Lecture Notes on Classical Mechanics for Physics 106ab

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Introduction

These notes were written during the Fall, 2004, and Winter, 2005, terms. They are indeed *lecture notes* – I literally lecture from these notes. They combine material from Hand and Finch (mostly), Thornton, and Goldstein, but cover the material in a different order than any one of these texts and deviate from them widely in some places and less so in others.

The reader will no doubt ask the question I asked myself many times while writing these notes: why bother? There are a large number of mechanics textbooks available all covering this very standard material, complete with worked examples and end-of-chapter problems. I can only defend myself by saying that all teachers understand their material in a slightly different way and it is very difficult to teach from someone else's point of view – it's like walking in shoes that are two sizes wrong. It is inevitable that every teacher will want to present some of the material in a way that differs from the available texts. These notes simply put my particular presentation down on the page for your reference.

These notes are not a substitute for a proper textbook; I have not provided nearly as many examples or illustrations, and have provided no exercises. They are a supplement. I suggest you skim them in parallel while reading one of the recommended texts for the course, focusing your attention on places where these notes deviate from the texts.

Contents

Chapter 1

Elementary Mechanics

This chapter reviews material that was covered in your first-year mechanics course – Newtonian mechanics, elementary gravitation, and dynamics of systems of particles. None of this material should be surprising or new. Special emphasis is placed on those aspects that we will return to later in the course. If you feel less than fully comfortable with this material, please take the time to review it now, before we hit the interesting new stuff!

The material in this section is largely from Thornton Chapters 2, 5, and 9. Small parts of it are covered in Hand and Finch Chapter 4, but they use the language of Lagrangian mechanics that you have not yet learned. Other references are provided in the notes.

1.1 Newtonian Mechanics

References:

- Thornton and Marion, *Classical Dynamics of Particles and Systems*, Sections 2.4, 2.5, and 2.6
- Goldstein, *Classical Mechanics*, Sections 1.1 and 1.2
- Symon, *Mechanics*, Sections 1.7, 2.1-2.6, 3.1-3.9, and 3.11-3.12
- any first-year physics text

Unlike some texts, we're going to be very pragmatic and ignore niceties regarding the equivalence principle, the logical structure of Newton's laws, etc. I will take it as given that we all have an intuitive understanding of velocity, mass, force, inertial reference frames, etc. Later in the course we will reexamine some of these concepts. But, for now, let's get on with it!

1.1.1 The equation of motion for a single particle

We study the implications of the relation between force and rate of change of momentum provided by Newton's second law.

Definitions

Position of a particle as a function of time: $\vec{r}(t)$

Velocity of a particle as a function of time: $\vec{v}(t) = \frac{d}{dt}\,\vec{r}(t)$. We refer to the magnitude of the velocity, $v = |\vec{v}|$, as the **speed**.

Acceleration of a particle as a function of time: $\vec{a}(t) = \frac{d}{dt}\,\vec{v}(t) = \frac{d^2}{dt^2}\,\vec{r}(t)$.

Momentum of a particle: $\vec{p}(t) = m(t)\,\vec{v}(t)$

Newton's second law

In inertial frames, it holds that

$$\vec{F}(t) = \frac{d}{dt}\,\vec{p}(t) \tag{1.1}$$

If the mass is not time-dependent, we have

$$\vec{F}(t) = m\frac{d}{dt}\,\vec{v}(t) = m\frac{d^2}{dt^2}\,\vec{r}(t) \tag{1.2}$$

We use the "dot" shorthand, defining $\dot{\vec{r}} = \frac{d}{dt}\,\vec{r}$ and $\ddot{\vec{r}} = \frac{d^2}{dt^2}\,\vec{r}$, which gives

$$\vec{F} = \dot{\vec{p}} = m\dot{\vec{v}} = m\ddot{\vec{r}} \tag{1.3}$$

Newton's second law provides the **equation of motion**, which is simply the equation that needs to be solved find the position of the particle as a function of time.

Conservation of Linear Momentum:

Suppose the force on a particle is $\vec{F}$ and that there is a vector $\vec{s}$ such that the force has no component along $\vec{s}$; that is

$$\vec{F} \cdot \vec{s} = 0 \tag{1.4}$$

Newton's second law is $\vec{F} = \dot{\vec{p}}$, so we therefore have

$$\dot{\vec{p}} \cdot \vec{s} = 0 \Longrightarrow \vec{p} \cdot \vec{s} = \alpha \tag{1.5}$$

where α is a constant. That is, there is conservation of the component of linear momentum along the direction $\vec{s}$ in which there is no force.

Solving simple Newtonian mechanics problems

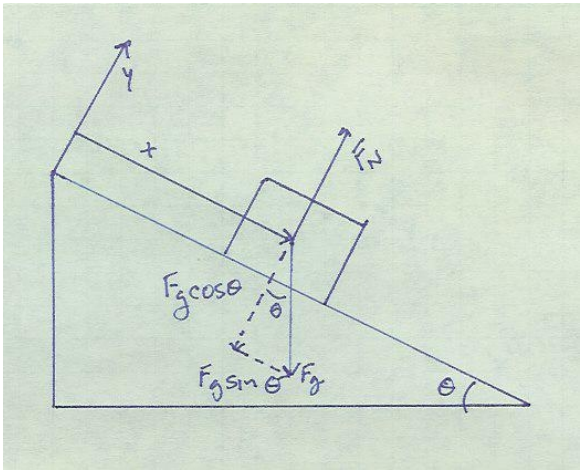
Try to systematically perform the following steps when solving problems:

- Sketch the problem, drawing all the forces as vectors.
- Define a coordinate system in which the motion will be convenient; in particular, try to make any constraints work out simply.
- Find the net force along each coordinate axis by breaking down the forces into their components and write down Newton's second law component by component.
- Apply the constraints, which will produce relationships among the different equations (or will show that the motion along certain coordinates is trivial).
- Solve the equations to find the acceleration along each coordinate in terms of the known forces.
- Depending on what result is desired, one either can use the acceleration equations directly or one can integrate them to find the velocity and position as a function of time, modulo initial conditions.
- If so desired, apply initial conditions to obtain the full solution.

Example 1.1

(Thornton Example 2.1) A block slides without friction down a fixed, inclined plane. The angle of the incline is $\theta = 30^\circ$ from horizontal. What is the acceleration of the block?

- Sketch:



  $\vec{F_g} = m\vec{g}$ is the gravitational force on the block and $\vec{F_N}$ is the normal force, which is exerted by the plane on the block to keep it in place on top of the plane.
- Coordinate system: x pointing down along the surface of the incline, y perpendicular to the surface of the incline. The constraint of the block sliding on the plane forces there to be no motion along y, hence the choice of coordinate system.

- Forces along each axis:

$$\begin{aligned} m\,\ddot{x} &= F_g \sin\theta \\ m\,\ddot{y} &= F_N - F_g \cos\theta \end{aligned}$$

- Apply constraints: there is no motion along the y axis, so $\ddot{y} = 0$, which gives $F_N = F_g \cos\theta$. The constraint actually turns out to be unnecessary for solving for the motion of the block, but in more complicated cases the constraint will be important.
- Solve the remaining equations: Here, we simply have the x equation, which gives:

$$\ddot{x} = \frac{F_g}{m} \sin\theta = g \sin\theta$$

  where $F_g = mg$ is the gravitational force
- Find velocity and position as a function of time: This is just trivial integration:

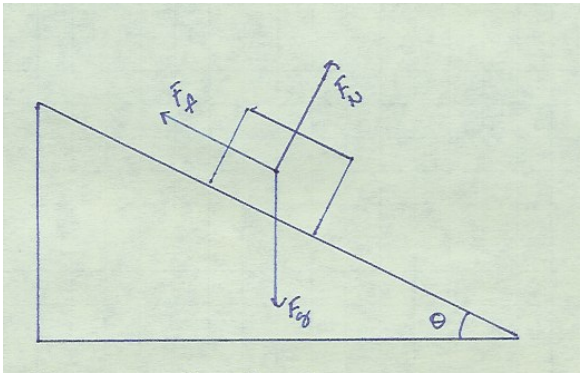
$$\begin{aligned} \frac{d}{dt}\,\dot{x} = g \sin\theta \Longrightarrow \dot{x}(t) &= \dot{x}(t=0) + \int_0^t dt'\, g \sin\theta \\ &= \dot{x}_0 + g\,t \sin\theta \\ \frac{d}{dt}\,x = \dot{x}(t=0) + g\,t \sin\theta \Longrightarrow x(t) &= x_0 + \int_0^t dt' \left[\dot{x}_0 + g\,t' \sin\theta\right] \\ &= x_0 + \dot{x}_0\,t + \frac{1}{2}\,g\,t^2 \sin\theta \end{aligned}$$

  where we have taken x_0 and $\dot{x}_0$ to be the initial position and velocity, the constants of integration. Of course, the solution for y is $y(t) = 0$, where we have made use of the initial conditions $y(t=0) = 0$ and $\dot{y}(t=0) = 0$.

Example 1.2

(Thornton Example 2.3) Same as Example 1.1, but now assume the block is moving (*i.e.*, its initial velocity is nonzero) and that it is subject to sliding friction. Determine the acceleration of the block for the angle $\theta = 30^\circ$ assuming the frictional force obeys $F_f = \mu_k\,F_N$ where $\mu_k = 0.3$ is the coefficient of kinetic friction.

- Sketch:



  We now have an additional frictional force F_f which points along the $-x$ direction because the block of course wants to slide to $+x$. Its value is fixed to be $F_f = \mu_k\,F_N$.

- Coordinate system: same as before.
- Forces along each axis:

$$\begin{aligned} m\,\ddot{x} &= F_g \sin\theta - F_f \\ m\,\ddot{y} &= F_N - F_g \cos\theta \end{aligned}$$

  We have the additional frictional force acting along $-x$.
- Apply constraints: there is no motion along the y axis, so $\ddot{y} = 0$, which gives $F_N = F_g \cos\theta$. Since $F_f = \mu_k F_N$, the equation resulting from the constraint can be used directly to simplify the other equation.
- Solve the remaining equations: Here, we simply have the x equation,

$$\begin{aligned} \ddot{x} &= \frac{F_g}{m} \sin\theta - \mu_k \frac{F_g}{m} \cos\theta \\ &= g\,[\sin\,\theta - \mu_k\,\cos\theta] \end{aligned}$$

  That is all that was asked for. For $\theta = 30^\circ$, the numerical result is

$$\ddot{x} = g\,(\sin 30^\circ - 0.3\cos 30^\circ) = 0.24\,g$$

Example 1.3

(Thornton Example 2.2) Same as Example 1.1, but now allow for static friction to hold the block in place, with coefficient of static friction $\mu_s = 0.4$. At what angle does it become possible for the block to slide?

