\right)$$

And since all other quantities are positive, $\ddot{a} < 0$

Therefore, a collapse is inevitable

Cosmological Constant (Λ) Dominated

Friedmann Equation: $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_\Lambda - \frac{kc^2}{a^2}$$

Which can be written as: $$\dot{a}^2 = C_0\rho a^2 - kc^2$$

Assuming that a is allowed to grow, then eventually $C_0 a^2$ dominates over $-kc^2$ no matter what value of $\boldsymbol{k}$

If $\rho_\Lambda > 0$ then $\dot{a} > 0$ → Universe expands for ever

(Note that if $\rho_\Lambda < 0$ things get more complicated)

Asymptotically the expansion becomes exponential:

$\dot{a}/a \sim const.$ $\quad da/a \sim dt$ $\quad a(t) \sim exp(t)$

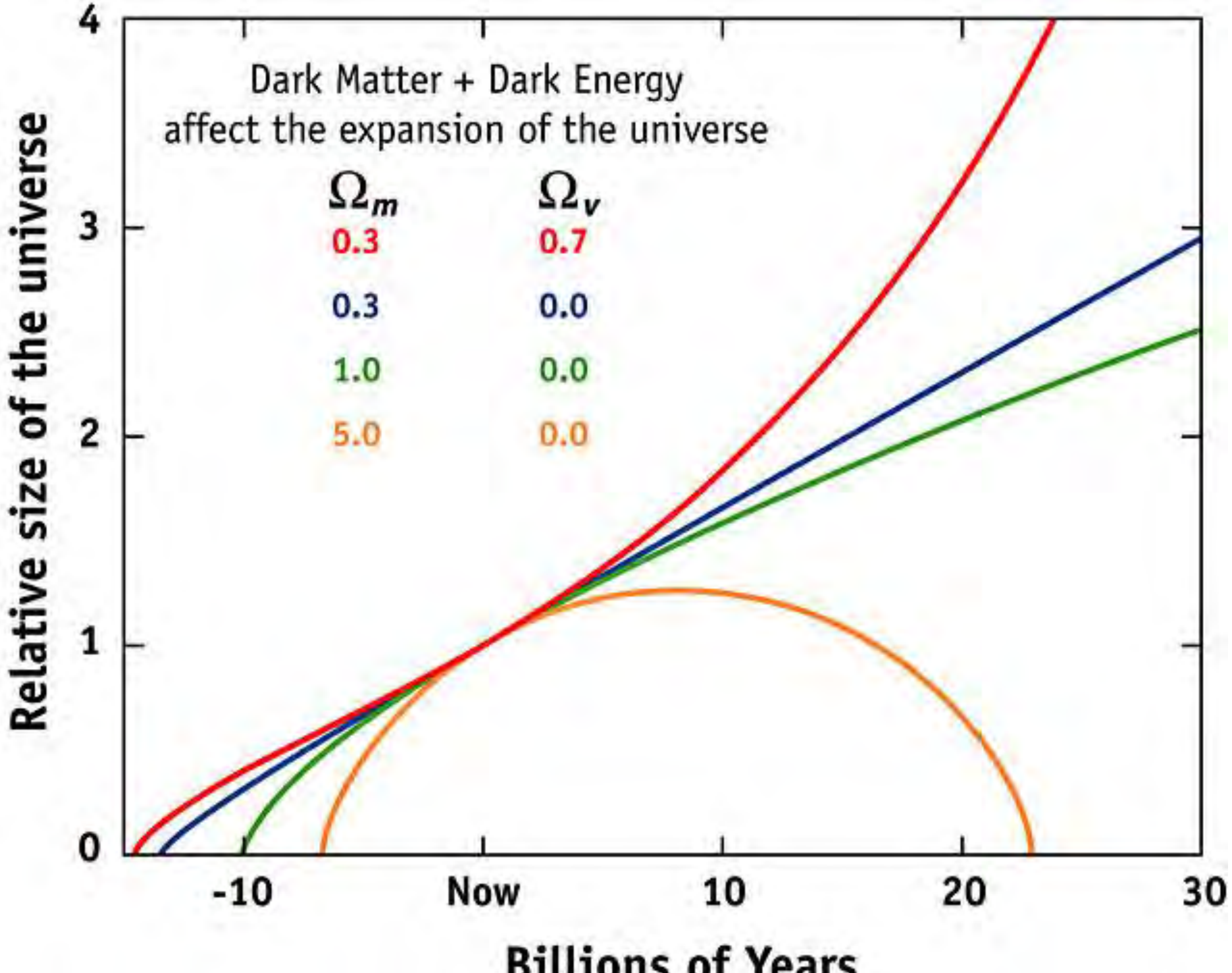
This is exactly what happens during the inflation era

Dynamics of the Universe

In general: $a(t) \sim t^{2/[3(w+1)]}$ (w = Equation of state parameter)

- **Matter dominated** ($w = 0$): $a \sim t^{2/3}$
  - Decelerating (because the power of t is < 1)
- **Radiation dominated** ($w = 1/3$): $a \sim t^{1/2}$
  - Decelerating (because the power of t is < 1)
- **Cosmological constant** ($w = -1$): $a \sim e^{\lambda t}$
  - Accelerating (exponentially, since the derivative of an exponential is also an exponential)
- Where is the transition?
  - $w > -1/3$ decelerating
  - $w < -1/3$ accelerating

