8.1 Structure Formation: Introduction and the Growth of Density Perturbations
Structure Formation and Evolution

From this \((\Delta \rho/\rho \sim 10^{-6})\)

to this
\((\Delta \rho/\rho \sim 10^2)\)

to this
\((\Delta \rho/\rho \sim 10^6)\)
Origin of Structure in the Universe

• Origin and evolution of the structure in the universe (galaxies, large-scale structures) is a central problem in cosmology

• Structure is generally thought to arise through a growth of density perturbations which originate in the early universe
  – We think they came from quantum fluctuations in the scalar field that caused inflation, and were then amplified by the exponential inflation of the universe

• What do we know about the early structure formation?
  – We see CMB fluctuations with $\delta T/T \sim 10^{-6} \sim \Delta \rho/\rho$, since radiation and baryons are coupled before recombination
  – High-$z$ objects: We observe galaxies and quasars at $z > 6$. A galaxy requires an overdensity of $\sim 10^6$ relative to the mean

• Can we get to the such large density enhancements required for galaxies, clusters, etc., by evolving the small fluctuations we see in the CMB?
The Astrophysics of Structure Formation

Messy! Must be done numerically
Linear Growth of Density Fluctuations

How do fluctuations in density evolve with time? For simplicity, consider a flat, matter-dominated universe with $\Omega_m = 1$ (a good approx. for the early times).

The Friedmann equation is:

$$H^2 - \frac{8}{3}\pi G \rho = -\frac{k}{R^2} = 0$$

Now consider a small area with a slight overdensity of matter, it will evolve slightly differently:

$$H^2 - \frac{8}{3}\pi G \rho' = -\frac{k}{R^2}$$

Subtract the two equations:

$$-\frac{8}{3}\pi G (\rho' - \rho) = -\frac{k}{R^2}$$

Rewrite this as:

$$\rho' - \rho = -\frac{3k}{8\pi G R^2}$$
We can define $\delta$ as the *fractional overdensity*:

$$\delta \equiv \left( \frac{\rho' - \rho}{\rho} \right) = -\frac{3k}{8\pi G R^2 \rho}$$

But note that:

$$\delta \sim \frac{1}{R^2 \rho} \sim \frac{1}{R^2 R^{-3}} \sim R$$

Since $R \sim (1+z)^{-1}$,

$$\delta \sim (1 + z)^{-1}$$

And this is the linear growth of density fluctuations.

Evolution of the density contrast between two redshifts is then

$$\frac{\delta_f}{\delta_i} = \frac{(1 + z)_i}{(1 + z)_f}$$
Linear Growth of Density Fluctuations

Let’s try it from recombination to $z = 5$: 

$$\delta_f = 10^{-5} \left( \frac{1 + 1000}{1 + 5} \right) \sim 0.002$$

Oops! It should be $\sim 10^6$ - now what?

We need to start with larger fluctuations to get galaxies, etc. We neglected the effects of non-baryonic dark matter. Before recombination, the radiation prevented the baryonic fluctuations of collapsing, but fluctuations in non-baryonic dark matter can start growing much earlier!

The fluctuations we see in the CMB (in the baryonic matter) are sitting on top of much stronger fluctuations in the non-baryonic matter. Once recombination occurs, the baryons can fall into the dark matter concentrations to form galaxies …
8.2 Collapse of Density Fluctuations
Evolution of the Spherical Top-Hat Model

Schematic evolution:
- Density contrast grows as universe expands.
- Perturbation “turns around” at $R = R_{\text{turn}}$, $t = t_{\text{turn}}$.
- If exactly spherical, collapses to a point at $t = 2 \ t_{\text{turn}}$.
- Realistically, bounces and virializes at radius $R = R_{\text{virial}}$. 

expansion of the flat background universe solution for a realistic overdensity with some aspherical perturbation

closed universe solution for exact spherical symmetry
Evolution of the Spherical Top-Hat Model