- Sketch: Same as before, except the distinction is that the frictional force F_f does not have a fixed value, but we know its maximum value is $\mu_s F_N$.
- Coordinate system: same as before.
- Forces along each axis:

$$\begin{aligned} m\,\ddot{x} &= F_g \sin\theta - F_f \\ m\,\ddot{y} &= F_N - F_g \cos\theta \end{aligned}$$

- Apply constraints: there is no motion along the y axis, so $\ddot{y} = 0$, which gives $F_N = F_g \cos\theta$. We will use the result of the application of the constraint below.
- Solve the remaining equations: Here, we simply have the x equation,

$$\ddot{x} = \frac{F_g}{m} \sin\theta - \frac{F_f}{m}$$

- Since we are solving a static problem, we don't need to go to the effort of integrating to find $x(t)$; in fact, since the coefficient of sliding friction is usually lower than the coefficient of static friction, the above equations become incorrect as the block begins to move. Instead, we want to figure out at what angle $\theta = \theta'$ the block begins to slide. Since F_f has maximum value $\mu_s F_N = \mu_s\, m\, g\, \cos\theta$, it holds that

$$\ddot{x} \geq \frac{F_g}{m} \sin\theta - \mu_s \frac{F_N}{m}$$

i.e.,

$$\ddot{x} \geq g\left[\sin\theta - \mu_s \cos\theta\right]$$

It becomes impossible for the block to stay motionless when the right side becomes positive. The transition angle θ' is of course when the right side vanishes, when

$$0 = \sin\theta' - \mu_s \cos\theta'$$

or

$$\tan\theta' = \mu_s$$

which gives $\theta' = 21.8°$.

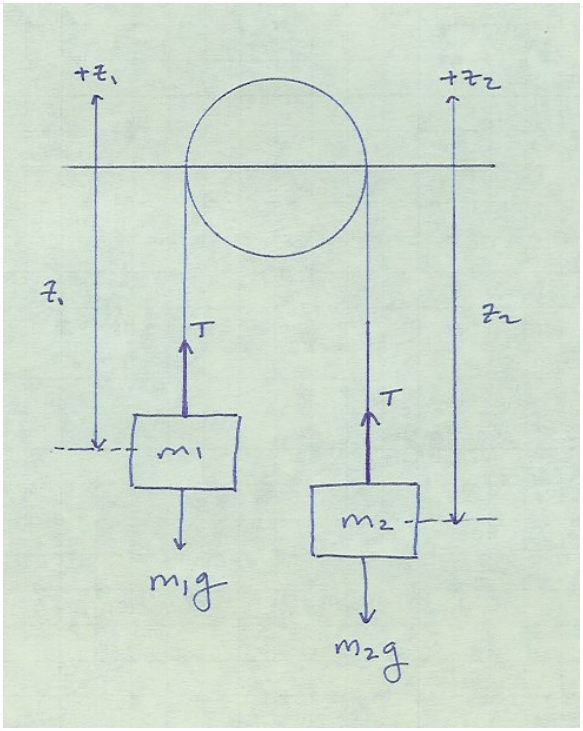
Atwood's machine problems

Another class of problems Newtonian mechanics problems you have no doubt seen before are Atwood's machine problems, where an Atwood's machine is simply a smooth, massless pulley (with zero diameter) with two masses suspended from a (weightless) rope at each end and acted on by gravity. These problems again require only Newton's second equation.

Example 1.4

(Thornton Example 2.9) Determine the acceleration of the two masses of a simple Atwood's machine, with one fixed pulley and two masses m_1 and m_2.

- Sketch:



- Coordinate system: There is only vertical motion, so use the z coordinates of the two masses z_1 and z_2.

- Forces along each axis: Just the z-axis, but now for two particles:

$$\begin{aligned} m_1 \ddot{z_1} &= -m_1 g + T \\ m_2 \ddot{z_2} &= -m_2 g + T \end{aligned}$$

  where T is the tension in the rope. We have assumed the rope perfectly transmits force from one end to the other.

- Constraints: The rope length l cannot change, so $z_1 + z_2 = -l$ is constant, $\dot{z}_1 = -\dot{z}_2$ and $\ddot{z}_1 = -\ddot{z}_2$.

- Solve: Just solve the first equation for T and insert in the second equation, making use of $\ddot{z}_1 = -\ddot{z}_2$:

$$\begin{aligned} T &= m_1(\ddot{z_1} + g) \\ -\ddot{z}_1 &= -g + \frac{m_1}{m_2}(\ddot{z_1} + g) \end{aligned}$$

  which we can then solve for $\ddot{z}_1$ and T:

$$\begin{aligned} -\ddot{z}_2 = \ddot{z}_1 &= -\frac{m_1 - m_2}{m_1 + m_2}\, g \\ T &= \frac{2\, m_1\, m_2}{m_1 + m_2}\, g \end{aligned}$$

  It is instructive to consider two limiting cases. First, take $m_1 = m_2 = m$. We have in this case

$$\begin{aligned} -\ddot{z}_2 = \ddot{z}_1 &= 0 \\ T &= m\, g \end{aligned}$$

  As you would expect, there is no motion of either mass and the tension in the rope is the weight of either mass – the rope must exert this force to keep either mass from falling. Second, consider $m_1 \gg m_2$. We then have

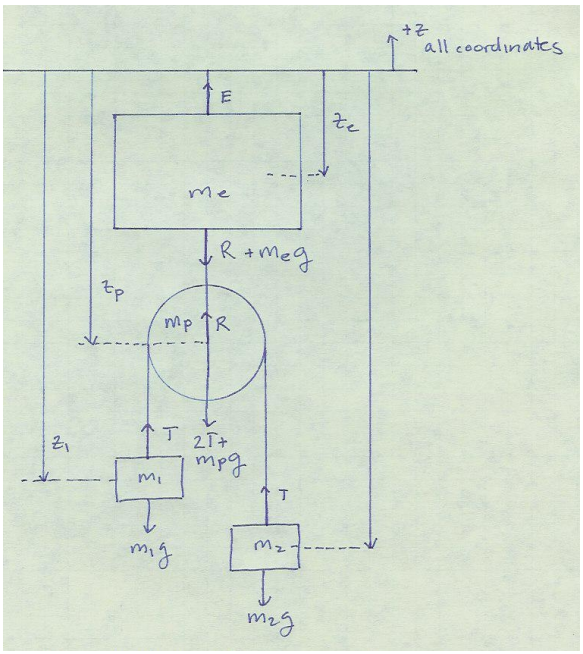
$$\begin{aligned} -\ddot{z}_2 = \ddot{z}_1 &= -g \\ T &= 2\, m_2\, g \end{aligned}$$

  Here, the heavier mass accelerates downward at the gravitational acceleration and the other mass accelerates upward with the same acceleration. The rope has to have sufficient tension to both counteract gravity acting on the second mass as well as to accelerate it upward at g.

Example 1.5

(Thornton Example 2.9) Repeat, with the pulley suspended from an elevator that is accelerating with acceleration a. As usual, ignore the mass and diameter of the pulley when considering the forces in and motion of the rope.

- Sketch:



  Obviously, we have the gravitational forces on each object. The pulley also has $2T$ acting downward on it (due to the force exerted by the rope on the pulley) and R acting upward (the force pulling upward on the pulley through the rope connected to the elevator). Similarly, the elevator has tension forces R acting downward and E upward. We include the forces on the pulley and elevator since, *a priori*, it's not obvious that they should be ignored. We will see that it is not necessary to solve for the forces on the pulley and elevator to find the accelerations of the masses, but we will be able to find these forces.

- Coordinate system: Remember that Newton's second law only holds in inertial reference frames. Therefore, we should reference the positions of the masses to the fixed frame rather than to the elevator. Again, denote the z coordinates of the two masses by z_1 and z_2. Let the z coordinates of the pulley and elevator be z_p and z_e.

- Forces along each axis: Just the z-axis

$$\begin{aligned} m_1\ddot{z_1} &= -m_1 g + T \\ m_2\ddot{z_2} &= -m_2 g + T \\ m_p\ddot{z_p} &= R - 2T - m_p g \\ m_e\ddot{z_e} &= E - R - m_e g \end{aligned}$$

  where T is the tension in the rope holding the two masses, R is the tension in the rope holding the pulley, and E is the force being exerted on the elevator to make it ascend or descend. Note especially the way we only consider the forces acting directly on an object; trying to unnecessarily account for forces is a common error. For example, even though gravity acts on m_1 and m_2 and some of that force is transmitted to and acts on the pulley, we do not directly include such forces; they are implicitly included by their effect on T. Similarly for the forces on the elevator.

- Constraints: Again, the rope length cannot change, but the constraint is more complicated because the pulley can move: $z_1 + z_2 = 2\,z_p - l$. The fixed rope between the pulley and the elevator forces $\ddot{z}_p = \ddot{z}_e = a$, so $\ddot{z}_1 + \ddot{z}_2 = 2\,a$
- Solve: Just solve the first equation for T and insert in the second equation, making use of the new constraint $\ddot{z}_1 = -\ddot{z}_2 + 2a$:

$$\begin{aligned} T &= m_1(\ddot{z}_1 + g) \\ 2a - \ddot{z}_1 &= -g + \frac{m_1}{m_2}(\ddot{z}_1 + g) \end{aligned}$$

which we can then solve for $\ddot{z}_1$ and T:

$$\begin{aligned} \ddot{z}_1 &= -\frac{m_1 - m_2}{m_1 + m_2}\,g + \frac{2m_2}{m_1 + m_2}\,a \\ \ddot{z}_2 &= \frac{m_1 - m_2}{m_1 + m_2}\,g + \frac{2m_1}{m_1 + m_2}\,a \\ T &= \frac{2\,m_1\,m_2}{m_1 + m_2}(g + a) \end{aligned}$$

We can write the accelerations relative to the elevator (*i.e.*, in the non-inertial, accelerating frame) by simply calculating $\ddot{z}_1' = \ddot{z}_1 - \ddot{z}_p$ and $\ddot{z}_2' = \ddot{z}_2 - \ddot{z}_p$:

$$\begin{aligned} \ddot{z}_1' &= \frac{m_2 - m_1}{m_2 + m_1}\,(g + a) \\ \ddot{z}_2' &= \frac{m_1 - m_2}{m_1 + m_2}\,(g + a) \end{aligned}$$

