Physics 106b – Problem Set 9 – Due Jan 20, 2006

Version 1 – Jan 14, 2006

We cover central force motion, Keplerian orbits, and scattering in this set. Chapter 4 of the lecture notes and Hand and Finch. Be sure to have checked the lecture notes errata! Problems 1 through 4 are required, problem 5 is extra credit and equal in weight to the first four problems.

1. Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with a period τ. If they are suddenly stopped in their orbits and allowed to gravitate toward each other, show that they will collide after a time $\frac{\tau}{4\sqrt{2}}$. You may find the following useful:

$$\int \frac{dx\,\sqrt{x}}{\sqrt{1-x}} = -2\int \sqrt{1-z^2}dz \qquad \text{with} \qquad z = \sqrt{1-x}$$
$$\int \sqrt{1-z^2}\,dz = \frac{1}{2}\left[z\,\sqrt{1-z^2} + \arcsin z\right]$$

2. Assume that the earth's orbit is circular with radius r_0. Let the sun's mass suddenly decrease by half. Which of the following quantities change at the instant of the decrease in the sun's mass, and what are the new values (if different from the old ones): radius, angular momentum, kinetic energy, potential energy, total energy? What orbit does the earth then have (energy, eccentricity, semimajor and semiminor axes, radial turning points)? Will the earth escape the solar system? You may make use of the approximation $M_E \ll M_S$ and any consequences thereof. You will find it useful to think about changes in the scale radius p and scale energy E_{scale}.

3. According to Yukawa's theory of nuclear forces, the attractive force between a neutron and a proton has the potential

$$V(r) = -\frac{K\,e^{-\alpha\,r}}{r}$$

In units with $c = 1$ and $\hbar = 1$, α is the mass of the pion[1], which was the particle Yukawa theorized was exchanged between protons and neutrons to bind nuclei together. The massiveness of the particle that mediates the force – in contrast to the masslessness of the photon or graviton – is what gives rise to the exponential factor. Answer the following:

   (a) Given that $\alpha = m_\pi \approx 135$ MeV/c^2, calculate the scale length $1/\alpha$ in meters.

   (b) Calculate the force law. Which term dominates for what range of radius? What is the form for $r \ll 1/\alpha$? Make a plot (by hand or using a computer graphing program).

[1] so $1/\alpha$ has units of 1/mass = 1/energy = 1/frequency (because $\hbar = 1$) = time = length (because $c = 1$).

(c) Write down the equation you would solve in order to find the radius of a circular orbit in this potential and reduce it to a transcendental equation (which is not analytically solvable). Use this equation to show that the existence of a circular orbit is equivalent to the condition

$$\frac{1}{\alpha} \geq \frac{l^2}{\mu\, K} \frac{\exp\left(\frac{1+\sqrt{5}}{2}\right)}{2+\sqrt{5}}$$

(Hint: Think about the qualitative shape of the graph of the transcendental equation.) What is the physical interpretation of this condition?

(d) Assume there is a circular orbit of radius a. Show that the angular momentum and energy of the particle in that orbit are

$$l = \sqrt{\mu\, K\, e^{-\alpha\, a}\, a\, (1+\alpha\, a)} \qquad E = -\frac{1}{2}\frac{K}{a}\, e^{-\alpha\, a}\, (1-\alpha\, a)$$

Show that the period of the orbit is

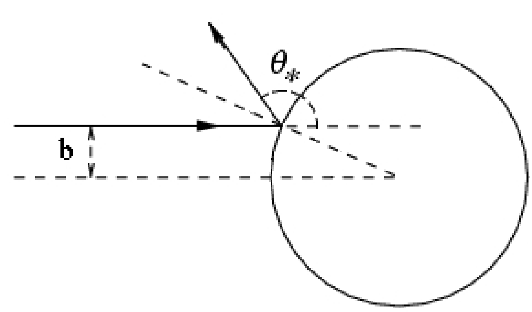
$$\tau = 2\,\pi \sqrt{\frac{\mu}{K} \frac{a^3\, e^{\alpha\, a}}{1+\alpha\, a}}$$

(e) Show that the period of small radial oscillations about the above circular orbit is given by

$$\tau_{osc} = 2\,\pi \sqrt{\frac{\mu}{K} \frac{a^3\, e^{\alpha\, a}}{1+\alpha\, a - \alpha^2 a^2}}$$

Why does the above imply that nearly circular orbits are almost closed for $a \ll \frac{1}{\alpha}$? (Think about a nearly circular orbit as a radial oscillation about a circular orbit.)

4. Hard sphere scattering. Calculate the differential cross-section $\frac{d\sigma}{d\Omega}(\theta_*)$ for scattering of a beam of point particles off of a hard sphere of radius a, as indicated below. Remember that the scattering angle θ_* is the angle between the incoming and outgoing velocity vectors, measured from incoming to outgoing. Integrate the differential cross-section to obtain the total cross-section (which will have an obvious form).



5. Hand and Finch 5-2. Notes:

- Remember that Hand and Finch's $k = G\,\mu\,M$ for the Kepler problem.
- Hand and Finch reference Equation 4.38, which fixes θ and uses ϕ as the angle variable for the orbit. In lecture, we fixed ϕ and used θ. You can do the problem either way, just be sure you are self-consistent (*e.g.*, if you fix θ, then your angular velocity should be $\dot{\phi}$).
- The last question "Is there a corresponding symmetry transformation?" is very non-trivial. You may regard this part of the problem as a small literature search project. You will run into discussions of how the LRL vector extends the symmetry group of the Kepler problem from SO(3) (rotations in three dimensions) to SO(4) (rotation group in four dimensions) or SO(3,1) (Lorentz group, rotations + boosts in three space and one time dimension). This will mesh with our discussion of rotation groups this week and the discussion of the Lorentz group that we will come back to in a couple of weeks.