We can use the virial theorem to derive the final radius of the collapsed perturbation. Let perturbation have mass $M$, kinetic energy $<KE>$, and gravitational potential energy $<PE>$:

**Virial theorem:** $<PE> + 2 <KE> = 0$ (in equilibrium)

**Energy conservation:** $<PE> + <KE> = \text{const.}$

\[-\frac{GM^2}{R_{\text{turn}}} = -\frac{GM^2}{R_{\text{virial}}} + \langle KE \rangle = -\frac{GM^2}{2R_{\text{virial}}}\]

Energy at turnaround

Energy when virialized

Conclude: $R_{\text{virial}} = \frac{1}{2} R_{\text{turn}}$
Evolution of the Spherical Top-Hat Model

We can apply the solution for a closed universe to calculate the final overdensity:

Friedmann equation: \( \dot{a}^2 = \frac{A^2}{a} - kc^2 \)

where \( A^2 = \left( \frac{8\pi G}{3} \right) \rho_0 a_0^3 \)

Solutions:

\[
a = \left( \frac{3A}{2} \right)^{2/3} t^{2/3} \quad k = 0 \quad \text{solution derived previously}
\]

\[
a = \frac{1}{2} \frac{A^2}{c^2} \left( 1 - \cos \Psi \right)
\]

\[
t = \frac{1}{2} \frac{A^2}{c^3} \left( \Psi - \sin \Psi \right) \quad \text{Parametric solution for a closed,} \\
\quad k = 1 \text{ universe (see the derivation elsewhere, e.g., in Ryden’s book)}
\]
Evolution of the Spherical Top-Hat Model

Now we can calculate the turnaround time for the collapsing sphere by finding when the size of the small closed universe has a maximum:

\[ \frac{da}{d\Psi} = \frac{1}{2} \frac{A^2}{c^2} \sin \Psi = 0 \rightarrow \Psi = 0, \pi, \ldots \]

At turnaround, \( \Psi = \pi \), which corresponds to time \( t_{\text{turn}} = \frac{1}{2} \frac{A^2}{c^3} \pi \)

The scale factors of the perturbation and of the background universe are:

\[ a_{\text{sphere}} = \frac{1}{2} \frac{A^2}{c^2} \left(1 - \cos \pi \right) = \frac{A^2}{c^2} \]

\[ a_{\text{background}} = \left( \frac{3A}{2} \right)^{2/3} \quad t_{\text{turn}}^{2/3} = \left( \frac{3\pi}{4} \right)^{2/3} \frac{A^2}{c^2} \]
Evolution of the Spherical Top-Hat Model

The density contrast at turnaround is therefore:

\[
\frac{\rho_{\text{sphere}}}{\rho_{\text{background}}} = \left( \frac{a_{\text{sphere}}}{a_{\text{background}}} \right)^{-3} = \frac{9\pi^2}{16}
\]

At the time when the collapsing sphere virialized, at \( t = 2 t_{\text{turn}} \):
- Its density has increased by a factor of 8 (since \( R_{\text{turn}} = 2 R_{\text{vir}} \))
- Background density has decreased by a factor of \((2^{2/3})^3 = 4\)

A collapsing object virializes when its density is greater than the mean density of the universe by a factor of \(18 \pi^2 \sim 180\) (and this is about right for the large-scale structure today)

First objects to form are small and dense (since they form when the universe is denser). These later merge to form larger structures: “bottom-up” structure formation
Non-Spherical Collapse

Real perturbations will not be spherical. Consider a collapse of an ellipsoidal overdensity:

- Perturbation first forms a "pancake"
- Then forms filaments
- Then forms clusters

The expansion turns into collapse along the shortest axis first …
Then along the intermediate axis
Then along the longest axis

This kinds of structures are seen both in numerical simulations of structure formation and in galaxy redshift surveys
How Long Does It Take?