Examples of Models



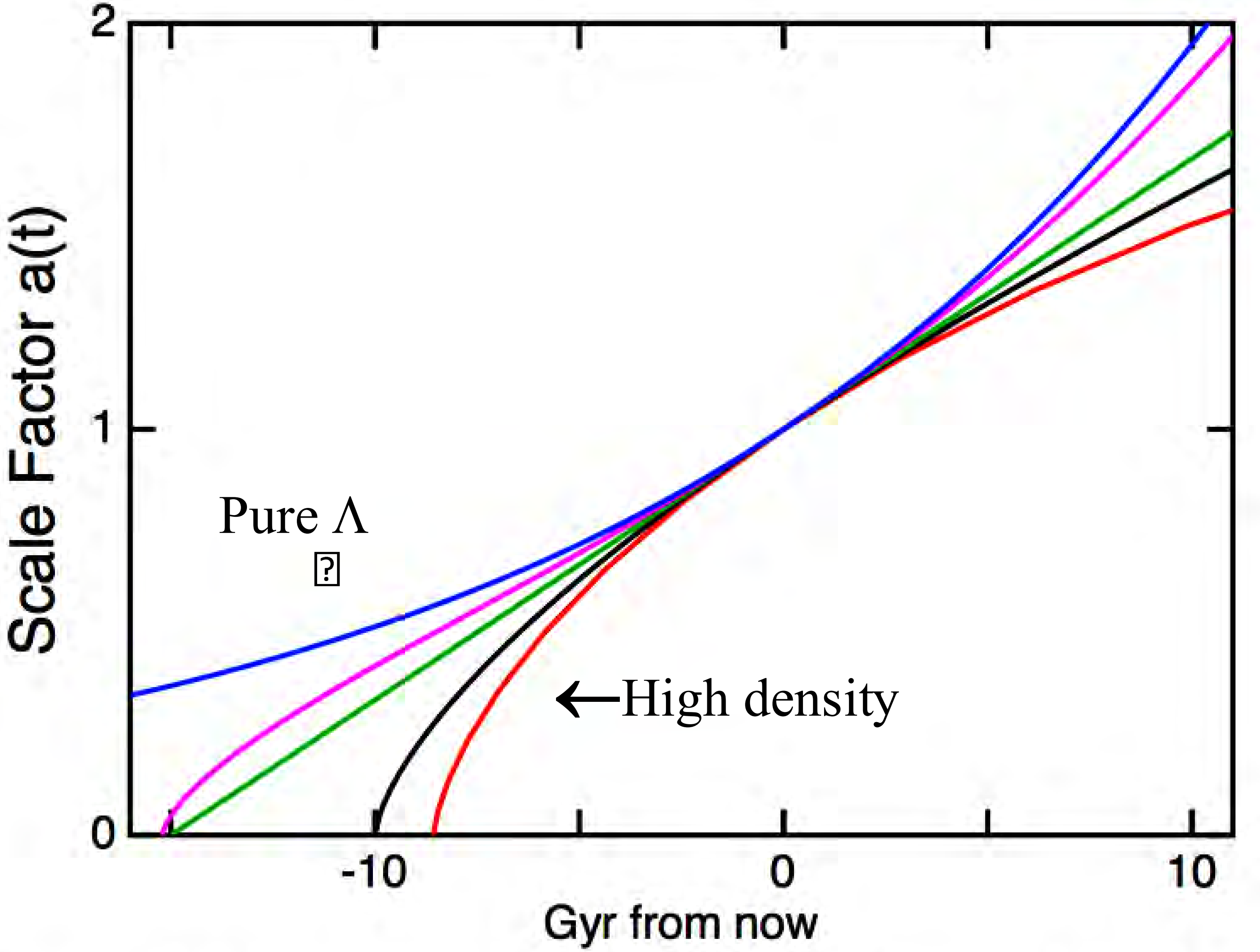
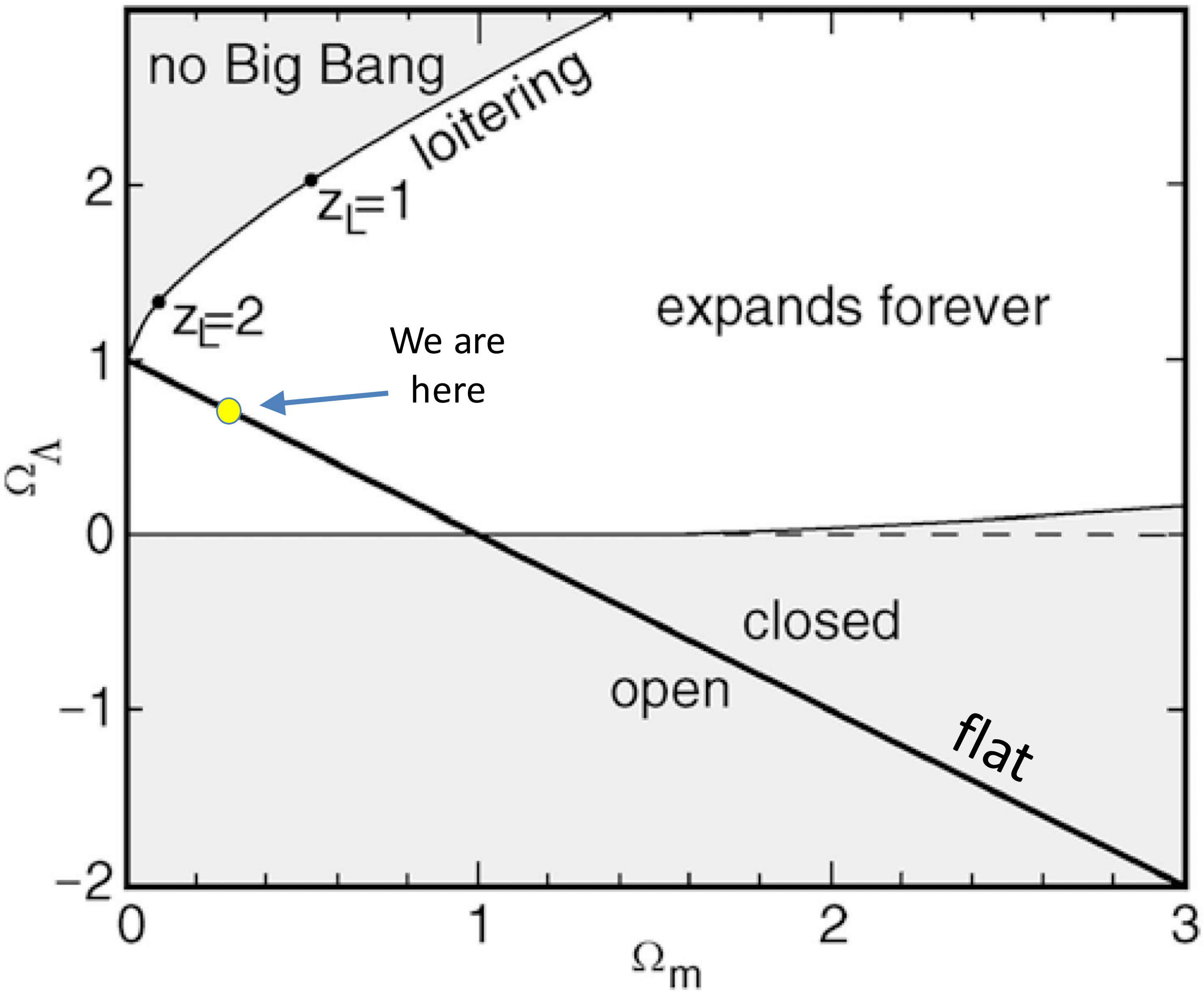


