Physics 106a/196a – Midterm Exam – Due Nov 4, 2005

Instructions

Material: All lectures through Oct 20, Sections 1, 2.1, 2.2.1, and 2.2.2 of lecture notes, Hand and Finch Chapter 1, Sections 2.1-2.5, and Sections 5.1 and 5.2. Only Ph196 students are responsible for the following topics:

- virtual work and generalized forces
- derivation of Euler-Lagrange equations via virtual work
- nonholonomic constraints
- Lagrangians for nonconservative forces

Review the material ahead of time, consult me, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: The exam consists of this page plus 4 pages of exam questions. Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one. Problems 1 and 2 are for 106 students only, 3 and 4 for 106 and 196 students, and 5 and 6 for 196 students only. Problem 4 has an additional piece for 196 students.

Time: 4 hrs, fixed time. You may take as many breaks as you like, but they may add up to no more than 30 minutes (2 x 15 minutes, 3 x 10 minutes, etc.).

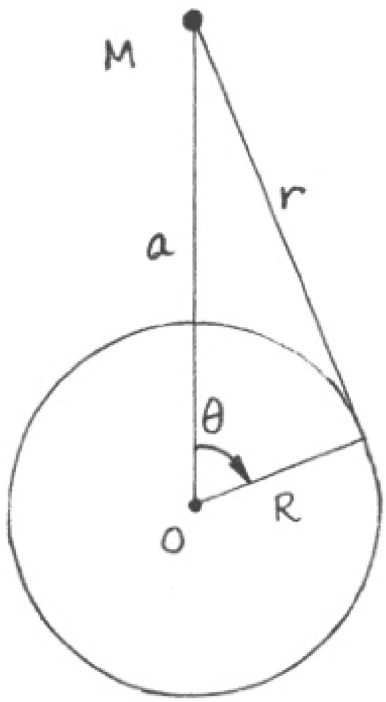
Reference policy: Hand and Finch, Thornton (including solutions manual), official class lecture notes and errata, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students. Calculators and symbolic manipulation programs are neither needed or allowed.

Due date: Friday, Nov 4, 5 pm, my office (311 Downs). 5 pm means 5 pm. Late exams will require extenuating circumstances; otherwise, no credit will be given.

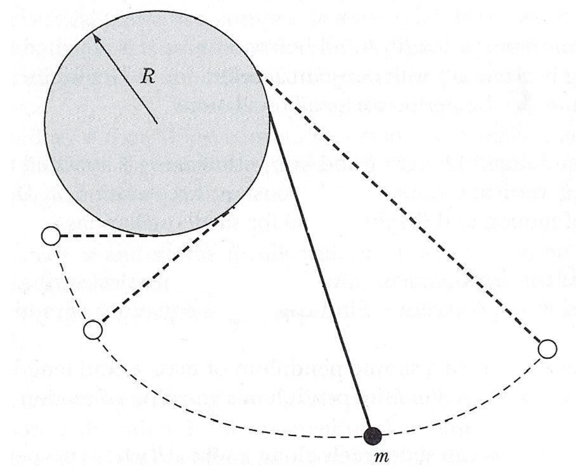
Grading: Each problem is 25 points out of 100. The exam is 30% of the class grade.

Suggestions on taking the exam:

- Go through and figure out roughly how to do each problem first; make sure you've got the physical concept straight before you start writing down formulae.
- Don't fixate on a particular problem. They are not all of equal difficulty. Come back to ones you are having difficulty with.
- Don't get buried in algebra (this really should not be an issue on this exam). Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra.