We see that, in the reference frame of the elevator, the accelerations are equal and opposite, as they must be since the two masses are coupled by the rope. Note that we never needed to solve the third and fourth equations, though we may now do so:

$$\begin{aligned} R &= m_p(\ddot{z}_p + g) + 2T \\ &= m_p(a + g) + \frac{4m_1m_2}{m_1 + m_2}\,(g + a) \\ &= \left[m_p + \frac{4m_1m_2}{m_1 + m_2}\right]\,(g + a) \\ E &= m_e(\ddot{z}_e + g) + R \\ &= \left[m_e + m_p + \frac{4m_1m_2}{m_1 + m_2}\right]\,(g + a) \end{aligned}$$

That these expressions are correct can be seen by considering limiting cases. First, consider the case $m_1 = m_2 = m$; we find

$$\begin{aligned} \ddot{z}_1 &= a \\ \ddot{z}_2 &= a \\ T &= m\,(g + a) \\ R &= [m_p + 2\,m]\,(g + a) \\ E &= [m_e + m_p + 2\,m]\,(g + a) \end{aligned}$$

That is, the two masses stay at rest relative to each other and accelerate upward with the elevator; there is no motion of the rope connecting the two (relative to the pulley)

because the two masses balance each other. The tensions in the rope holding the pulley and the elevator cable are determined by the total mass suspended on each. Next, consider the case $m_1 \gg m_2$. We have

$$\begin{aligned} \ddot{z}_1 &= -g \\ \ddot{z}_2 &= g + 2\,a \\ T &= 0 \\ R &= m_p\,(g+a) \\ E &= [m_e + m_p]\;(g+a) \end{aligned}$$

m_1 falls under the force of gravity. m_2 is pulled upward, but there is a component of the acceleration in addition to just g because the rope must unwind over the pulley fast enough to deal with the accelerating motion of the pulley. R and E no longer contain terms for m_1 and m_2 because the rope holding them just unwinds, exerting no force on the pulley.

The mass combination that appears in the solutions, $m_1 m_2/(m_1 + m_2)$, is the typical form one finds for transforming continuously between these two cases $m_1 = m_2$ and $m_1 \gg m_2$ (or vice versa), as you will learn when we look at central force motion later.

Retarding Forces

(See Thornton 2.4 for more detail, but these notes cover the important material)

A next level of complexity is introduced by considering forces that are not static but rather depend on the velocity of the moving object. This is interesting not just for the physics but because it introduces a higher level of mathematical complexity. Such a force can frequently be written as a power law in the velocity:

$$\vec{F}_r = \vec{F}_r(v) = -k\,v^n \frac{\vec{v}}{v} \tag{1.6}$$

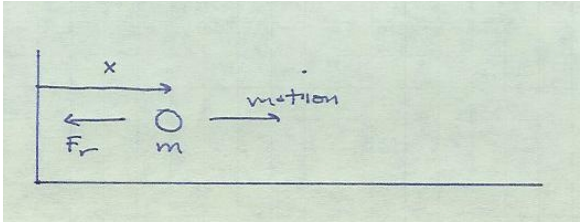
k is a constant that depends on the details of the problem. Note that the force is always directed opposite to the velocity of the object.

For the simplest power law retarding forces, the equation of motion can be solved analytically. For more complicated dependence on velocity, it may be necessary to generate the solution numerically. We will come back to the latter point.

Example 1.6

(Thornton Example 2.4). Find the velocity and position as a function of time for a particle initially having velocity v_0 along the $+x$ axis and experiencing a linear retarding force $F_r(v) = -k\,v$.

- Sketch:



- Coordinate system: only one dimension, so trivial. Have the initial velocity $\dot{x}_0$ be along the $+x$ direction.

- Forces along each axis: Just the x-axis.

$$m\,\ddot{x} = -k\,\dot{x}$$

- Constraints: none

- Solve: The differential equation for x is

$$\frac{d}{dt}\,\dot{x} \;=\; -\frac{k}{m}\,\dot{x}(t)$$

This is different than we have seen before since the right side is not fixed, but depends on the left side. We can solve by separating the variables and integrating:

$$\begin{aligned}
\frac{d\dot{x}}{\dot{x}} &= -\frac{k}{m}\,dt \\
\int_{\dot{x}_0}^{\dot{x}(t)} \frac{dy}{y} &= -\frac{k}{m}\int_0^t dt' \\
\log\dot{x}(t) - \log\dot{x}_0 &= -\frac{k}{m}\,t \\
\dot{x}(t) &= \dot{x}_0\,\exp\left(-\frac{k}{m}\,t\right)
\end{aligned}$$

That is, the velocity decreases exponentially, going to 0 as $t \to \infty$. The position is easily obtained from the velocity:

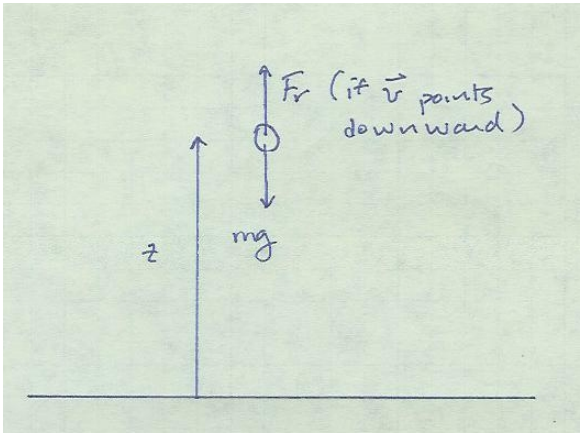
$$\begin{aligned}
\frac{d}{dt}\,x &= \dot{x}_0\,\exp\left(-\frac{k}{m}\,t\right) \\
x(t) &= x_0 + \dot{x}_0\int_0^t dt'\,\exp\left(-\frac{k}{m}\,t'\right) \\
x(t) &= x_0 + \frac{m\,\dot{x}_0}{k}\left[1 - \exp\left(-\frac{k}{m}\,t\right)\right]
\end{aligned}$$

The object asymptotically moves a distance $m\,\dot{x}_0/k$.

Example 1.7

(Thornton Example 2.5). Repeat Example 1.6, but now for a particle undergoing vertical motion in the presence of gravity.

- Sketch:



- Coordinate system: only one dimension, so trivial. Have the initial velocity $\dot{z}_0$ be along the $+z$ direction.
- Forces along each axis: Just the z-axis.

$$m\,\ddot{z} = -m\,g - k\,\dot{z}$$

- Constraints: none
- Solve: The differential equation for z is

$$\frac{d}{dt}\,\dot{z} \;=\; -g - \frac{k}{m}\,\dot{z}(t)$$

Now we have both constant and velocity-dependent terms on the right side. Again, we solve by separating variables and integrating:

$$\begin{aligned}
\frac{d\dot{z}}{g + \frac{k}{m}\dot{z}} &= -dt \\
\int_{\dot{z}_0}^{\dot{z}(t)} \frac{dy}{g + \frac{k}{m}\,y} &= -\int_0^t dt' \\
\frac{m}{k}\int_{g+\frac{k}{m}\,\dot{z}_0}^{g+\frac{k}{m}\,\dot{z}(t)} \frac{du}{u} &= -\int_0^t dt' \\
\log\left(g + \frac{k}{m}\,\dot{z}(t)\right) - \log\left(g + \frac{k}{m}\,\dot{z}_0\right) &= -\frac{k}{m}\,t \\
1 + \frac{k}{m\,g}\dot{z}(t) &= \left(1 + \frac{k}{m\,g}\right)\exp\left(-\frac{k}{m}\,t\right) \\
\dot{z}(t) &= -\frac{m\,g}{k} + \left(\frac{m\,g}{k} + \dot{z}_0\right)\exp\left(-\frac{k}{m}\,t\right)
\end{aligned}$$

We see the phenomenon of terminal velocity: as $t \to \infty$, the second term vanishes and we see $\dot{z}(t) \to -m\,g/k$. One would have found this asymptotic speed by also solving the equation of motion for the speed at which the acceleration vanishes. The position as a function of time is again found easily by integrating, which yields

$$z(t) \;=\; z_0 - \frac{m\,g}{k}\,t + \left(\frac{m^2\,g}{k^2} + \frac{m\,\dot{z}_0}{k}\right)\left[1 - \exp\left(-\frac{k}{m}\,t\right)\right]$$

The third term deals with the portion of the motion during which the velocity is changing, and the second term deals with the terminal velocity portion of the trajectory.

Retarding Forces and Numerical Solutions

Obviously, for more complex retarding forces, it may not be possible to solve the equation of motion analytically. However, a natural numerical solution presents itself. The equation of motion is usually of the form

$$\frac{d}{dt}\dot{x} \quad = \quad F_s + F(\dot{x})$$

This can be written in discrete form

$$\Delta\dot{x} \quad = \quad [F_s + F(\dot{x})]\,\Delta t$$

where we have discretized time into intervals of length Δt. If we denote the times as $t_n = n\Delta t$ and the velocity at time t_n by $\dot{x}_n$, then we have

$$\dot{x}_{n+1} \quad = \quad \dot{x}_n + [F_s + F(\dot{x}_n)]\,\Delta t$$

The above procedure can be done with as small a step size as desired to obtain as precise a solution as one desires. It can obviously also be extended to two-dimensional motion. For more details, see Thornton Examples 2.7 and 2.8.

1.1.2 Angular Motion

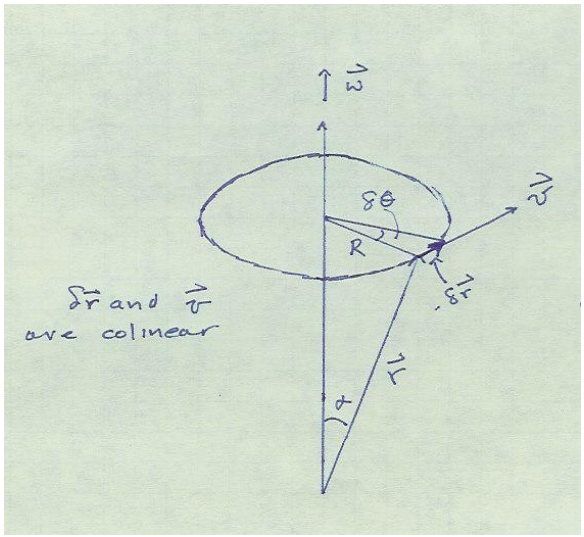
We derive analogues of linear momentum, force, and Newton's second law for angular motion.