Fig. 10.— Scale factor *vs.* time for 5 different models: from top to bottom having $(\Omega_{m\circ}, \Omega_{v\circ}) = (0, 1)$ in blue, (0.25,0.75) in magenta, (0,0) in green, (1,0) in black and (2,0) in red. All have $H_\circ = 65$.

Classification of the Models



(Ignoring Ω_{rad}, since it is negligible for most of the history of the universe)

Distances in Cosmology

A convenient unit is the **Hubble distance *or* radius,**

$$D_H = c\,/\,H_0 = 4.283\ h_{70}^{-1}\ \text{Gpc} = 1.322\times 10^{28}\ h_{70}^{-1}\ \text{cm}$$

and the corresponding **Hubble time**,

$$t_H = 1\,/\,H_0 = 13.98\ h_{70}^{-1}\ \text{Gyr} = 4.409\times 10^{17}\ h_{70}^{-1}\ \text{s}$$

At low z' s, distance $D \approx z\, D_H$. But more generally, the comoving distance to a redshift z is:

$$D_{\rm C} = D_{\rm H} \int_0^z \frac{dz'}{E(z')}$$

where

$$E(z) \equiv \sqrt{\Omega_{\rm r}\,(1+z)^4 + \Omega_{\rm m}\,(1+z)^3 + \Omega_{\rm k}\,(1+z)^2 + \Omega_{\Lambda}}$$

$$\Omega_{\rm k} = 1 - \Omega_{\rm r} - \Omega_{\rm m} - \Omega_{\Lambda}$$

In general, this integral is not solvable analytically

Distances in Cosmology

But the quantity really useful in computing the various physical quantities of interest is the "transverse comoving distance", where we account for the curvature:

$$D_M = \begin{cases} D_H \frac{1}{\sqrt{\Omega_k}} \sinh\left[\sqrt{\Omega_k}\, D_C/D_H\right] & \text{for } \Omega_k > 0 \\ D_C & \text{for } \Omega_k = 0 \\ D_H \frac{1}{\sqrt{|\Omega_k|}} \sin\left[\sqrt{|\Omega_k|}\, D_C/D_H\right] & \text{for } \Omega_k < 0 \end{cases}$$

$$\Omega_M \equiv \frac{8\pi G \rho_0}{3 H_0^2} \qquad \Omega_M + \Omega_\Lambda + \Omega_k = 1$$

$$\Omega_\Lambda \equiv \frac{\Lambda c^2}{3 H_0^2}$$

And usually we can neglect Ω_r

Distances in Cosmology

In general this is non-analytic. In a special case of a $\Lambda = 0$ Universe, we have $q_0 = \Omega_0 / 2$, and:

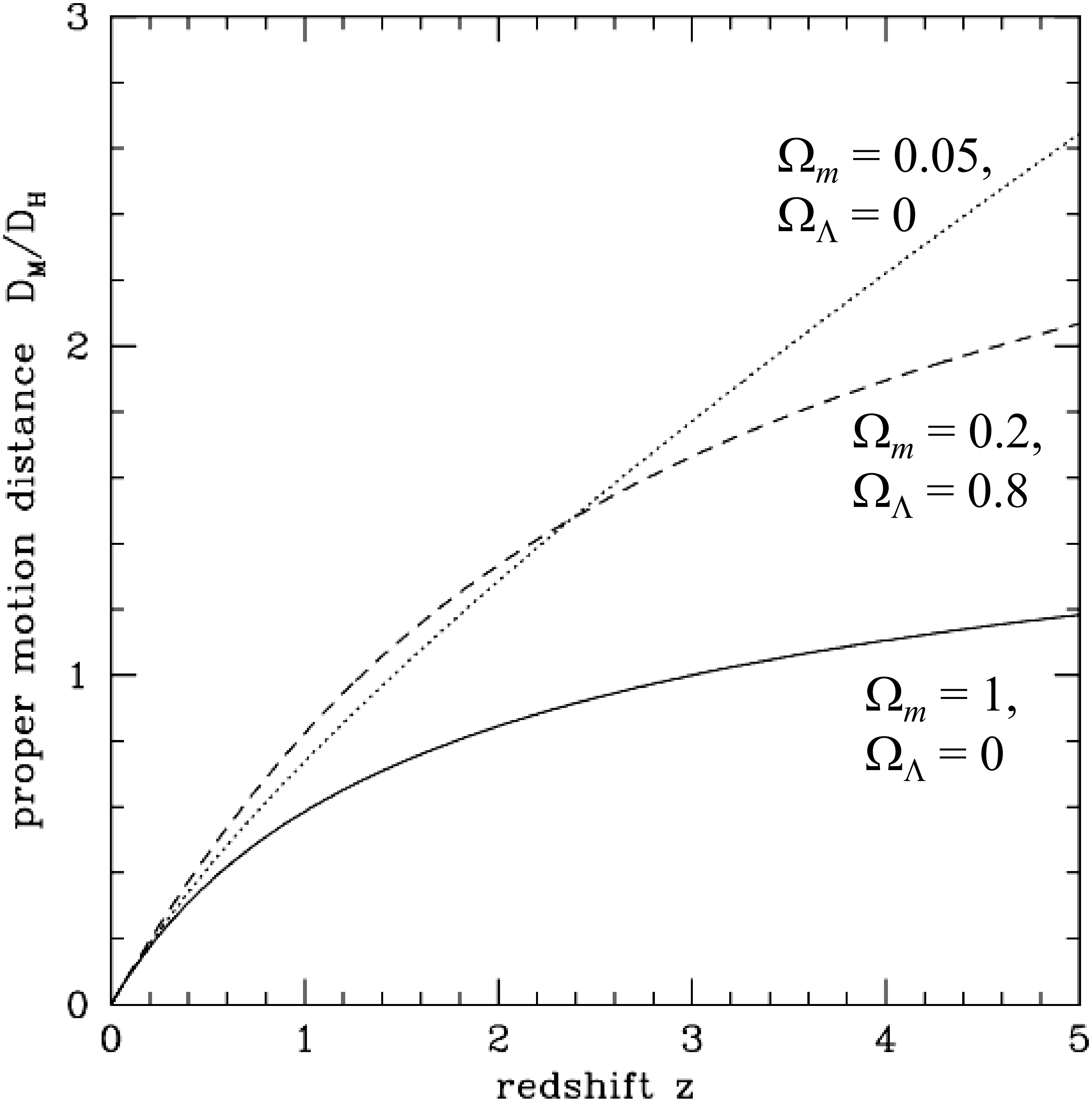
$$d_p = \frac{c}{H_0 q_0^2 (1+z)} \left\{ q_0 z + (q_0 - 1)\left[(2q_0 z + 1)^{1/2} - 1 \right] \right\}$$

For a non-zero Λ universe:

$$d_p = |\Omega_k|^{-\frac{1}{2}} \sinh \left\{ |\Omega_k|^{\frac{1}{2}} \int_0^z \left\{ (1+z)^2 (1 + \Omega_M z) - \Omega_\Lambda z (2+z) \right\}^{\frac{1}{2}} dz \right\}$$

If $\Omega_k > 0$ then the *sinh* becomes a *sin* and if $\Omega_k = 0$ then the *sinh* and the Ω_k drop out and all that's left is the integral, which has to be evaluated numerically.

Comoving Distance



Luminosity Distance

In relativistic cosmologies, observed flux (bolometric, or in a finite bandpass) is:

$$f = L \,/\, [\,(4\pi\, D^2)\,(1+z)^2\,]$$

One factor of $(1+z)$ is due to the energy loss of photons, and one is due to the time dialation of the photon rate.

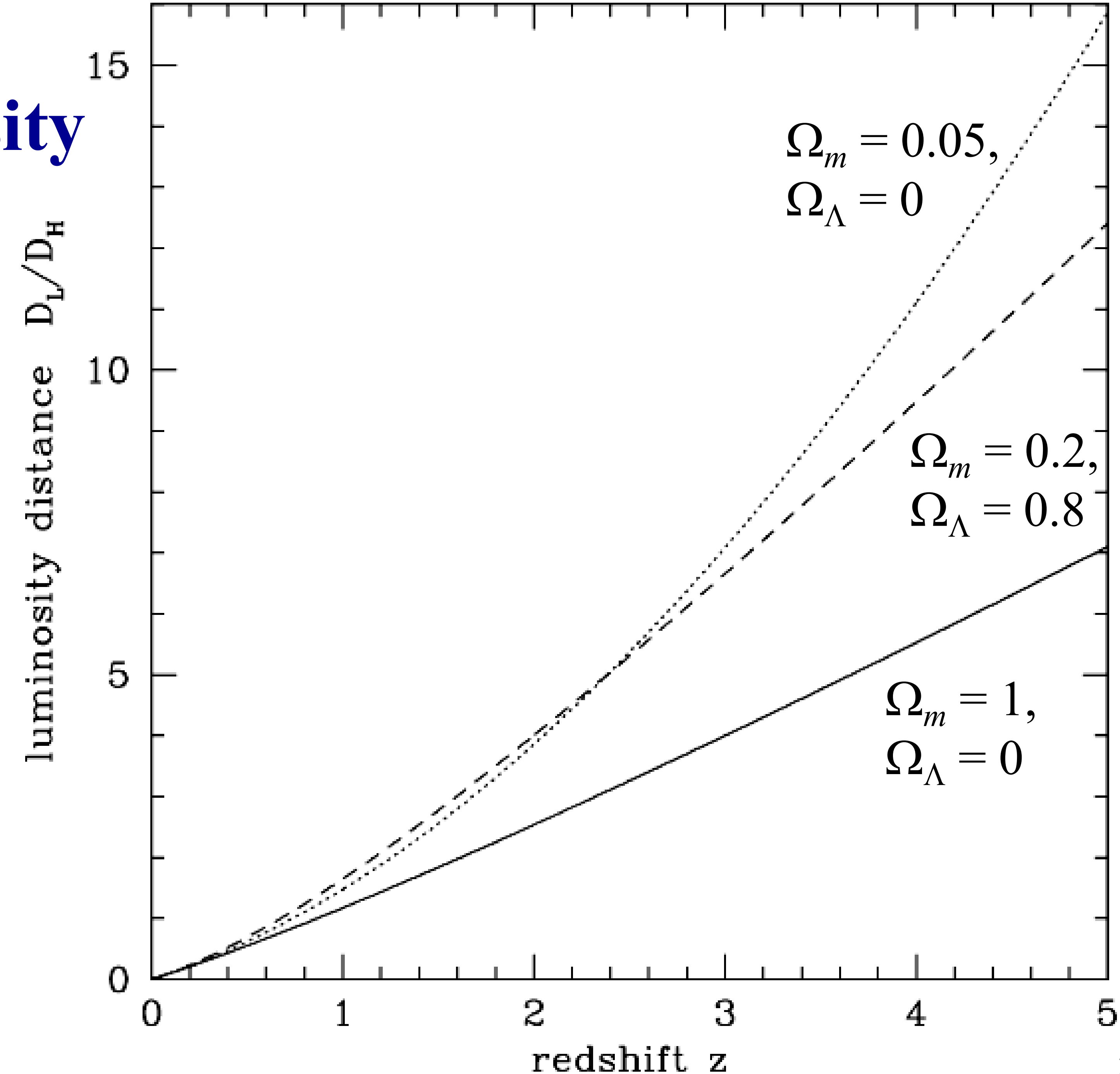
A **luminosity distance** is defined as $D_L = D\,(1+z)$, so that $f = L \,/\, (4\pi\, D_L{}^2)$.

