Physics 106a/196a – Problem Set 7 – Due Dec 2, 2005

Version 3, Nov 27, 2005

In this set we finish up the SHO and study coupled oscillations/normal modes and waves. Problems 1, 2, and 3 are for 106a students only, 4, 5, and 6 for 106a and 196a students, and 7, 8, and 9 for 196a students only.

IMPORTANT NOTES:

- You have **2 weeks** to do this set (albeit, with Thanksgiving in between). It is **50% longer** than the usual set – 6 problems for each student instead of 4. You will have most of the material you need by the end of the Nov 22 lecture, and the remainder certainly on Nov 29. **Please get an early start on this set!**
- Keep a xerox of your solutions. We will likely not get the graded problem sets back in time for you to use on your final exam.

Changes since v. 1: More hints added on various problems, state what ω_0 and ω_0' are in Problem 3, correct exchanged mode vectors in Problem 5, make gauge choice in Problem 9.

Changes since v. 2: Add a drawing for Problem 5, add a hint for Problm 7, correct some errors in H&F in Problem 8.

1. (106) A woofer in a sealed box ("acoustic suspension") is the simplest type of speaker to analyze. The motion of the speaker cone of mass m is governed by the equation

$$m\,\ddot{x} + b\,\dot{x} + k\,x = F_0 \cos\omega\,t$$

where F_0 is constant if the amplifier output resistance or voice coil resistance is excessive (not a typical assumption, but one we make here for simplicity). The average sound intensity is proportional to the average (acceleration)^2 of the cone. The damping factor b is proportional to the strength of the magnetic "motor" – the magnet and voice coil assembly. The restoring force characterized by spring constant k is provided by the gas pressure in the box (when the speaker moves, it changes the volume, which changes the pressure in the box relative to outside, creating a pressure differential on the cone that pushes it back toward its equilibrium position.).

   (a) Try to think up a simple mechanical test that you can perform in the showroom to see whether the cone is underdamped or overdamped. (Don't overthink this one!).

   (b) Suppose the assembly has resonant frequency of $\nu_0 = 50$ Hz and $Q = 1$. (These are typical specifications for a medium quality home-use speaker.) At what frequency does one obtain the maximum sound intensity? (Remember, sound intensity $\propto$ (acceleration)^2, not (amplitude)^2.) What is the sound intensity at 25 Hz, 50 Hz, and 100 Hz relative to that maximum (one of these may be the frequency of the maximum)?

(c) Suppose you consider a bigger speaker that is driven with the same amplitude drive function. The various parameters scale as follows: m increases as D^2 where D is the cone diameter. k increases as D^2 also because the pressure inside the box is unchanged by increasing it size, so the restoring force PA increases as A. The damping parameter b does not change. How do the natural frequency, the resonsnt frequency, and the Q change? Does the larger speaker have a sharper or flatter frequency response?

2. (106) Hand and Finch 9-3. In addition to the given problem, do the following:

   (a) Assume the initial condition $x_1(t=0) = 0$, $x_2(t=0) = \alpha$ and both masses at rest at time $t = 0$ and find the mode coefficients $\{A_i\}$ and write down the full solution $\vec{\phi}(t)$.

   (b) Let $k = 0.05$ and plot your result (feel free to use Mathematica or any other numerical program). You should see the "beat" phenomenon. Use one of the very useful trigonometric identities below to rewrite your result in a form that provides directly the beat period. Check that it matches your numerical calculation (factors of 2 can trip you up here).

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A-B}{2}\sin\frac{A+B}{2}$$

Reminders: 1) Remember to be careful when constructing **t** and **v** to do your derivatives and factors of 1/2 correctly. 2) Don't forget to normalize the normal mode vectors correctly in the end.

3. (106) Consider a wave packet in which the amplitude distribution is given by

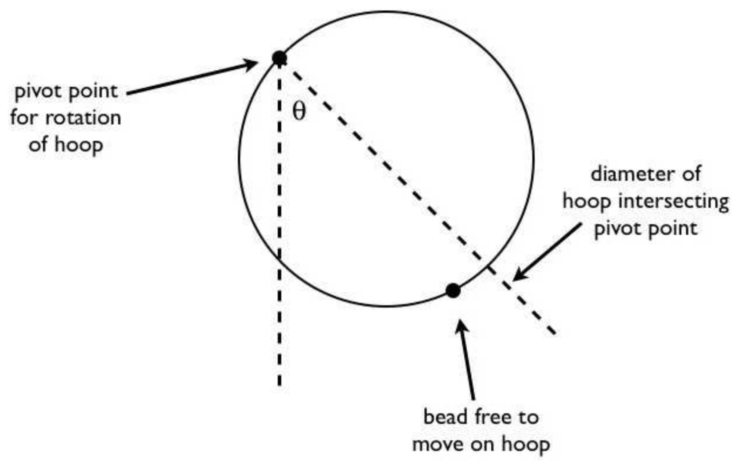
$$\alpha(k) = \begin{cases} 1 & |k - k_0| < \Delta k \\ 0 & \text{elsewhere} \end{cases}$$