Definitions

Angular velocity of a particle as a function of time with respect to a particular origin:

$$\vec{v}(t) = \vec{\omega}(t) \times \vec{r}(t) \tag{1.7}$$

This is an implicit definition that is justified by considering a differential displacement:



This can be written mathematically as

$$\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$$

where $\delta\vec{\theta}$ points along the axis of the motion and $\times$ indicates a vector cross-product. The cross-product gives the correct direction for the displacement $\delta\vec{r}$ (perpendicular to the axis and $\vec{r}$) and the correct amplitude ($|\delta\vec{r}| = R\,\delta\theta = r\,\delta\theta\sin\alpha$). If we then divide by the time δt required to make this displacement, we have

$$\frac{\delta\vec{r}}{\delta t} = \frac{\delta\vec{\theta}}{\delta t} \times \vec{r} \Longrightarrow \vec{v} = \vec{\omega} \times \vec{r}$$

Angular momentum of a particle (relative to a particular origin):

$$\vec{L}(t) = \vec{r}(t) \times \vec{p}(t) \tag{1.8}$$

The cross-product implies that $\vec{L}$ is a vector perpendicular to the plane formed by $\vec{r}$ and $\vec{p}$, with its direction set by the right-hand rule. $\vec{L}$ is defined as a cross-product between $\vec{r}$ and $\vec{p}$ so that a particle constrained to move in a circle with constant speed v (though with changing $\vec{v}$) by a central force (one pointing along $-\vec{r}$) has constant angular momentum. The sign is set by the right-hand rule.

We can rewrite in terms of $\vec{\omega}$ by using the implicit definition o f $\vec{\omega}$:

$$\begin{aligned} \vec{L} &= \vec{r} \times (m\,\vec{\omega} \times \vec{r}) \\ &= m\,[(\vec{r}\cdot\vec{r})\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r}] \end{aligned}$$

where we have used a vector identity to expand the triple cross-product (see Section A.3). For the simple case where $\vec{r}$ and $\vec{v}$ are perpendicular (and hence $\vec{\omega}$ is perpendicular to both of them), this simplifies to

$$\vec{L} = mr^2\vec{\omega}$$

i.e., $\vec{L}$ points along $\vec{\omega}$.

Torque exerted by a force $\vec{F}$ (relative to a particular origin): $\vec{N}(t) = \vec{r}(t) \times \vec{F}(t)$. We shall see below that this is the natural definition given the way angular momentum was defined above.

Note: Angular velocity, angular momentum, and torque depend on the choice of origin!

Newton's second law, angular momentum, and torque

From the definitions of angular momentum $\vec{L}$ and torque $\vec{N}$, it is trivial to see that Newton's second law implies a relation between them:

$$\begin{aligned} \frac{d}{dt}\,\vec{L}(t) &= \frac{d}{dt}\,[\vec{r}(t) \times \vec{p}(t)] \\ &= \dot{\vec{r}}(t) \times \vec{p}(t) + \vec{r}(t) \times \dot{\vec{p}}(t) \\ &= m\,\dot{\vec{r}}(t) \times \dot{\vec{r}}(t) + \vec{r}(t) \times \vec{F}(t) \\ &= \vec{N}(t) \end{aligned}$$

where we have used the definition of momentum and Newton's second law in going from the second line to the third line, and where the first term in the next-to-last line vanishes because the cross-product of any vector with itself is zero.

Conservation of Angular Momentum

Just as we proved that linear momentum is conserved in directions along which there is no force, one can prove that angular momentum is conserved in directions along which there is no torque. The proof is identical, so we do not repeat it here.

Choice of origin

There is a caveat: angular momentum and torque depend on the choice of origin. That is, in two frames 1 and 2 whose origins differ by a constant vector $\vec{o}$ such that $\vec{r}_2(t) = \vec{r}_1(t) + \vec{o}$, we have

$$\vec{L}_2(t) = \vec{r}_2(t) \times \vec{p}(t) = \vec{r}_1(t) \times \vec{p}(t) + \vec{o} \times \vec{p}(t) = \vec{L}_1(t) + \vec{o} \times \vec{p}(t)$$
$$\vec{N}_2(t) = \vec{r}_2(t) \times \vec{F}(t) = \vec{r}_1(t) \times \vec{F}(t) + \vec{o} \times \vec{F}(t) = \vec{N}_1(t) + \vec{o} \times \vec{F}(t)$$

where we have used the fact that $\vec{p}$ and $\vec{F}$ are the same in the two frames ($\vec{p}$ because it involves a time derivative; $\vec{F}$ via its relation to $\vec{p}$ by Newton's second law). Thus, while Newton's second law and conservation of angular momentum certainly hold regardless of choice of origin, angular momentum may be constant in one frame but not in another because a torque that vanishes in one frame may not vanish in another! In contrast, if linear momentum is conserved in one frame it is conserved in any displaced frame. Thus, angular momentum and torque are *imperfect* analogues to linear momentum and force. Let's consider this in more detail.

We first solve the linear equations of motion for a particle moving in a circle at fixed speed as shown in the previous figure. Choose the origin of the system to be at the center of the circle and the motion to be in the xy plane. Clearly, in this frame, the particle's position and velocity as a function of time are

$$\begin{aligned} \vec{r}_1(t) &= R\,(\hat{x}\,\cos\omega t + \hat{y}\,\sin\omega t) \\ \vec{v}(t) &= \omega\,R\,(-\hat{x}\,\sin\omega t + \hat{y}\,\sin\omega t) \end{aligned}$$

where we obtained the velocity by simple differentiation. We do not subscript $\vec{v}$ because it is independent of the choice of origin. The mass is fixed so the momentum is just $\vec{p}(t) = m\,\vec{v}(t)$. The force is, by Newton's second law,

$$\begin{aligned} \vec{F}(t) &= \frac{d\vec{p}}{dt} \\ &= -m\,\omega^2\,R\,(\hat{x}\,\cos\omega t + \hat{y}\,\sin\omega t) \\ &= -m\,\omega^2\,R\,\hat{r}_1(t) \\ &= -m\,\frac{v^2}{R}\,\hat{r}_1(t) \end{aligned}$$

where $\hat{r}_1(t)$ is a unit vector pointing along $\vec{r}_1(t)$. Clearly, the force is back along the line to center of the circle, has magnitude $F = m\,v^2/R = m\,\omega^2\,R$, and is perpendicular to the velocity. The velocity, momentum, and force are independent of the choice of origin.

Let's determine the angular momentum and torque. First consider the same coordinate system with position vector $\vec{r}_1(t)$. Since $\vec{F}$ points back along $\vec{r}_1$, it holds that the torque $\vec{N} = \vec{r}_1 \times \vec{F}$ vanishes. The angular momentum vector is $\vec{L}_1 = \vec{r}_1 \times \vec{p} = mv\,R\,\hat{z}$. Since v is fixed, $\vec{L}_1$ is fixed, as one would expect in the absence of torque.

Next, consider a frame whose origin is displaced from the center of the particle orbit along the z axis, as suggested by the earlier figure. Let $\vec{r}_2$ denote the position vector in this frame. In this frame, the torque $\vec{N}_2$ is nonzero because $\vec{r}_2$ and $\vec{F}$ are not colinear. We can write out the torque explicitly:

$$\begin{aligned}
\vec{N}_2(t) &= \vec{r}_2(t) \times \vec{F}(t) \\
&= r_2 \left[\sin\alpha(\hat{x}\cos\omega t + \hat{y}\sin\omega t) + \hat{z}\cos\alpha\right] \times F\left(-\hat{x}\cos\omega t - \hat{y}\sin\omega t\right) \\
&= r_2 F \left[-\sin\alpha(\hat{x}\times\hat{y}\cos\omega t\sin\omega t + \hat{y}\times\hat{x}\sin\omega t\cos\omega t)\right. \\
&\quad \left. -\cos\alpha(\hat{z}\times\hat{x}\cos\omega t + \hat{z}\times\hat{y}\sin\omega t)\right] \\
&= r_2\, F\cos\alpha\,(\hat{x}\sin\omega t - \hat{y}\cos\omega t)
\end{aligned}$$

where, between the second and third line, terms of the form $\hat{x}\times\hat{x}$ and $\hat{y}\times\hat{y}$ were dropped because they vanish.

Let's calculate the angular momentum in this system:

$$\begin{aligned}
\vec{L}_2(t) &= \vec{r}_2(t) \times \vec{p}(t) \\
&= r_2 \left[\sin\alpha(\hat{x}\cos\omega t + \hat{y}\sin\omega t) + \hat{z}\cos\alpha\right] \times mv\left(-\hat{x}\sin\omega t + \hat{y}\cos\omega t\right) \\
&= r_2\, mv\left[\sin\alpha(\hat{x}\times\hat{y}\cos^2\omega t - \hat{y}\times\hat{x}\sin^2\omega t) + \cos\alpha(-\hat{z}\times\hat{x}\sin\omega t + \hat{z}\times\hat{y}\cos\omega t)\right] \\
&= \hat{z}\, mv\, r_2\sin\alpha + mv\, r_2\cos\alpha\,(\hat{y}\sin\omega t - \hat{x}\cos\omega t)
\end{aligned}$$

So, in this frame, we have a time-varying component of $\vec{L}_2$ in the plane of the orbit. This time derivative of $\vec{L}_2$ is due to the nonzero torque $\vec{N}_2$ present in this frame, as one can demonstrate directly by differentiating $\vec{L}_2(t)$ and using $F = mv^2/R = mv^2/(r_2\cos\alpha)$ and $v = R\,\omega = r_2\,\omega\,\cos\alpha$. The torque is always perpendicular to the varying component of the angular momentum, so the torque causes the varying component of the angular momentum to precess in a circle.

One can of course consider even more complicated cases wherein the origin displacement includes a component in the plane of the motion. Clearly, the algebra gets more complicated but none of the physics changes.

1.1.3 Energy and Work

We present the concepts of kinetic and potential energy and work and derive the implications of Newton's second law on the relations between them.

Work and Kinetic Energy

We define the **work** done on a particle by a force $\vec{F}(t)$ in moving it from $\vec{r}_1 = \vec{r}(t_1)$ to $\vec{r}_2 = \vec{r}(t_2)$ to be

$$W_{12} = \int_{t_1}^{t_2} \vec{F}\cdot d\vec{r} \tag{1.9}$$

The integral is a *line integral*; the idea is to integrate up the projection of the force along the instantaneous direction of motion. We can write the expression more explicitly to make this clear:

$$\begin{aligned}
W_{12} &= \int_{t_1}^{t_2} \vec{F}(t)\cdot\frac{d\vec{r}}{dt}\,dt \\
&= \int_{t_1}^{t_2} \vec{F}(t)\cdot\vec{v}(t)\,dt
\end{aligned}$$

The value of this definition is seen by using Newton's second law to replace $\vec{F}$:

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} \frac{d\vec{p}}{dt} \cdot \frac{\vec{p}}{m}\, dt \\ &= \frac{1}{2m} \int_{\vec{p}_1}^{\vec{p}_2} d(\vec{p} \cdot \vec{p}) \\ &= \frac{p_2^2}{2m} - \frac{p_1^2}{2m} \\ &\equiv T_2 - T_1 \end{aligned} \tag{1.10}$$

where we have defined the **kinetic energy** $T = p^2/2m = mv^2/2$. Thus, the work the force does on the particle tells us how much the particle's kinetic energy changes. This is kind of deep: it is only through Newton's second law that we are able to related something external to the particle – the force acting on it – to a property of the particle – its kinetic energy.