The (dissipationless) gravitational collapse timescale is on the order of the free-fall time, \( t_{ff} \):

The outermost shell has acceleration \( g = GM/R^2 \)
It falls to the center in:

\[
t_{ff} = \left(\frac{2R}{g}\right)^{1/2} = \left(\frac{2R^3}{GM}\right)^{1/2} \approx \left(\frac{2}{G\rho}\right)^{1/2}
\]

Thus, low density lumps collapse more slowly than high density ones. More massive structures are generally less dense, take longer to collapse. For example:

For a galaxy: \( t_{ff} \sim 600 \) Myr \( (R/50\text{kpc})^{3/2} (M/10^{12}M_\odot)^{-1/2} \)

For a cluster: \( t_{ff} \sim 9 \) Gyr \( (R/3\text{Mpc})^{3/2} (M/10^{15}M_\odot)^{-1/2} \)

So, we expect that galaxies collapsed early (at high redshifts), and that clusters are still forming now. This is as observed!
8.3 The Power Spectrum of Density Fluctuations
Quantifying the Density Field

- Consider the overall fluctuating density field as a superposition of waves with different wavelengths, phases, and amplitudes.

- Then we can take a Fourier transform and measure the Power on different scales, expressed either as wavelengths $l$ or frequencies or wave numbers $k = 1/l$.

- **Density fluctuations field**: $\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}$

- **Fourier Transform of density field**: $\delta_k = \sum \delta e^{-ik \cdot r}$

- **Its Power Spectrum**: $P(k) = \langle |\delta_k|^2 \rangle$

It measures the power of fluctuations on a given scale $k$.

Sometimes one uses the “mass scale”; $M \sim k^{-3 \left(3+n\right) / 4}$.
Density Fluctuation Spectrum

• A common assumption is that the fluctuations have the same amplitude $\delta \sim 10^{-4}$ when they enter the horizon.
• This gives a *scale-free* or *Harrison-Zeldovich* spectrum.
• Such spectrum is also predicted in the inflationary scenario.
• More generally, it can be a power-law:

$$P(k) = \langle |\delta_k|^2 \rangle \propto k^n$$

(for Harrison-Zeldovich, $n = 1$)

• The fluctuations grow under self-gravity, so the power increases (if all scales grow equally, the spectrum just shifts up in log), but they can be also erased or damped through various processes - the shape of the spectrum changes.
Types of Primordial Fluctuations

- **Adiabatic**: Corresponding to changes in volume in the early universe. Fluctuations in the number density of photons and matter particles are equal, but their mass densities change differently
  - Can be described as “1/f noise”; an example is the Harrison-Zeldovich spectrum, with equal power on all mass scales (not frequencies). This is also what we see in the CMB

- **Isocurvature**: Start with no perturbations in the total mass/energy density field, but with fluctuations in the matter opposed to the radiation $\delta_\gamma = -\delta_m$

- **Isothermal**: Radiation field is unperturbed, fluctuations in matter only (rarely considered nowadays)
  - Can be described as “white noise” (equal power on all frequencies, so most power on small mass scales)
Dark Matter and Damping of Fluctuations

- Different types of dark matter form structure differently
- Baryonic dark matter is coupled to radiation, so it does not help in forming structure prior to the recombination
- Fluctuations can be erased or damped by sound waves (this is also called the Meszaros effect). This is important for slowly moving DM particles, i.e., cold dark matter (CDM)
- They can be erased by free streaming of relativistic particles, i.e., hot dark matter (HDM); diffusion of photons, which then “drag along” the baryons in the radiation-dominated era, does the same thing (this is also called the Silk damping)
- Thus HDM vs. CDM make very different predictions for the evolution of structure in the universe!
- In any case, the smaller fluctuations are always erased first
Damping of Fluctuations

The primordial (H-Z) spectrum

Sound waves diminish the strength of small scale fluc’s for the CDM case

Relativistic streaming and photon diffusion erase them completely for the HDM case
Damping of Fluctuations

The same, but represented in terms of the mass scales

Note that the cutoff mass is \( \sim \) galaxy cluster mass
Structure Formation in the HDM Scenario

- HDM particles are relativistic, their speed means they can escape from small density fluctuations. This removes mass from the fluctuation and essentially smooths out any small fluctuations.