Show that the wave function is

$$y(x,t) = 2\frac{\sin\left[\left(\omega_0' t - x\right)\Delta k\right]}{\omega_0' t - x}\exp\left[i\left(\omega_0 t - k_0 x\right)\right]$$

Recall from the notes that $\omega_0 = \omega(k_0)$ and $\omega_0' = \frac{d\omega}{dk}\big|_{k_0}$. Sketch the shape of the wave packet at $t = 0$. Suppose you take the distance between the first zeroes of the envelope (on each side of $x = 0$) of $y(x, t = 0)$ to be the characteristic position spread Δx. What happens to Δx when Δk is changed? Can you find a relation analogous to Liouville's theorem or the uncertainty principle relating Δx and Δk? At what speed does the peak of the wave packet propagate away from the origin?

4. (106/196) Hand and Finch 3-21. When it asks "For what time is the response maximized?", you may find this numerically (doing it analytically results in a transcendental equation).



Problem 5

5. (106/196) A thin hoop of radius R and mass M oscillates in its own plane hanging from a single fixed point. Attached to the hoop is a small mass M constrained to move (in a frictionless manner) along the hoop. (See figure.) Consider only small oscillations, and show the following:

   (a) The kinetic and potential energy matrices are

   $$\mathbf{t} = \frac{1}{2}\,M\,R^2 \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix} \qquad \mathbf{v} = \frac{1}{2}\,M\,g\,R \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

   (b) The normal mode frequencies are

   $$\omega_1 = \sqrt{2}\sqrt{\frac{g}{R}} \qquad \omega_2 = \frac{1}{\sqrt{2}}\sqrt{\frac{g}{R}}$$

   (c) The normal mode vectors are (up to normalization)

   $$\vec{\Phi}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \qquad \vec{\Phi}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

   Recall that the orthonormalization relation is $\vec{\Phi}_i^T \mathbf{t} \vec{\Phi}_j = \delta_{ij}$, and $\mathbf{t}$ is definitely not a multiple of the identity matrix in this case.

   (d) Find the two sets of initial conditions that result in oscillation in one or the other normal mode only. Describe qualitatively what these modes look like.

   The challenges in this problem are: a) setting it up correctly (getting the right T and V) and b) being very careful in calculating the $\mathbf{t}$ and $\mathbf{v}$ matrices. The intermediate results are provided so you can complete the problem correctly even if you can't demonstrate a particular step.

6. (106/196) Consider an infinitely long continuous string in which the tension is τ. A mass M is attached to the string at $x = 0$. If a sinusoidal wave train with velocity $v = \omega/k$ is incident from the left, analyze the reflection and transmission that occur at $x = 0$. Find the reflected and transmitted power coefficients R and T analogous to those defined in Section 3.3.3 of the

lecture notes (this case is easier because the string has the same parameters on both sides of the interface). Show that

$$R = \sin^2\theta \qquad T = \cos^2\theta \qquad \text{where} \qquad \tan\theta = \frac{M\,\omega^2}{2\,k\,\tau}$$

To do this problem, you will need to carefully set up the boundary conditions at $x = 0$. You will find it useful to review Section 3.3.3 of the lecture notes, where we derived the wave equation by calculating forces. How does adding M change that derivation?

7. (196) Treat the problem of wave propagation along a string loaded with particles of two different masses, m_a and m_b, which alternate in placement; that is $m_i = m_a$ for i odd and $m_i = m_b$ for i even. Obtain the $\omega - k$ curve – the dispersion relation – and show that it has two branches in this case. Hint: because there are two different types of masses, their displacements may not have the same amplitude given that the tension is fixed. How can you accomodate this in your assumed form for the solution?