For a specific flux, however,

$$S_\lambda = \frac{1}{(1+z)} \frac{L_{\lambda/(1+z)}}{L_\lambda} \frac{L_\lambda}{4\pi\, D_{\mathrm{L}}^2}$$

(since Angstroms are also stretched by 1+z)

Luminosity Distance



Angular Diameter Distance

Angular diameter of an object with a fixed ***comoving*** size X is by definition

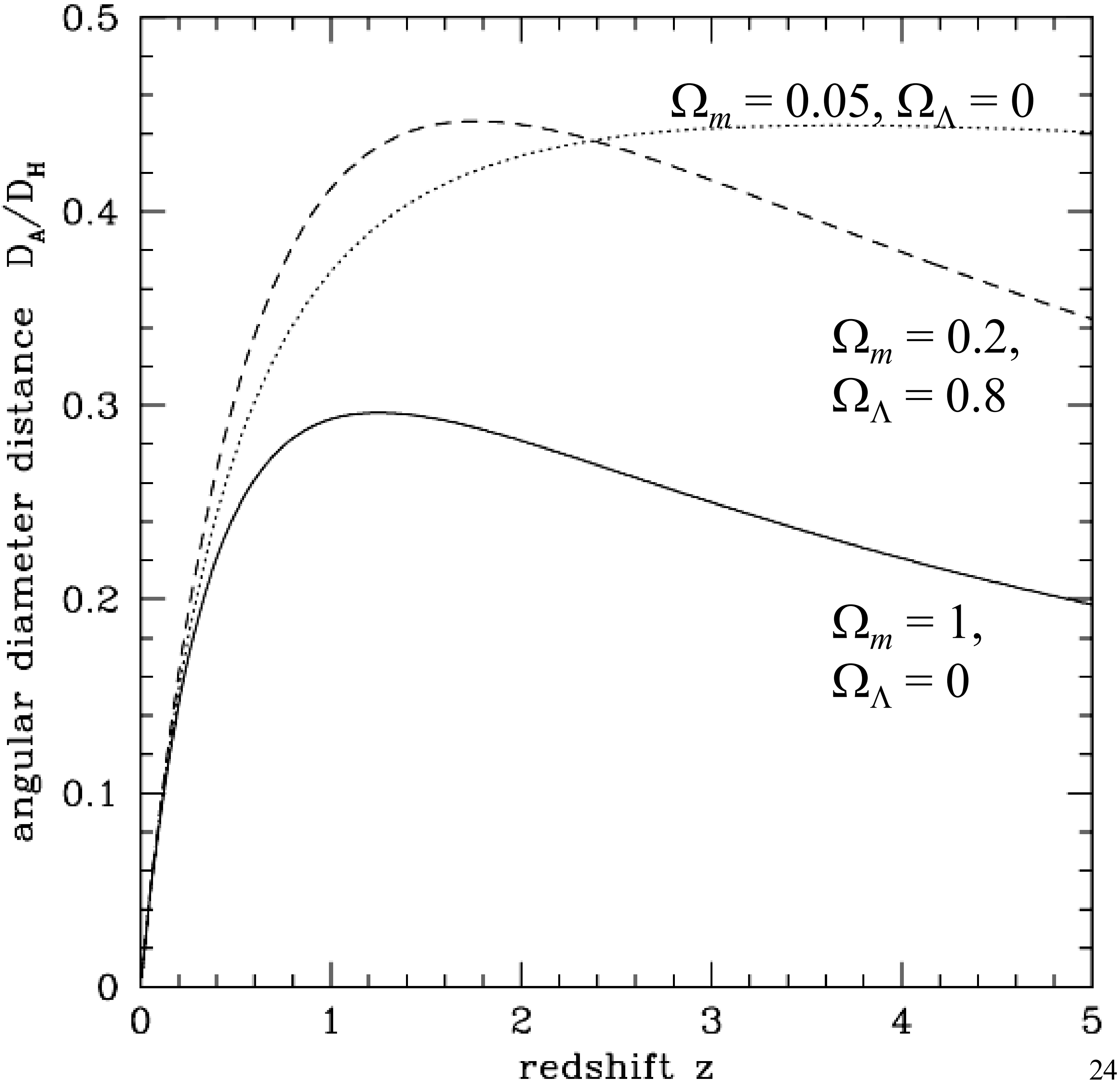
$$\theta = X / D$$

However, an object with a fixed ***proper*** size X is $(1+z)$ times larger than in the comoving coordinates, so its apparent angular diameter will be

$$\theta = (1+z)\, X / D$$

Thus, we define the **angular diameter distance** $D_A = D / (1+z)$, so that the angular diameter of an object whose size is fixed in proper coordinates is $\theta = X / D_A$