Note here that, to demonstrate the connection between work and kinetic energy, we have had to specialize to consider the **total force** on the particle; Newton's second law applies only to the total force. For example, consider an elevator descending at constant velocity. Two forces act on the elevator: a gravitational force pointing downward and tension force in the elevator cable pointing upward. If the elevator is to descend at constant velocity, the net force vanishes. Thus, no work is done on the elevator, as is evinced by the fact that its speed (and therefore its kinetic energy) are the same at the start and end of its motion. One could of course calculate the work done by the gravitational force on the elevator and get a nonzero result. This work would be canceled exactly by the negative work done by the cable on the elevator.

Potential Energy, Conservation of Energy, and Conservative Forces

Consider forces that depend only on position $\vec{r}$ (no explicit dependence on t, $\vec{v}$). Counterexample: retarding forces.

Furthermore, consider forces for which W_{12} is *path-independent*, *i.e.*, depends only on $\vec{r}_1$ and $\vec{r}_2$. Another way of saying this is that the work done around a closed path vanishes: pick any two points 1 and 2, calculate the work done in going from 1 to 2 and from 2 to 1. The latter will be the negative of the former if the work done is path-independent. By Stokes' Theorem (see Appendix A), we then see that path-independence of work is equivalent to requiring that $\vec{\nabla} \times \vec{F} = 0$ everywhere. (Do there exist position-dependent forces for which this is not true? Hard to think of any physically realized ones, but one can certainly construct force functions with nonzero curl.)

Then it is possible to define the **potential energy** as a function of position:

$$U(\vec{r}) = U(0) - \int_0^{\vec{r}} \vec{F}(\vec{r}_1) \cdot d\vec{r}_1 \tag{1.11}$$

The potential energy is so named because it indicates the amount of kinetic energy the particle *would* gain in going from $\vec{r}$ back to the origin; the potential energy says how much work the force $\vec{F}$ *would* do on the particle.

The offset or origin of the potential energy is physically irrelevant since we can only measure changes in kinetic energy and hence differences in potential energy. That is,

$$U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r} = -W_{12} = T_1 - T_2$$

If we define the total energy E as the sum of potential and kinetic energies (modulo the aforementioned arbitrary offset $U(0)$) and rewrite, we obtain **conservation of energy**:

$$E_2 = U(\vec{r}_2) + T_2 = U(\vec{r}_1) + T_1 = E_1 \tag{1.12}$$

i.e., the total energy E is conserved. Forces for which conservation of energy holds – *i.e.*, those forces for which the work done in going from $\vec{r}_1$ to $\vec{r}_2$ is *path-independent* – are accordingly called **conservative forces**.

For conservative forces, we may differentiate Equation 1.11 with respect to time to obtain an alternate relation between force and potential energy:

$$\begin{aligned} \frac{d}{dt}U &= -\vec{F}\cdot\frac{d\vec{r}}{dt} \\ \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} &= -\vec{F}\cdot\vec{v} \\ -\vec{\nabla}U\cdot\vec{v} &= \vec{F}\cdot\vec{v} \end{aligned}$$

where $\vec{\nabla}$ is the gradient operator you are familiar with. Now, since the initial velocity is arbitrary, the above must hold for any velocity $\vec{v}$, and the more general statement is

$$-\vec{\nabla}U = \vec{F} \tag{1.13}$$

That is, a conservative force is the negative of the gradient of the potential energy function that one can derive from it. Recall the physical meaning of the gradient: given a function $U(\vec{r})$, the gradient of U at a given point $\vec{r}$ is a vector perpendicular to the surface passing through $\vec{r}$ on which U is constant. Such surfaces are called **equipotential surfaces**. The force is normal to such equipotential surfaces, indicating that particles want to move away from equipotential surfaces in the direction of decreasing U.

Example 1.8

Calculate the work done by gravity on a particle shot upward with velocity $\vec{v} = v_0\,\hat{z}$ in the time 0 to t_f. Demonstrate that the work equals the change in kinetic energy. Also calculate the change in potential energy, demonstrate that the change in potential energy equals the negative of the work done, and demonstrate conservation of energy.

- First, calculate the motion of the particle. This is straightforward, the result is

$$\begin{aligned} \vec{v}(t) &= \hat{z}\,(v_0 - g\,t) \\ \vec{r}(t) &= \hat{z}\,(z_i + v_0\,t - \frac{1}{2}g\,t^2) \end{aligned}$$

  The time at which the particle reaches its maximum high is $t_m = v_0/g$ and the maximum height is $z_m = z_i + v_0^2/2g$.

- Calculate the work done:

$$\begin{aligned} W(t_f) &= \int_{z_i\hat{z}}^{z_f\hat{z}} \vec{F}(t)\cdot d\vec{r} \\ &= \int_{z_i}^{z_m} (-m\,g)\,dz + \int_{z_m}^{z_f} (-m\,g)\,dz \\ &= m\,g\,[-(z_m - z_i) - (z_f - z_m)] \\ &= m\,g\,(z_i - z_f) \end{aligned}$$

We explicitly split the integral into two pieces to show how to deal with change of sign of the direction the particle moves. Now, $\vec{r} = \hat{x}\,x + \hat{y}\,y + \hat{z}\,z$, so $d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$. This is to be left as-is – do not mess with the sign of dz. Rather, realize that the limits of integration for the second part of the path go from z_m to z_f because the limits follow along the path of the particle.. This provides the sign flip needed on the second term to give the result we expect. Alternatively, one could have flipped the sign on dz and the limits simultaneously, integrating over $(-dz)$ from z_f to z_m; but that doesn't make much sense. So, the general rule is – your limits of integration should follow the chronological path of the particle, and your line element $d\vec{r}$ should be left untouched. We could also calculate the work using the other form:

$$\begin{aligned} W(t_f) &= \int_0^t \vec{F}(t)\cdot\vec{v}(t)\,dt \\ &= \int_0^{t_f} (-m\,g)\,(v_0 - g\,t)\,dt \\ &= -m\,g\,(v_0\,t_f - \frac{1}{2}\,g\,t_f^2) \\ &= -m\,g\,(-z_i + z_i + v_0\,t_f - \frac{1}{2}\,g\,t_f^2) \\ &= m\,g\,(z_i - z_f) \end{aligned}$$

- Check that the work is equal to the change in kinetic energy:

$$\begin{aligned} T_f - T_i &= \frac{1}{2}\,m\,(v_f^2 - v_0^2) \\ &= \frac{1}{2}\,m\,\left[(v_0 - g\,t_f)^2 - v_0^2\right] \\ &= \frac{1}{2}\,m\,(g^2 t_f^2 - 2\,v_0\,g\,t_f) \\ &= m\,g\,(-v_0\,t_f + \frac{1}{2}\,g\,t_f^2) \\ &= m\,g\,(z_i - z_i - v_0\,t_f + \frac{1}{2}\,g\,t_f^2) \\ &= m\,g\,(z_i - z_f) \end{aligned}$$

- Check that the change in potential energy is the negative of the work:

$$\begin{aligned} U(z_f) - U(z_i) &= m\,g\,z_f - m\,g\,z_i \\ &= -m\,g\,(z_i - z_f) \\ &= -W(t_f) \end{aligned}$$

- And check that energy is conserved:

$$\begin{aligned} E_i = U(z_i) + T_i &= m\,g\,z_i + \frac{1}{2}\,m\,v_0^2 \\ E_f = U(z_f) + T_f &= m\,g\,z_f + \frac{1}{2}\,m\,[v(t_f)]^2 \\ &= m\,g\,(z_i + v_0\,t_f - \frac{1}{2}\,g\,t_f^2) + \frac{1}{2}m\,(v_0^2 + -2v_0\,g\,t_f + g^2\,t_f^2 \\ &= m\,g\,z_i + \frac{1}{2}\,m\,v_0^2 \\ &= E_i \end{aligned}$$

Nonconservative Forces, Mechanical vs. Thermal Energy

Of course, there are many forces that are not conservative. We consider the example of a particle falling under the force of gravity and air resistance, and launched downward with the terminal velocity. The particle's velocity remains fixed for the entire fall. Let's examine the concepts of work, kinetic energy, potential energy, and conservation of energy for this case.

- **Work:** The net force on the particle vanishes because the air resistance exactly cancels gravity. The particle's speed remains fixed. Thus, no work is done on the particle.
- **Kinetic Energy:** Since the particle's speed remains fixed, its kinetic energy is also fixed, consistent with the fact that no work is done.
- **Potential Energy:** Clearly, the particle's potential energy decreases as it falls.
- **Conservation of Energy:** So, where does the potential energy go? Here we must make the distinction between **mechanical energy** and **thermal energy**. We demonstrated earlier that the sum of the kinetic and potential energy is conserved when a particle is acted upon by conservative forces. The kinetic and potential energy are the total **mechanical energy** of the particle. For extended objects consisting of many particles, there is also **thermal energy**, which is essentially the random kinetic energy of the particles making up the extended object; the velocities of these submotions cancel, so they correspond to no net motion of the object. Of course, we have not considered thermal energy because we have only been talking about motion of pointlike particles under the influence of idealized forces. Even in the presence of nonconservative forces, the sum of the mechanical and thermal energy is conserved. The potential energy lost by the falling particle in our example is converted to thermal energy of the falling particle and the surrounding air.

We will be able to rigorously prove the conservation of total energy later when we consider the dynamics of **systems of particles**.

Calculating Motion from the Potential Energy

For particles acting under conservative forces, we have seen that mechanical energy is conserved. We can use this fact to deduce the dynamics purely from the potential energy function.