- For example, large amounts of neutrinos will dissolve away mass fluctuations smaller than $10^{15} \ M_\odot$ before recombination. Temperature fluctuations in CMB should have large angular wavelengths and large amplitudes.

- Thus, only big lumps survive to collapse. These lumps are on the scale of clusters of galaxies, with relatively low overdensities, and thus collapse slowly. After the big structures have collapsed, fragmentation into smaller structures (like galaxies) can occur. Structure forms slowly, “top-down”. Galaxies form very late in the universe’s history (~ now).

- This isn’t what we see, so HDM doesn’t work!
Structure Formation in the CDM Scenario

- CDM particles don’t diffuse out of small lumps. So lumps exist on all scales, both large and small.
- Small lumps collapse first, big things collapse later. The larger overdensities will incorporate smaller things as they collapse, via merging.
- Structure forms early with CDM, and it forms “bottom-up.” Galaxies form early, before clusters, and clusters are still forming now.
- This picture is known as “hierarchical structure formation.”
- This closely matches what we observe.
- It also produces the right kind of CMB fluctuations.
- Thus, while we know that massive neutrinos (which would constitute HDM) do exist, most of the dark matter must be cold!
A Comparison of Structure Formation Using Numerical Simulations

Box size = 100 h\(^{-1}\) Mpc

(Jenkins & Frenk 2004)
8.4 The Growth of Density Fluctuations
The Growth of Fluctuations

- Prior to matter-radiation equality, perturbations are prevented from growing due to radiation pressure.
- Pressure opposes gravity effectively for all wavelengths below the Jeans Length:

\[
\lambda_J = c_s \sqrt{\frac{\pi}{G\rho}}
\]

where \( c_s \) is the speed of sound and the equation of state is:

\[
c_s^2 = \frac{\partial P}{\partial \rho}
\]

- Jeans Length is the scale at which sound waves can cross an object in about the time for gravitational collapse.
- In a radiation-dominated universe, Jeans Length is close to the horizon size.
- At matter-radiation equality, the sound speed starts to drop, and fluctuations can grow.
The Evolution of the Jeans Mass

\[ \log \left( \frac{M}{M_\odot} \right) \]

Epoch of equality of matter and radiation energy densities

Epoch of recombination

\[ \log \text{(scale factor)} \]
The Growth of Fluctuations

• *Horizon scale at matter-radiation equality defines a particular scale of fluctuations*

• After matter-radiation equality baryons still affected by photon pressure due to Thomson scattering and perturbations oscillate as sound waves. These are responsible for the Doppler peaks we observe in the CMB.

• After $z_{\text{rec}}$ the fluctuations can grow, and any fluctuations or potential wells in the dark matter dominated density field will gravitationally attract baryons. So quickly the density fields will be similar again.
The Evolution of Fluctuations

DM fluctuations grow, but baryons don’t follow

Baryons collapse into potential wells of DM

Radiation dominated

Matter dominated

Post-recombination
The Evolution of Fluctuations

![Graph showing the evolution of fluctuations with logarithmic scales for $|\Delta|$ and $\log_{10} R$.]

- $\Delta_D$
- $\Delta_B$
- $\Delta_{rad}$
The Power Spectra

\[ \log(P(k)/h^3 \text{Mpc}^{-3}) \]

\[ \log(k/h \text{ Mpc}^{-1}) \]
And this is what we observe in CMBR:
Post-Recombination Universe

• Fluctuations continue to grow, and soon enter the non-linear regime, at progressively ever larger scales
• This evolution can only be followed numerically
• Once the gas infalling into the DM potential wells (dark halos) is compressed enough, *dissipative effects* begin to play a significant role (shocks, star formation, feedback). These are even difficult to simulate numerically!
• Energy dissipation accelerates collapse, and leads to higher densities which cannot be achieved by the dissipationless collapse (a factor of 8…). This is often called “*cooling*”
• A good mechanism is inverse Compton scattering of CMB photons on hot electrons; this process is effective only at \( z < 100 \) or so, since the CMB is too hot at higher \( z \)’s
Cooling and Structure Formation

Define the *cooling time* as:

\[ t_{cool} = E / |dE/dt| = 3 \, n \, k \, T / \left[ n^2 \, \Lambda(T) \right] \]

where \( n \) is the particle number density, \( k \) is the Boltzmann constant, and \( \Lambda(T) \) is a cooling rate function which depends on the chemical composition of the plasma.