Angular Diameter Distance



Volume Element

$$dV_C = D_H \frac{(1+z)^2 D_A^2}{E(z)} \, d\Omega \, dz$$

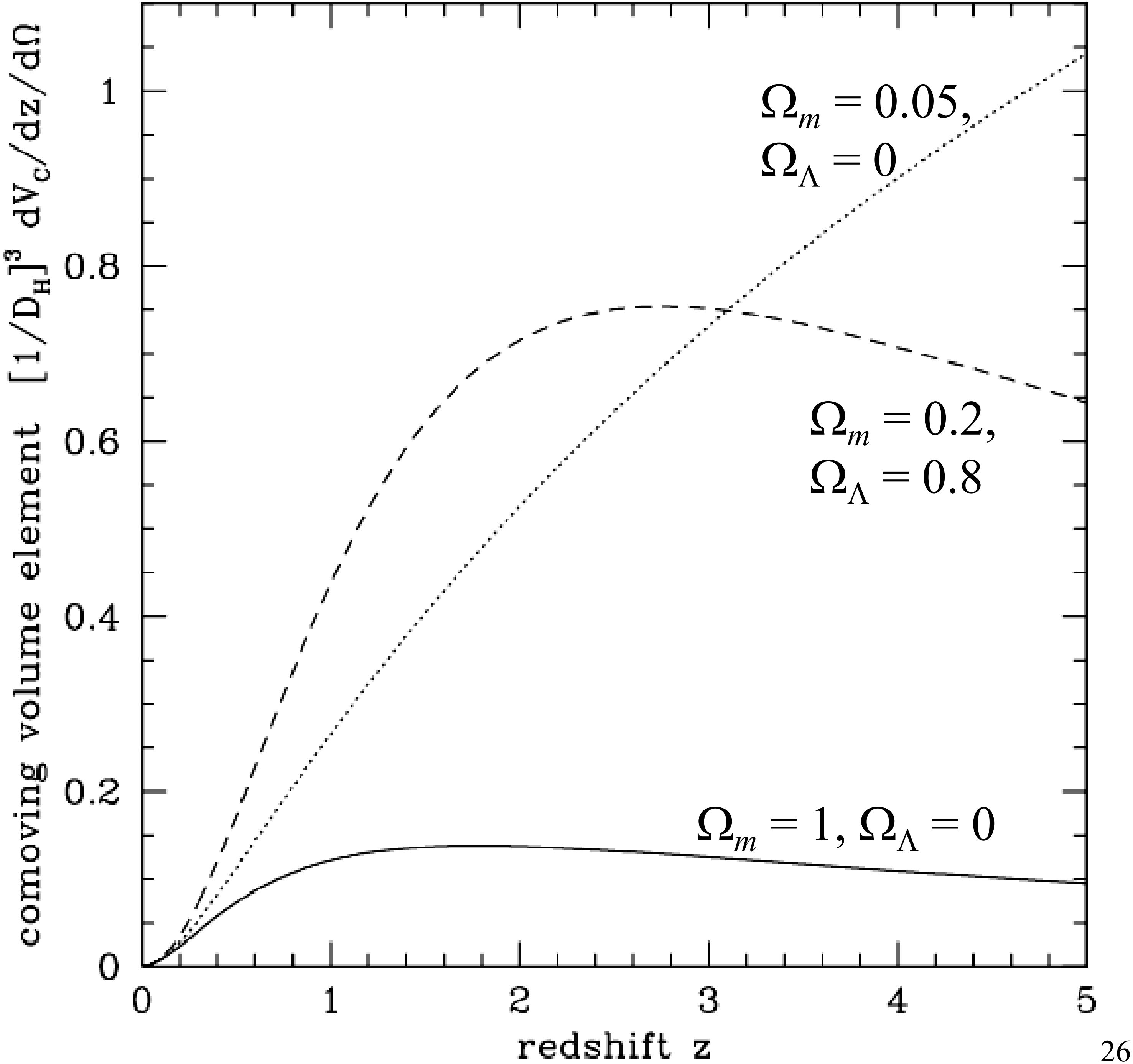
This is useful, e.g., when computing the source counts.

Generally, it has to be evaluated numerically.

The total volume out to some *z*, over the whole sky, is:

$$V_C = \begin{cases} \left(\frac{4\pi D_H^3}{2\Omega_k}\right) \left[\frac{D_M}{D_H}\sqrt{1+\Omega_k \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_k|}} \operatorname{arcsinh}\left(\sqrt{|\Omega_k|}\,\frac{D_M}{D_H}\right)\right] & \text{for } \Omega_k > 0 \\ \frac{4\pi}{3} D_M^3 & \text{for } \Omega_k = 0 \\ \left(\frac{4\pi D_H^3}{2\Omega_k}\right) \left[\frac{D_M}{D_H}\sqrt{1+\Omega_k \frac{D_M^2}{D_H^2}} - \frac{1}{\sqrt{|\Omega_k|}} \arcsin\left(\sqrt{|\Omega_k|}\,\frac{D_M}{D_H}\right)\right] & \text{for } \Omega_k < 0 \end{cases}$$