- **Solving for the motion using the potential energy**
  Conservation of energy tells us that there is a constant E such that

$$E = T + U = \frac{1}{2}\, m\, v^2 + U(x)$$

  Rearranging, we have

$$\frac{dx}{dt} = v = \pm\sqrt{\frac{2}{m}\left[E - U(x)\right]}$$

  Formally, we can integrate to find

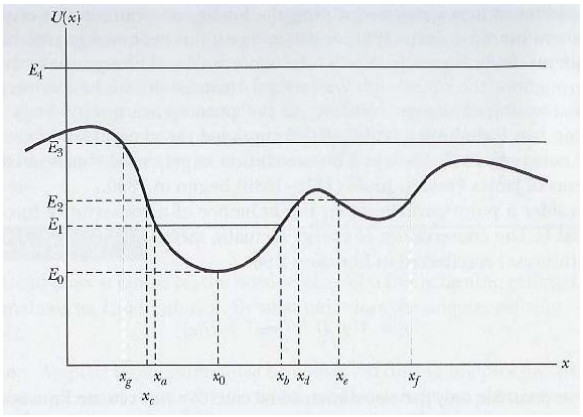
$$t - t_0 = \int_{x_0}^{x} \frac{\pm\, dx'}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}}$$

Given $U(x)$ it is possible to find $x(t)$. In some cases, it is possible to do this analytically, in others numerical integration may be required. But the fundamental point is that $U(x)$ determines the motion of the particle.

- **Is the motion bounded?**

  $T \geq 0$ always holds. But $\vec{v}$ may go through zero and change sign. If this happens for both signs of $\vec{v}$, then the motion is **bounded**.

  Consider the abstract potential energy curve shown in the following figure (Thornton Figure 2.14):



  $\vec{v}$ goes to zero when T vanishes. T can vanish if there are points x such that $U(x) \geq E$. Thus, if the particle begins at a point x_0 such that there are points x_a and x_b, $x_a < x_0 < x_b$, such that $U(x_a) \geq E$ and $U(x_b) \geq E$, then T vanishes at those points and the velocity vanishes. In order to make the velocity change sign, there must be a force continuing to accelerate the particle at these endpoints. The force is $\vec{F} = -\vec{\nabla}U = -\hat{x}\, dU/dx$ in this one-dimensional example; *i.e.*, if U has a nonzero derivative with the appropriate sign at x_a and x_b, then the particle turns around and the motion is bounded. A particle with energy E_1 as indicated in the figure has bounded motion.

  There can be multiple regions in which a particle of a given energy can be bounded. In the figure, a particle with energy E_2 could be bounded between x_a and x_b or x_e and x_f. Which one depends on the initial conditions. The particle cannot go between the two bounded regions.

  The motion is of course **unbounded** if E is so large that x_a and x_b do not exist. The motion can be bounded on only one side and unbounded on the other. For example, a particle with energy E_3 as indicated in the figure is bounded on the left at x_g but unbounded on the right. A particle with energy E_4 is unbounded on both sides.

- **Equilibria**

  A point with $\vec{\nabla}U = 0$ is an **equilibrium** point because the force vanishes there. Of course, the particle must have zero velocity when it reaches such a point to avoid going past it into a region where the force is nonzero. There are three types of equilibrium points.

  A **stable equilibrium** point is an equilibrium point at which d^2U/dx^2 is positive. The potential energy surface is concave up, so any motion away from the equilibrium

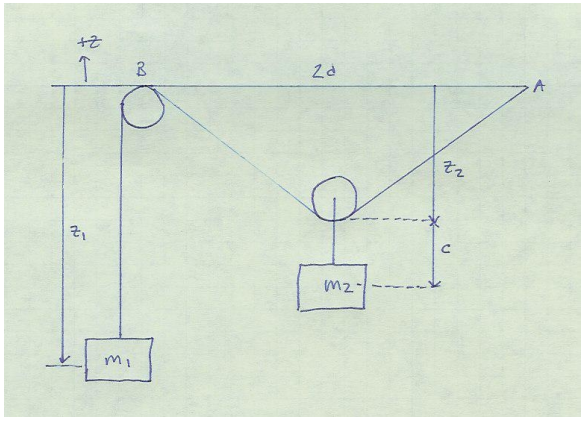
point pushes the particle back toward the equilibrium. In more than one dimension, this corresponds to $(\vec{s}\cdot\vec{\nabla})^2U$ being positive for all constant vectors $\vec{s}$ regardless of direction.

An **unstable equilibrium** point is an equilibrium point at which d^2U/dx^2 is negative. The potential energy surface is concave down, so any motion away from the equilibrium point pushes the particle away from the equilibrium. In more than one dimension, this corresponds to $(\vec{s}\cdot\vec{\nabla})^2U$ being negative for all constant vectors $\vec{s}$ regardless of direction.

A **saddle point** is an equilibrium point at which d^2U/dx^2 vanishes. Higher-order derivatives must be examined to determine stability. The point may be stable in one direction but not another. In more than one dimension, a saddle point can occur if there are some directions $\vec{s}$ with $(\vec{s}\cdot\vec{\nabla})^2U < 0$ and others with $(\vec{s}\cdot\vec{\nabla})^2U > 0$. For smooth U, this means that there is some direction $\vec{s}$ with $(\vec{s}\cdot\vec{\nabla})^2U = 0$.

Example 1.9

Consider the system of pulleys and masses shown in the following figure. The rope is of fixed length b, is fixed at point A, and runs over a pulley at point B a distance $2d$ away. The mass m_1 is attached to the end of the rope below point B, while the mass m_2 is held onto the rope by a pulley between A and B. Assume the pulleys are massless and have zero size. Find the potential energy function of the following system and the number and type of equilibrium positions.



Let the vertical coordinates of the two masses be z_1 and z_2, with the z-axis origin on the line AB and $+z$ being upward. The potential energy is, obviously

$$U = m_1\,g\,z_1 + m_2\,g\,(z_2 - c)$$

The relation between z_1 and z_2 is

$$-z_2 = \sqrt{\left[\frac{b+z_1}{2}\right]^2 - d^2}$$

So the simplified potential energy is

$$U \;=\; m_1\,g\,z_1 - m_2\,g\sqrt{\left[\frac{b+z_1}{2}\right]^2 - d^2} - m_2\,g\,c$$

Differentiate with respect to z_1 and set the result to 0:

$$\begin{aligned}
\left.\frac{dU}{dz_1}\right|_0 = m_1\,g - m_2\,g\,\frac{1}{4}\frac{b+z_1}{\sqrt{\left[\frac{b+z_1}{2}\right]^2 - d^2}} &= 0 \\
16\,m_1^2\left[\left[\frac{b+z_1}{2}\right]^2 - d^2\right] &= m_2^2\,(b+z_1)^2 \\
\left(4\,m_1^2 - m_2^2\right)(b+z_1)^2 &= 16\,m_1^2\,d^2 \\
-z_1 &= b - \frac{4\,m_1\,d}{\sqrt{4\,m_1^2 - m_2^2}}
\end{aligned}$$

where we have chosen the sign of the square root to respect the string length constraint. There is an equilibrium if $m_1 > m_2/2$ (so that the square root is neither zero nor imaginary) and if b and d are such that the resulting value of $z_1 < 0$: m_1 is not allowed to go above point B.

Is the equilibrium stable? The second derivative is

$$\begin{aligned}
\frac{d^2U}{dz_1^2} &= \frac{m_2\,g}{16}\frac{(b+z_1)^2}{\left[\left[\frac{b+z_1}{2}\right]^2 - d^2\right]^{3/2}} - \frac{m_2\,g}{4}\frac{1}{\left[\left[\frac{b+z_1}{2}\right]^2 - d^2\right]^{1/2}} \\
&= \frac{m_2\,g}{4}\frac{d^2}{\left[\left[\frac{b+z_1}{2}\right]^2 - d^2\right]^{3/2}} \\
\left.\frac{d^2U}{dz_1^2}\right|_0 &= \frac{m_2\,g}{4\,d}\frac{1}{\left[\frac{4\,m_1^2}{4\,m_1^2 - m_2^2} - 1\right]^{3/2}} \\
&= \frac{m_2\,g}{4\,d}\left[\frac{4\,m_1^2 - m_2^2}{m_2^2}\right]^{3/2} \\
&= \frac{\left(4\,m_1^2 - m_2^2\right)^{3/2}}{4m_2^2}\,\frac{g}{d}
\end{aligned}$$

Since we have already imposed the condition $m_1 > m_2/2$ to give an equilibrium, it holds that $d^2U/dz_1^2\big|_0 > 0$ and the equilibrium is stable if it exists.

From a force point of view, we see that what is happening is that m_2 sinks low enough so that the projection of the rope tension along the z axis is enough to cancel the gravitational force on m_2. As m_2 sinks lower, the projection of the tension grows, but the maximum force that can be exerted on m_2 is $2T$. Since T is supplied by m_1, the maximum value of the upward force on m_2 is $2T = 2\,m_1\,g$; hence the condition that $m_1 > m_2/2$.

1.2 Gravitation

References:

- Thornton and Marion, *Classical Dynamics of Particles and Systems*, Chapter 5
- Symon, *Mechanics*, Chapter 6.
- any first-year physics text

We define gravitational force and potential, prove Newton's Iron Sphere theorem and demonstrate the gravitational potential satisfies Poisson's equation.

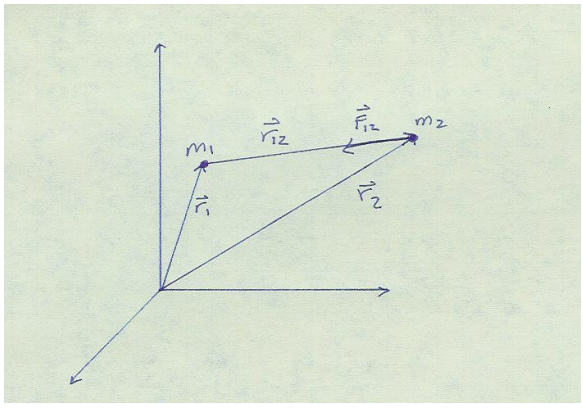
1.2.1 Gravitational Force

Force between two point masses

Given two particles with positions $\vec{r}_1$ and $\vec{r}_2$ and masses m_1 and m_2, the gravitational force exerted by particle 1 on particle 2 is

$$\vec{F}_{21}(\vec{r}_1, \vec{r}_2) = -G\,\frac{m_1\,m_2}{r_{21}^2}\,\hat{r}_{21}$$

where $\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$ is the vector from m_1 to m_2, $r_{21} = |\vec{r}_{21}|$ and $\hat{r}_{21} = \vec{r}_{21}/r_{21}$. The force is indicated in the following figure.