The key question is the relation between the free-fall time \( t_{ff} \), cooling time, \( t_{cool} \), and the Hubble time, \( t_H \):

**If** \( t_{cool} < t_{ff} \) the cooling dominates the contraction, objects collapse faster and to smaller radii and higher densities; and vice versa.

**If** \( t_{cool} > t_H < t_{ff} \) objects cannot form.

The position of objects in the “cooling diagram” plane \( \{T,n\} \) thus determines their fate! Note also that \( M_{Jeans} = f(T,n) \).
Fluctuations which cool faster than they fall together under gravity alone are subject to Jeans instability and fragmentation.
The Cooling Diagram

Objects here cool faster than freefall time

$\tau_{cool} = \tau_{grav}$

$\tau_{cool} = \tau_{Hubble}$

$\tau_{grav} = \tau_{Hubble}$
And indeed, we see that the cooling curve separates the dissipative structures (galaxies) from the dissipationless ones (groups and clusters)

(from J. Silk)
Structure Formation Theory: A Summary of the Key Ideas (1)

- Structure grows from initial density perturbations in the early universe, via gravitational infall and hierarchical merging.
- Initial conditions described by the primordial density (Fourier power) spectrum $P(k)$, often assumed to be a power-law, e.g., $P(k) \sim k^n$, $n = 1$ is called a Harrison-Zeldovich spectrum.
- Dark matter (DM) plays a key role: fluctuations can grow prior to the recombination; after the recombination, baryons fall in the potential wells of DM fluctuations (proto-halos).
- Damping mechanisms erase small-scale fluctuations; how much, depends on the nature of the DM: HDM erases too much of the high-freq. power, CDM fits all the data.
- Collapse occurs as blobs $\rightarrow$ sheets $\rightarrow$ filaments $\rightarrow$ clusters.
Structure Formation Theory:  
A Summary of the Key Ideas (2)

- Pure gravitational infall leads to overdensities of \( \sim 200 \) when the virialization is complete.
- Free-fall time scales imply galaxy formation early on (\( t_{ff} \sim \) a few \( \times 10^8 \) yrs), clusters are still forming (\( t_{ff} \sim \) a few \( \times 10^9 \) yrs).
- Characteristic mass for gravitational instability is the Jeans mass; it grows before the recombination, then drops precipitously, from \( \sim 10^{16} M_\odot \), to \( \sim 10^5 M_\odot \).
- Cooling is a key concept:
  - Galaxies cool faster than the free-fall time: formation dominated by the dissipative processes, achieve high densities;
  - Groups and clusters cool too slowly: formation dominated by self gravity, lower densities achieved;
  - The cooling curve separates them.
8.5 Numerical Simulations of Structure Formation
Numerical Simulations in Cosmology

• Many problems in astrophysics lack the symmetries that generally allow analytic solution. A full 3D numerical treatment is required

• For the Newtonian gravity, there are no analytical solutions beyond the 2-body (Kepler) problem

• Numerical simulations thus play a key role in cosmology

• In cosmology know both the governing equations (Newtonian gravity, cosmological expansion) and can assume the initial conditions (primordial density field)

• Pure gravity simulations are dissipationless; adding gas and radiation makes the problem much harder, due to dissipation, hydrodynamical effects, radiation transfer…

• The output of simulations can be compared to observations
Galaxy Encounters and Mergers

Toomre & Toomre (1972)