   This problem is of quite general applicability in condensed matter physics. Our original dispersion relation for a loaded string with all masses the same,

$$\omega_n = 2\sqrt{\frac{\tau}{m\,d}}\left|\sin\left(\frac{1}{2}\,\frac{n\,\pi}{M+1}\right)\right| = 2\sqrt{\frac{\tau}{m\,d}}\left|\sin\left(\frac{k\,d}{2}\right)\right|$$

   implies there is a maximum mode energy, given when $k\,d = \pi$. This is approximately the dispersion relation seen for phonons in a monatomic crystal. This problem considers crystals with two different types of atoms – such as ionic crystals – and the two branches of the dispersion relation correspond to different kinds of phonons. They are conventionally called "acoustic" (lower branch) and "optical" (higher branch).

8. (196) Hand and Finch 9-7. There are some errors/ambiguities in this problem:

   - Part (b): It is somewhat confusing for the problem to state that you should prove that $(\mathbf{1} \pm \mathbf{R})\vec{\phi}$ are modes and also that either $(\mathbf{1} + \mathbf{R})\vec{\phi} = 0$ or $(\mathbf{1} - \mathbf{R})\vec{\phi} = 0$. The apparent contradiction is resolved by realizing that, if some vector ψ is identically equal to zero, then it will satisfy the equation of motion for a particular normal mode frequency and hence could be considered a "mode"; of course, it is a trivial one that we never consider. Also, when they say "Hence, prove that the modes must either be odd or even in terms of this reflection," you are simply being asked to prove that $\mathbf{R}\vec{\phi} = \pm\vec{\phi}$.
   - Part (c): The diagram given in Figure 9.13 for the u mode is very misleading, as it does not appear to leave the center-of-mass position unchanged. Don't rely on it too much to determine what the u mode is. Also, the description of the atom displacements in terms of the u, v, and w mode amplitudes seems to have completely exchanged all the $\sin\alpha$ and $\cos\alpha$ factors; that is, the expansions should be:

$$\delta\vec{r}_1 = (-\,(u+w)\sin\alpha - v\,\cos\alpha, v\,\sin\alpha - (u+w)\cos\alpha)$$
$$\delta\vec{r}_2 = (-\,(u-w)\sin\alpha + v\,\cos\alpha, v\,\sin\alpha + (u-w)\cos\alpha)$$
$$\delta\vec{r}_0 = 2\,\frac{M_H}{M_O}\,(u\,\sin\alpha, -v\,\sin\alpha + w\,\cos\alpha)$$

     Finally, your mode definitions for u, v, and w may differ from those of Hand and Finch by a sign, so any or all the mode coefficients may pick up negative signs. If you want to match their signs, check that the sign of your mode's displacement for $\delta\vec{r}_1$ matches Figure 9.13.

9. (196) We saw in previous homework and on the midterm how the calculus of variations can be used to derive a variety of "field equations." We will pursue two exercises of this form here. Both of these problems make use of multiple independent variables (space and time), so consult your midterm exam and Section 2.9 of Hand and Finch for a reminder of how to deal with such situations.

   (a) Determine a Lagrangian $\mathcal{L}$ that will yield the wave equation (in one dimension) when the Euler-Lagrange procedure is performed.

   (b) The Lagrangian density (per unit volume) for a charge density $\rho(\vec{r}, t)$ and current density $\vec{j}(\vec{r}, t)$ in the presence of an electromagnetic field is

$$\mathcal{L} = \frac{E^2 - B^2}{8\,\pi} - \rho\,\phi + \frac{1}{c}\,\vec{j} \cdot \vec{A}$$

   The first term is the Lagrangian density corresponding to the self-energy of the free field, and the latter terms represent the interaction between fields and charges. The self-energy of the individual (point) charges is infinity in classical theory and is omitted. As usual, the vector potential is defined by

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

   (Gaussian units are used here.) The homogeneous (charge and current independent) Maxwell equations follow directly from the equations relating $\vec{E}$ and $\vec{B}$ and the potentials. To complete the picture, using ϕ and the three components of $\vec{A}$ as four generalized (field) coordinates, apply the Euler-Lagrange equations to $\mathcal{L}$ to obtain the two inhomogeneous Maxwell equations

$$\vec{\nabla} \cdot \vec{E} = 4\,\pi\,\rho$$
$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\,\pi}{c}\,\vec{j}$$

   You should make the gauge choice $\vec{\nabla} \cdot \vec{A} = 0$. Hint: you will find it easiest to do this problem using the Einstein summation convention and index notation; using vector identities will work, but it gets messy notationally. You will need to make use of some of the indexing and vector identities in Appendix A.3 of the notes (you can translate the vector identities into index notation).