# Physics 106b – Problem Set 8 – Due Jan 13, 2006

Version 2 – Jan 9, 2006

We cover special relativity in this set, Chapter 6 of the lecture notes and Chapter 12 of Hand and Finch. Problems 1 through 4 are required, problem 5 is extra credit and equal in weight to the first four problems.

**Changes since version 1:** Correction to sign error in expression for $|a^\mu|^2$ in Problem 2.

1. Hand and Finch 12-4. This is a classic problem that is not too different from the discussion of length contraction in the lecture notes. Doing the problem, though, will hopefully help you deepen your understanding.

2. Hand and Finch 12-6 (relativistic acceleration). This problem is useful for calculating the motion of a charged particle in various accelerators (cyclotron, synchrotron, linear accelerator). In addition to what is asked, do the following:

   - Show that
   $$|a^\mu|^2 = -\frac{\gamma^6}{\gamma_\perp^2}\left|\frac{d\vec{\beta}}{dt}\right|^2$$
   where
   $$\gamma_\perp^2 = \left(1-\beta_\perp^2\right)^{-1} = \left(1-\left[\beta^2-\left(\vec{\beta}\cdot\frac{\frac{d\vec{\beta}}{dt}}{\left|\frac{d\vec{\beta}}{dt}\right|}\right)^2\right]\right)^{-1}$$
   is the $\gamma$ factor due to the lab-frame velocity perpendicular to the lab-frame acceleration. (It's less complicated than it looks.)
   - Show that the above formula reduces to the two that you found for circular motion and parallel acceleration.

   Notes:

   - Equation 12.116 erroneously implies that the three-acceleration $\vec{a}$ is given by the space components of $a^\mu$. The three-acceleration $\vec{a}$ is always $\frac{d\vec{\beta}}{dt}$. Just as the space components of the four velocity $u^\mu$ are not just the three-velocity $\vec{\beta}$ but are rather $\gamma\,\vec{\beta}$, the space components of the four-acceleration are not just the three-acceleration $\vec{a}$. There would be no ambiguity if H&F simply dropped the $(a^0,\vec{a})$ part of Equation 12.116.
   - As in the definition of four-velocity, you will find it necessary to convert $\frac{d}{d\tau}$ to $\frac{d}{dt}$ to obtain the desired explicit form. In our demonstration that $u^\mu = \gamma\left(1,\vec{\beta}\right)$, we used $\tau = \sqrt{|x^\mu|^2}$ and took the derivative with respect to $t$. We implicitly assumed there that

$\vec{\beta}$ was constant. Since we are no longer making that assumption, the original relation for $\tau(t)$ may no longer hold. What we can be sure of, though, is that the infinitesimal version of the relation holds:

$$\begin{aligned}(d\tau)^2 &= (dt)^2 - (d\vec{x} \cdot d\vec{x})^2 \\ d\tau &= dt\,\sqrt{1-\beta^2} = \frac{dt}{\gamma}\end{aligned}$$

where $\vec{\beta}$ may now be a function of time. You will want to use the differential relation.[1]

- There is a minor error in the problem: it should ask you to demonstrate that "the relation between the laboratory acceleration of a particle undergoing circular motion at constant speed and the acceleration in the instantaneous rest frame of the particle is $a_{rest} = \gamma^2\, a_{lab}$." That this is a typo is confirmed by Goldstein derivation 7.7 and the generic formula for $|a^\mu|^2$ given above.

3. Hand and Finch 12-17 (Compton scattering). Notes:

   - In class we have not used the notation $k_\mu\, k^\mu$ or $p_\mu\, p^\mu$. In general, for any four-vector, the expression $a_\mu\, a^\mu$ is the invariant norm $|a^\mu|^2 = (a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2$. The formal meaning of lowered indices is discussed in the lecture notes, but you don't need to know that for this problem.
   - The $k^\mu$ defined in this problem is different from the $k^\mu$ defined in class by a factor $\hbar$. You can basically forget about the $k^\mu$ defined in class when doing this problem – just take it as given that $k^\mu$ is the four-momentum of the photon, with the time component being the energy and the space components being the spatial momentum, and that $|k^\mu|^2 = 0$ because the photon rest mass vanishes.
   - A "backscattered" photon is one with outgoing angle $\theta' = \pi$.

4. Relativistic harmonic oscillator (clearer version of Hand and Finch 12-21). The Lagrangian for the relativistic simple harmonic oscillator is ($c = 1$ as usual):

$$L = -m\,\sqrt{1-\beta^2} - V(x) \qquad\qquad V(x) = \frac{1}{2}\,k\,x^2$$

   Let $a$ be the amplitude of the motion. Use the usual Euler-Lagrange procedure to obtain the equation of motion for the particle, realizing that $\beta = \dot{x}$. Integrate the equation of motion to obtain the following expression for the energy:

$$E = \gamma\, m + \frac{1}{2}\,k\,x^2$$

   which, when evaluated at $x = a$ (when $\dot{x} = 0$), can be written

$$E = m + \frac{1}{2}\,k\,a^2$$

[1] Two finer points: 1) with the differential relation, we could of course have derived the four-velocity expression without the assumption of constant velocity; hence, it holds even when the velocity is not constant. 2) If we integrate the differential relation, we find

$$\tau(t) = \int_0^t d\tilde{t}\sqrt{1 - \left[\beta(\tilde{t})\right]^2}$$

which may in general be different from $\sqrt{|x^\mu|^2} = \sqrt{t^2 - \vec{x}\cdot\vec{x}}$, hence the distinction between using the differential and integral relations.

One can calculate the period of motion via the integral

$$\tau = 4 \int_{t=0}^{t=\tau/4} dt = 4 \int_{x=0}^{x=a} \frac{dx}{\dot{x}}$$

Show that this integral is

$$\tau = \frac{2\,a}{\kappa} \int_{u=0}^{u=1} \frac{1 + 2\,\kappa^2 \left(1 - u^2\right)}{\sqrt{1 + \kappa^2 \left(1 - u^2\right)}} \frac{du}{\sqrt{1 - u^2}}$$

where $u = x/a$ and the dimensionless parameter $\kappa^2 = \frac{k\,a^2}{4\,m}$. We see that the period is now amplitude-dependent via $\kappa^2$, which is half the the ratio of the maximum potential energy to the particle rest mass. Show that this integral reduces to the nonrelativistic period $\tau_{NR} = 2\,\pi\sqrt{m/k}$ when calculated to zeroth order in $\kappa^2$. Show that, to first order in $\kappa^2$, the period is

$$\tau = \tau_{NR}\left[1 + \frac{3}{4}\kappa^2\right] = \tau_{NR}\left[1 + \frac{3}{8}\frac{V(a)}{m}\right]$$

Hints:

- For help with integrating the equation of motion, refer back to the relativistic calculation of the relation between work and kinetic energy under the topic **The Energy-Momentum Four-Vector** in Section 6.1.4 of the lecture notes.
- To do the period integral, you will need to rewrite $\dot{x}$ in terms of $x$ using the energy.

5. Hand and Finch 12-18. The expression given for $\partial^\mu$ is incorrect, it should be $\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$. You'll have to read Section 6.1.3 of the lecture notes to make sense of the tensor notation.