Tidal tails, bridges, and other distortions
Merge 2 nearly equal mass disk galaxies; in a few dynamical times, the remnant looks just like an elliptical galaxy.
In the same merger, gas quickly loses energy (since it is dissipative), and sinks towards the center of the remnant, where it can fuel a starburst, or an AGN if a massive black hole is present.
We are using simulations to:

• Understand the evolution of galaxy clustering
  – Compare with redshift surveys
  – Mock galaxy catalogues, galaxy formation models

• Make predictions for weak lensing and SZ surveys
  – Model non-linear evolution of DM
  – Simulated observations for experiments
  – Basic theory

• Quantify merger rates of galaxies and black holes
  – Main gravity wave source for LIGO, LISA, …

• Elucidate the structure of our dark matter halos
  – Direct detection experiments

• Study small-scale structure (Ly-α forest)
  – Strong limits on DM properties (& dark energy?)
Types of Physics Included:

• Gravity
  – All cosmology codes have this! Dominates the forces above the scale of galaxies

• Hydrodynamics
  – Adiabatic physics
  – Cooling
  – Star “formation” and feedback

• Magnetic fields (MHD)
  – Not so common in cosmology, but may be important

• Radiative transfer
  – Still early days … resolution is a problem (photons are much smaller than galaxies)
Gravity: the N-body Method

- Original N-body work was for stellar systems (a true “N-body problem”, not doable analytically)
- Taken over to cosmology – change in emphasis: modelling DM dominated or pure stellar systems as a fluid
- This is for dissipationless (i.e., gravity only) interaction: good for DM and stars only, but not for gas
- Various approximations used, and various methods:

<table>
<thead>
<tr>
<th>Method</th>
<th>Complexity</th>
<th>Practicality</th>
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<tbody>
<tr>
<td>Direct summation</td>
<td>O(N²)</td>
<td>Practical for N&lt;10⁴</td>
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<td></td>
<td>[Special hardware]</td>
<td>[up to 10⁷]</td>
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<tr>
<td>Particle mesh</td>
<td>O(N logN)</td>
<td>Uses FFTs to invert Poisson equation</td>
</tr>
<tr>
<td>Tree codes</td>
<td>O(N logN)</td>
<td>Multipole expansion</td>
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(Usually, some combination of these is used)
Hydro Simulation (Dissipative)

• Much harder than gravity alone
  – Yet crucial for the realistic astrophysical problems

• Resolution is a big problem: how many resolution elements of what size can you do?
  – We know that structure and physics in gas occur at all scales we can observe

• Physics is also often uncertain
  – For example, we do not really understand star formation

• Science examples:
  – Galaxy formation, including star formation, feedback by AGN and SNe, etc.
  – Supernova explosions
  – Star and planet formation
  … etc. etc.
A “Cosmic Cube” Simulation by A. Kravtsov
A Group Formation Simulation by A. Kravtsov
A Cluster Formation Simulation by J. Dubinski
Snapshots from the Virgo Consortium’s “Millenium” simulation

Credit: Volker Springel
Zoom-in on a cluster (DM)
Simulations make qualitatively and quantitatively different predictions about the large-scale structure at different redshifts.

This can be compared directly with the observations.
Merger Tree

At small scales, galaxy merging is important. Each galaxy derives from a hierarchical sequence of mergers of smaller fragments.

Figure 6. A schematic representation of a “merger tree” depicting the growth of a halo as the result of a series of mergers. Time increases from top to bottom in this figure and the widths of the branches of the tree represent the masses of the individual parent halos. Slicing through the tree horizontally gives the distribution of masses in the parent halos at a given time. The present time $t_0$ and the formation time $t_f$ are marked by horizontal lines, where the formation time is defined as the time at which a parent halo containing in excess of half of the mass of the final halo was first created.
The Milky Way Formation Simulation by Diemand et al. (Dark Matter Only)
Hydro simulations
(DM + gas)
Disk Galaxy Merger Simulation With DM, Gas, and Black Holes (AGN Feedback) by V. Springel et al.