Problem 1

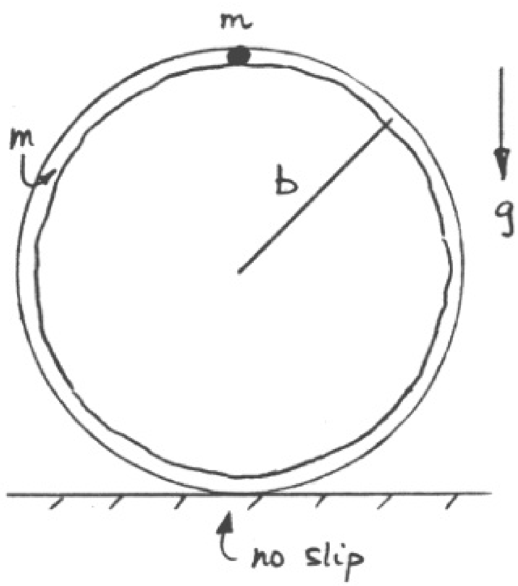


Problem 2

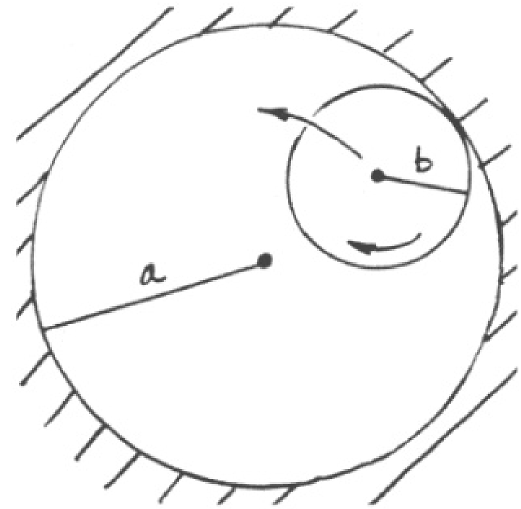
1. (106) Consider a point mass M that lies outside a spherical surface. Let $\phi(\vec{r})$ be the gravitational potential due to the point mass. Show that the average value of ϕ taken over the surface of the sphere is equal to its value at the center of the sphere. Hint: there is an easy way to do this problem that does not require explicitly doing the integral to calculate the average of ϕ over the sphere. But you can do it the hard way, too.

   (Since the potential due to an arbitrary mass distribution is the sum of the potentials due to point masses, this statement is also true for the gravitational potential due to an arbitrary mass distribution lying outside a spherical surface.)

2. (106) A pendulum is constructed by attaching a mass m to an extensionless string of length l. The upper end of the string is connected to the uppermost point on a vertical disk of radius R ($R < l/\pi$) as in the figure. Define an appropriate set of generalized coordinates, write down the Lagrangian, and obtain the pendulum's equation of motion. Rewrite the EOM in simple harmonic oscillator form for *small* oscillations about the equilibrium point and find the frequency of small oscillations. Even if you are unable to find the equation of motion for arbitrary oscillation amplitude, you can directly obtain the equation of motion for small oscillations by making appropriate approximations when you set up the problem.



Problem 3



Problem 4

3. (106/196) A bead of mass m moves inside a thin hoop-shaped pipe of average radius b and also of mass m. The pipe has a frictionless interior, so that the bead moves freely within the circumference of the hoop. You will consider two cases for the relationship between the pipe and the floor:

   (a) No friction between the pipe and floor.

   (b) A coefficient of friction between the hoop's exterior and the floor large enough that the hoop rolls on the floor without slipping.

   The bead is released from rest at the top of the hoop (it is given a small but negligible nudge in one direction). When the bead has fallen halfway to the floor, how far to the side will the hoop have moved for the two cases? Hint: you can obtain the answer based on simple arguments without obtaining the equations of motion. But it's perfectly fine to do it with the equations of motion.

4. (106/196) A hoop of radius b and mass m rolls without slipping within a circular hole of radius $a > b$ and is subject to gravity. Define an appropriate set of generalized coordinates, write down the Lagrangian, and find the Euler-Lagrange equation(s) of motion. Make the approximation that the motion about the equilibrium point is small and rewrite the equation of motion in simple harmonic oscillator form. What is the frequency of these small oscillations?