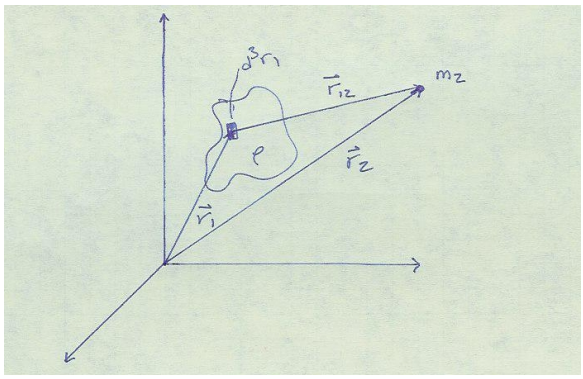


Force exerted on a point mass by an extended mass distribution

Since the gravitational force is linear in the two masses, we can calculate the gravitational force exerted on a point mass by an extended mass distribution $\rho(\vec{r})$:

$$\vec{F}_{21} = -G\,m_2 \int_{V_1} d^3r_1\,\frac{\rho(\vec{r}_1)}{r_{21}^2}\,\hat{r}_{21}$$

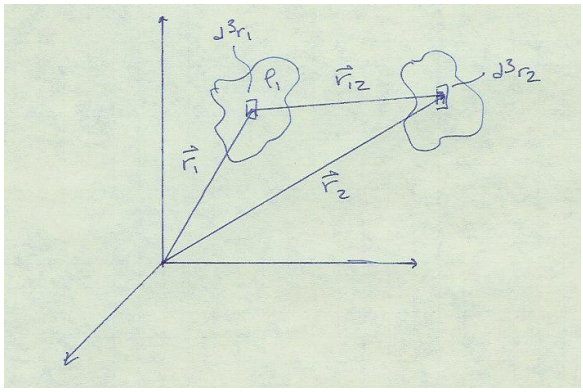
where the integral is a volume integral over the extended mass distribution.



Note that the relative position vector $\vec{r}_{21}$ depends on $\vec{r}_1$ and thus varies.

Force exerted on an extended mass distribution by and extended mass

We can further generalize, allowing m_2 to instead be an extended mass distribution. The two distributions are denoted by $\rho_1(\vec{r})$ and $\rho_2(\vec{r})$.



The force between the two mass distributions is now

$$\vec{F}_{21} = -G \int_{V_2} \int_{V_1} d^3r_2 \, d^3r_1 \, \frac{\rho_1(\vec{r}_1)\, \rho_2(\vec{r}_2)}{r_{21}^2} \, \hat{r}_{21}$$

Again, note that $\vec{r}_{21}$ varies with $\vec{r}_1$ and $\vec{r}_2$. The order of integration does not matter.

Gravitational vector field

Since the gravitational force is proportional to the mass being acted upon, we can define a **gravitational vector field** by determining the force that would act on a point mass m_2:

$$\begin{aligned} \vec{g}(\vec{r}_2) &= \frac{\vec{F}_{21}}{m_2} \\ &= -G \int_{V_1} d^3r_1 \, \frac{\rho(\vec{r}_1)}{r_{21}^2} \, \hat{r}_{21} \end{aligned}$$

The gravitational field is of course independent of m_2. Note that $\vec{g}$ has units of force/mass = acceleration.

1.2.2 Gravitational Potential

The gravitational force is conservative, gravitational potential energy

During our discussion of conservative forces, we argued that, for a force $\vec{F}$ for which the work done in going from $\vec{r}_1$ to $\vec{r}_2$ is independent of path, it holds that one can define a potential energy $U(\vec{r})$ and that $\vec{F} = -\vec{\nabla}U$. We can easily demonstrate that the gravitational force between two point masses is conservative. For simplicity, place one mass at the origin. The work done on the particle in moving from $\vec{r}_i$ to $\vec{r}_f$ is

$$\begin{aligned} W &= \int_{\vec{r}_i}^{\vec{r}_f} \vec{F}(\vec{r}) \cdot d\vec{r} \\ &= -G\, m_1\, m_2 \int_{r_i}^{r_f} \frac{dr}{r^2} \\ &= G\, m_1\, m_2 \left(\frac{1}{r_i} - \frac{1}{r_f} \right) \end{aligned}$$

where the line integral has been simplified to an integral along radius because any azimuthal motion is perpendicular to $\vec{F}$ and therefore contributes nothing to the integral. We can therefore define the gravitational potential energy of the two-mass system

$$U(\vec{r}_{21}) = -G\, \frac{m_1\, m_2}{r_{21}}$$

where we have chosen the arbitrary zero-point of the potential energy so that $U(\vec{r}_{21})$ vanishes as $\vec{r}_{21} \to \infty$.

Because U is linear in the two masses, we can of course determine the potential energy of a system of two extended masses:

$$U = -G \int_{V_2} \int_{V_1} d^3 r_2\, d^3 r_1\, \frac{\rho_1(\vec{r}_1)\, \rho_2(\vec{r}_2)}{r_{21}}$$

The gravitational potential

Clearly, m_2 is just a constant in the above discussion, so we can abstract out the gravitational potential

$$\begin{aligned} \Psi(\vec{r}_2) &= \frac{U(\vec{r}_{21})}{m_2} \\ &= -G\, \frac{m_1}{\vec{r}_{21}} \end{aligned}$$

If m_1 is extended, we have

$$\Psi(\vec{r}_2) = -G \int_{V_1} d^3 r_1\, \frac{\rho_1(\vec{r}_1)}{r_{21}}$$

It is obvious that, just in the way that the gravitational vector field $\vec{g}(\vec{r})$ is the force per unit mass exerted by the mass distribution giving rise to the field, the gravitational potential scalar field $\Psi(\vec{r})$ gives the work per unit mass needed to move a test mass from one position to another.

One of the primary advantages of using the gravitational potential is to greatly simplify calculations. The potential is a scalar quantity and thus can be integrated simply over a mass distribution. The gravtiational field or force is of course a vector quantity, so calculating it for extended mass distributions can become quite complex. Calculation of the potential followed by taking the gradient $\vec{g} = -\vec{\nabla}\Psi$ is usually the quickest way to solve gravitational problems, both analytically and numerically.

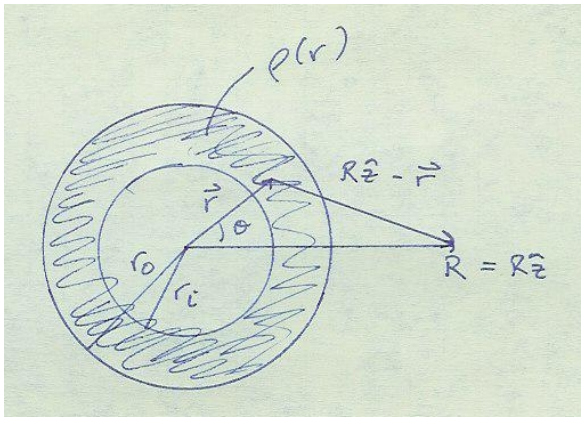
Newton's iron sphere theorem

Newton's iron sphere theorem says that the gravitational potential of a spherically symmetric mass distribution at a point outside the distribution at radius R from the center of the distribution is is the same as the potential of a point mass equal to the total mass enclosed by the radius R, and that the gravitational field at a radius R depends only on mass enclosed by the radius R. We prove it here.

Assume we have a mass distribution $\rho(\vec{r}) = \rho(r)$ that is spherically symmetric about the origin. Let r_i and r_o denote the inner and outer limits of the mass distribution; we allow $r_i = 0$ and $r_o \to \infty$. We calculate the potential at a point P that is at radius R from the origin. Since the distribution is spherically symmetric, we know the potential depends only on the radius R and not on the azimuthal and polar angles. Without loss of generality, we choose P to be at $R\,\hat{z}$. The potential is

$$\Psi(P = R\,\hat{z}) \;\; = \;\; -G \int_V d^3r \frac{\rho(r)}{|R\,\hat{z} - \vec{r}|}$$

Obviously, we should do the integral in spherical coordinates as indicated in the sketch below.



Spherical coordinates are defined in Appendix A.2. Writing out the integral gives

$$\begin{aligned}
\Psi(P = R\,\hat{z}) &= -G\int_{r_i}^{r_o} r^2\,dr\int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi} d\phi\,\frac{\rho(r)}{\sqrt{R^2+r^2-2\,R\,r\,cos\theta}} \\
&= -2\,\pi\,G\int_{r_i}^{r_o} r^2\,dr\int_0^{\pi}\sin\theta\,d\theta\frac{\rho(r)}{\sqrt{R^2+r^2-2\,R\,r\,cos\theta}} \\
&= -\frac{\pi\,G}{R}\int_{r_i}^{r_o} r\,\rho(r)\,dr\int_0^{\pi}\frac{2\,R\,r\,\sin\theta\,d\theta}{\sqrt{R^2+r^2-2\,R\,r\,cos\theta}} \\
&= -\frac{2\,\pi\,G}{R}\int_{r_i}^{r_o} r\,\rho(r)\,dr\left[\sqrt{R^2+r^2-2\,R\,r\,cos\theta}\right]\Big|_0^{\pi} \\
&= -\frac{2\,\pi\,G}{R}\int_{r_i}^{r_o} r\,\rho(r)\,dr\,[(R+r)-|R-r|]
\end{aligned}$$

Let's consider the solution by case:

- $R > r_o$: In this case, $|R-r| = R-r$ and we have

$$\begin{aligned}
\Psi(R) &= -\frac{4\,\pi\,G}{R}\int_{r_i}^{r_o} r^2\,\rho(r)\,dr \\
&= -\frac{G}{R}\int_{r_i}^{r_o} 4\,\pi\,\rho(r)\,r^2\,dr \\
&= -\frac{G}{R}\,M(r_o)
\end{aligned}$$

  where $M(r_o)$ is the mass enclosed at the radius r_o; *i.e.*, the total mass of the distribution. This is the first part of the iron sphere theorem.

- $R < r_i$: Then we have $|R-r| = r-R$ and

$$\Psi(R) = -G\int_{r_i}^{r_o}\frac{4\,\pi\,r^2\,\rho(r)}{r}\,dr = -G\int_{r_i}^{r_o} 4\,\pi\,r\,\rho(r)\,dr$$

  The potential is independent of R and is just the potential at the center of the mass distribution (which is easy to calculate thanks to the symmetry of the problem).

- $r_i < R < r_o$: The integral is broken into two pieces and we have

$$\begin{aligned}
\Psi(R) &= -\frac{G}{R}\int_{r_i}^{R} 4\,\pi\,r^2\,\rho(r)\,dr - G\int_R^{r_o}\frac{4\,\pi\,r^2\,\rho(r)}{r}\,dr \\
&= -\frac{G}{R}\,M(R) - G\int_R^{r_o} 4\,\pi\,r\,\rho(r)\,dr
\end{aligned}$$

Note how the potential is naturally continuous, as it ought to be since it is a line integral. The complicated form of the potential in the intermediate region is due to the requirement of continuity.