Volume Element



Age and Lookback Time

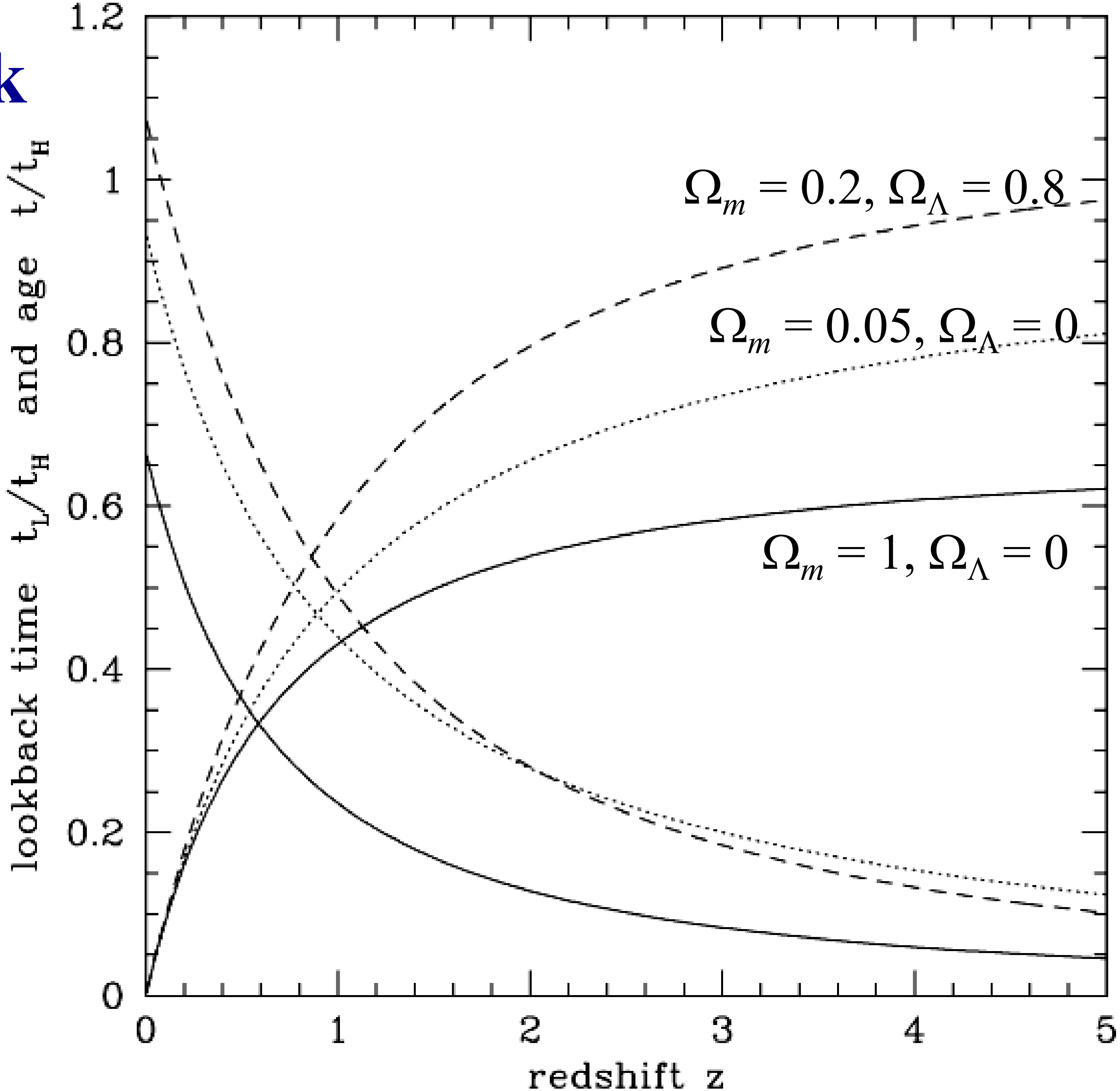
The time elapsed since some redshift *z* is:

$$t_{\rm L} = t_{\rm H} \int_0^z \frac{dz'}{(1+z')\,E(z')}$$

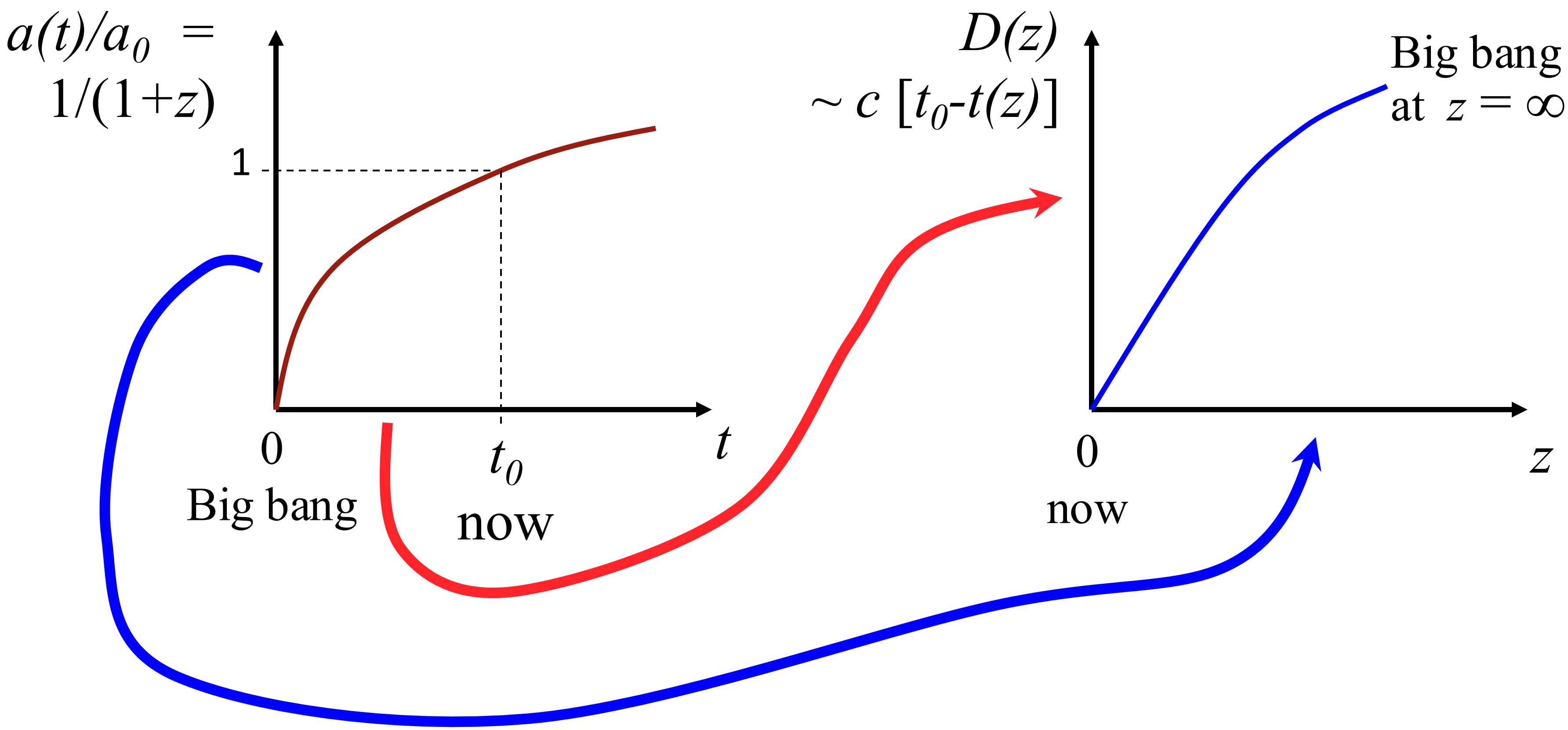
Generally it has to be integrated numerically, except in some special cases, such as $\Lambda = 0$.