   (196) In addition, consider the generalization of this system to three dimensions, with the hoop replaced by a hollow ball and the circular hole replaced by a spherical hole. Write down the constraints obtained from requiring that the ball roll without slipping on the spherical surface of the hole. Show which of the constraints are nonholonomic by the usual procedure of showing that, if the constraint were holonomic, some partial derivatives of the constraint function would not commute.

5. (196) A point particle moves in space under the influence of a force derivable from a velocity-dependent potential of the form

$$U(\vec{r}, \vec{v}) = E\,z + \vec{\sigma} \cdot \vec{L}$$

where $\vec{L}$ is the angular momentum about the origin and $\vec{\sigma} = \sigma\,\hat{z}$ is a constant vector pointing along the z axis.

(a) Calculate the generalized force from this potential in Cartesian coordinates using the equation from the lecture notes the implicitly defines velocity-dependent potentials:

$$\mathcal{F}_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

(b) Obtain the components in cylindrical coordinates using the definition of generalized force in terms of forces in Cartesian coordinates:

$$\mathcal{F}_k \equiv \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

(c) Obtain the generalized equations of motion in cylindrical coordinates.

(d) What kind of motion is obtained if the initial conditions are $\dot{\rho}(t=0) = 0, \dot{\theta}(t=0) = \sigma$? What about if $\dot{\theta}(t=0) = \sigma/2$?

6. (196) The quantum mechanics of a one-dimensional system is described by the Schrodinger equation for the complex wave function $\psi(x,t)$:

$$-\frac{\hbar^2}{2\,m}\,\frac{\partial^2 \psi}{\partial x^2} + V(x)\,\psi = i\,\hbar\,\frac{\partial \psi}{\partial t}$$

where m is the particle mass and $V(x)$ the potential energy (it is real, not complex). Find a variational principle for quantum mechanics using the two dependent variables ψ and its complex conjugate ψ^* and the two independent variables x and t. You can treat ψ and ψ^* as two independent coordinates since the real and imaginary parts of ψ are independent variables. Hints:

- You will try to make the variation of the double integral of the form below vanish:

$$0 = \delta \int dx \int dt\, \mathcal{L}\left(\psi, \psi^*, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial t}, \frac{\partial \psi^*}{\partial x}, \frac{\partial \psi^*}{\partial t}, x, t\right)$$

  See if you can guess the correct form for $\mathcal{L}$ such that the Euler-Lagrange equations lead to the Schrodinger equation and its complex conjugate. $\mathcal{L}$ must be real. It may obviously have pieces of the form $V(x)\,\psi^*\psi$, $\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}$. Any combination of ψ, ψ^* and derivatives thereof that is real is allowed. For example, $\psi^*\,\frac{\partial \psi}{\partial t}$ is not necessarily real, but $\left(\psi^*\,\frac{\partial \psi}{\partial t} + \psi\,\frac{\partial \psi^2*}{\partial t}\right)$ is real, as is $i\left(\psi^*\,\frac{\partial \psi}{\partial t} - \psi\,\frac{\partial \psi^*}{\partial t}\right)$.

- The extension of the calculus of variations to two independent variables is straightforward and given in Hand and Finch Section 2.9. For one dependent variable $y(x,t)$, the Lagrangian is now $\mathcal{L}(y, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, x, t)$. The Euler-Lagrange equation for any unconstrained dependent variable $y(x,t)$ becomes

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial y}{\partial x}}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \frac{\partial y}{\partial t}}\right) = 0$$

  Some care must be taken with the partial derivatives: t is held fixed for $\frac{\partial}{\partial x}$ and x is held fixed for $\frac{\partial}{\partial t}$, so these derivatives act on y and derivatives of y. On the other hand, for $\frac{\partial}{\partial y}$, we hold fixed x, t, and derivatives of y; for $\frac{\partial}{\partial \frac{\partial y}{\partial x}}$ we hold fixed x, t, y, and $\frac{\partial y}{\partial t}$, and similarly for $\frac{\partial}{\partial \frac{\partial y}{\partial t}}$.