Integrating to infinity gives the age of the universe, and the difference of the two is the age at a given redshift.

Lookback Time and Age

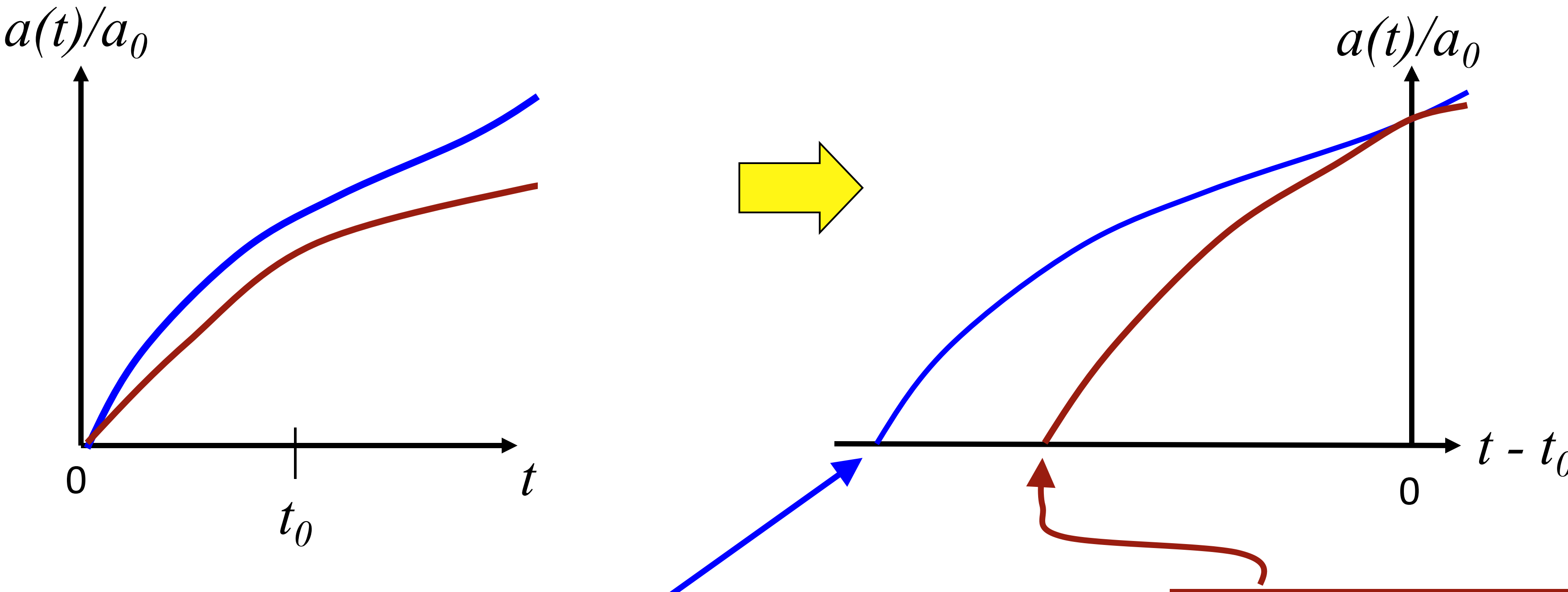


The Basis of Cosmological Tests



All cosmological tests essentially consist of comparing some measure of (relative) distance (or look-back time) to redshift. Absolute distance scaling is given by the H_0.

Cosmological Tests: Expected Generic Behavior of Various Models



Models with a lower density and/or positive Λ expand faster, are thus larger, older today, have more volume and thus higher source counts, at a given *z* sources are further away and thus appear fainter and smaller

Models with a higher density and lower Λ behave exactly the opposite

Key Concepts From Today

- To solve the Friedman eq. we need to know how the densities of different components change with the expansion factor
  - That determines what is dominant factor when
  - Usually expressed through the equation of state, $p = w\,\rho$
- Solving the simple (single component) models of *a(t)*
  - Einstein–de Sitter model, $\Omega_{m+r} = 1$, is a good approximation for the early universe
  - Pure $\Lambda > 0$ model expands exponentially
- Definition of different distances (comoving, luminosity, angular diameter)
- Expected general behavior of cosmological models as a function of density parameters