It is interesting to also calculate the gravitational field in the three regions using $\vec{g}(R) = -d\Psi/dR$:

- $R > r_o$:

$$\vec{g}(R) = -\left.\frac{d\Psi}{dR}\right|_R = -\frac{G}{R^2}\,M(r_o)\,\hat{R}$$

  where $\hat{R}$ is the unit vector pointing out from the R origin.

- $R < r_i$:

$$\vec{g}(R) = -\left.\frac{d\Psi}{dR}\right|_R = 0$$

- $r_i < R < r_o$:

$$\begin{aligned}\vec{g}(R) = -\left.\frac{d\Psi}{dR}\right|_R &= -\frac{G}{R^2}\,M(R)\,\hat{R} - \frac{G}{R}\,\frac{dM}{dR}\,\hat{R} + G\,\frac{4\,\pi\,R^2\,\rho(R)}{R}\,\hat{R} \\ &= -\frac{G}{R^2}\,M(R)\,\hat{R}\end{aligned}$$

Here we see the second component of the theorem, that the gravitational field at a radius R is determined only by the mass inside that radius. The potential is affected by the mass outside the radius because the potential is the line integral of the field and thus cares about what happens on a path in from $R = \infty$.

Example 1.10

Calculate the gravitational potential and field for a mass distribution that is uniform with density ρ between radii r_i and r_o and zero elsewhere.

We simply have to calculate the integrals given in the above discussion. Split by the three regions:

- $R > r_o$: As explained above, the potential is just that due to the total mass, which is $M(r_o) = \frac{4}{3}\,\pi\,\rho\,(r_o^3 - r_i^3)$, which gives

$$\Psi(R) = -\frac{4\,\pi\,\rho\,G}{3\,R}\left(r_o^3 - r_i^3\right)$$

- $R < r_i$: And, finally, the internal solution:

$$\Psi(R) = -2\,\pi\,G\,\rho\left(r_o^2 - r_i^2\right)$$

- $r_i < R < r_o$: Here, we calculate the two term solution:

$$\Psi(R) = -\frac{4\,\pi\,G\,\rho}{3\,R}\left(R^3 - r_i^3\right) - 2\,\pi\,G\,\rho\left(r_o^2 - R^2\right)$$

The gravitational field is easily calculated from the earlier formulae:

- $R > r_o$:

$$\vec{g}(R) = -\frac{4\,\pi\,\rho\,G}{3\,R^2}\left(r_o^3 - r_i^3\right)\hat{R}$$

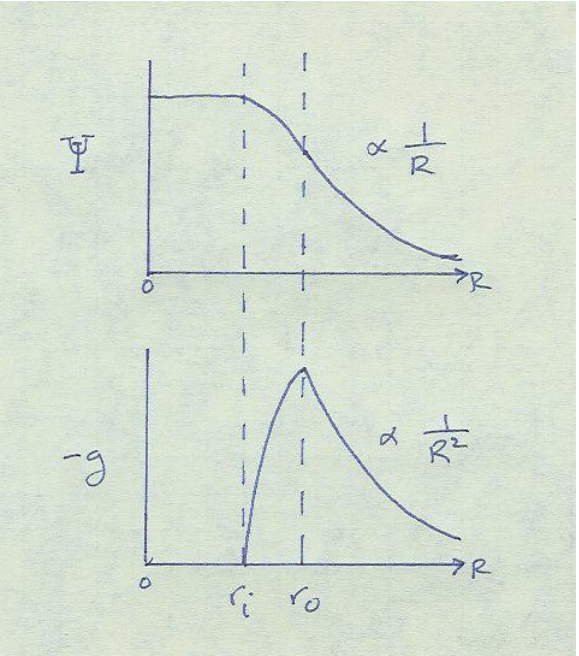
- $R < r_i$:

$$\vec{g}(R) = 0$$

- $r_i < R < r_o$:

$$\vec{g}(R) = -\frac{4\,\pi\,G\,\rho}{3\,R^2}\left(R^3 - r_i^3\right)\hat{R}$$

The solution is sketched below.



Poisson's Equation

A final application of the gravitational potential and field is to prove Poisson's Equation and Laplace's Equation.

Suppose we have a mass distribution $\rho(\vec{r})$ that sources a gravitational potential $\Psi(\vec{r})$ and gravitational field $\vec{g}(\vec{r})$. Consider a surface S enclosing a volume V; ρ need not be fully contained by S. Let us calculate the flux of the field through the surface S by doing an area integral over the surface of the projection of $\vec{g}$ along the surface normal vector $\hat{n}$:

$$\begin{aligned}
\Phi &= \int_S d^2r\, \hat{n}(\vec{r}) \cdot \vec{g}(\vec{r}) \\
&= -\int_S d^2r_2\, \hat{n}(\vec{r}_2) \cdot \int d^3r_1 \frac{G\,\rho(\vec{r}_1)}{r_{21}^2} \hat{r}_{21} \\
&= -G \int d^3r_1\, \rho(\vec{r}_1) \int_S d^2r_2\, \frac{\hat{n}(\vec{r}_2) \cdot \hat{r}_{21}}{r_{21}^2}
\end{aligned}$$

The integrand of the second integral is the solid angle subtended by the area element d^2r_2 as seen from $\vec{r}_1$ (the r_{21} in the denominator cancels a similar factor in the numerator to turn area into solid angle). There is a sign defined by whether the surface normal has a positive or negative projection along $\hat{r}_{21}$; *i.e.*, whether the surface normal at $\vec{r}_2$ points away from or toward $\vec{r}_1$. If $\vec{r}_1$ is inside the surface S, then the integral is just 4π because the surface normal always points away and 4π is the solid angle subtended by an enclosing surface. If $\vec{r}_1$ is outside the surface S, then the integrated solid angle subtended vanishes because positive contributions from the part of the surface with $\hat{n}$ pointing away from $\vec{r}_1$ are canceled by negative contributions from sections with $\hat{n}$ pointing toward $\vec{r}_1$. Therefore, we have

$$\begin{aligned}
\Phi &= -4\,\pi\,G \int_V d^3r_1\, \rho(\vec{r}_1) \\
&= -4\,\pi\,G\,M_V
\end{aligned}$$

where M_V is the mass enclosed by the surface S in the volume V. Gauss's divergence theorem (see Appendix A.4) tells us that

$$\Phi = \int_S d^2r\, \hat{n}(\vec{r}) \cdot \vec{g}(\vec{r}) = \int_V d^3r\, \vec{\nabla} \cdot \vec{g}(\vec{r})$$

i.e., that

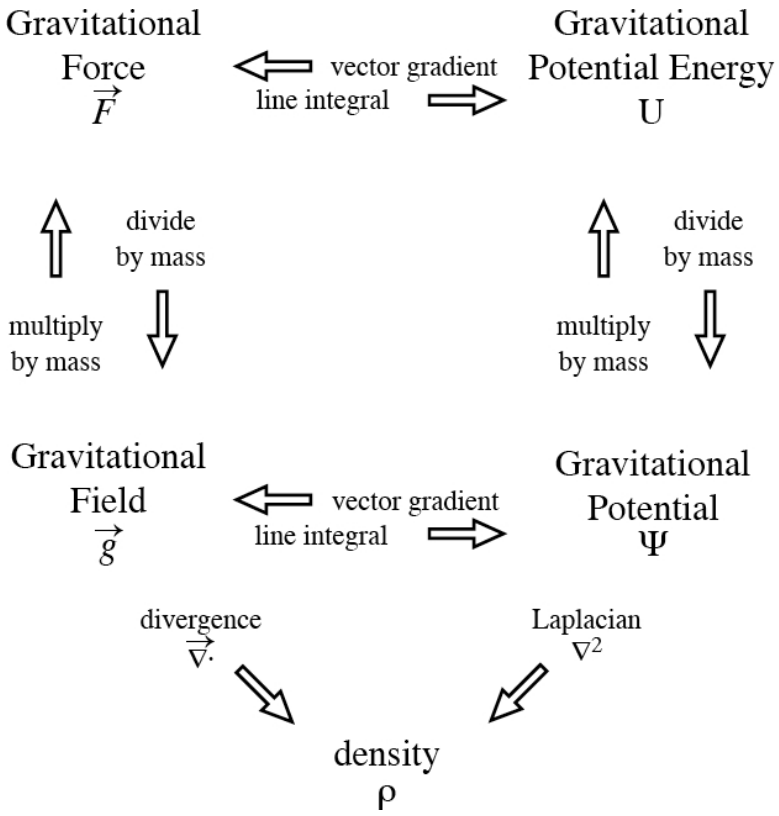
$$\int_V d^3r\, \vec{\nabla} \cdot \vec{g}(\vec{r}) = -4\,\pi\, G \int_V d^3r\, \rho(\vec{r})$$

The surface S (and enclosed volume V) we chose were arbitrary, so it holds that

$$\begin{aligned} \vec{\nabla} \cdot \vec{g}(\vec{r}) &= -4\,\pi\, G\, \rho(\vec{r}) \\ \nabla^2 \Psi(\vec{r}) &= 4\,\pi\, G\, \rho(\vec{r}) \end{aligned}$$

The latter relation is **Poisson's Equation**, which relates the derivatives of the potential to the mass density. Notice that it is a local relation; it was not necessarily obvious that such a relation would hold given the way the potential is defined as a line integral. When $\rho(\vec{r}) = 0$, the relation is called **Laplace's Equation**. There are analogous relations in electromagnetism, relating the electric potential to the electric charge density.

Summary of Relationships among Gravititational Quantities



1.3 Dynamics of Systems of Particles

References:

- Thornton and Marion, *Classical Dynamics of Particles and Systems*, Chapter 9
- Goldstein, *Classical Mechanics*, Section 1.2
- Symon, *Mechanics*, Chapter 4
- any first-year physics text

We introduce Newton's third law, define center of mass, and explore the concepts of momentum, angular momentum, and energy for systems of particles. Collisions of two-particle systems are considered.

1.3.1 Newtonian Mechanical Concepts for Systems of Particles

Newton's third law

We have so far considered motion of a single particle, which required Newton's second law. In considering systems of particles, we now require Newton's third law. There are two forms:

weak form

The forces exerted by two particles a and b on each other are equal in magnitude and opposite in direction. That is, if $\vec{f}_{ab}$ is the force exerted on particle a by particle b, then

$$\vec{f}_{ab} = -\vec{f}_{ba} \tag{1.14}$$

strong form

In addition to the above, the force between the two particles a and b is a function of only the difference in the two particles' positions and is directed along the vector between them:

$$\vec{f}_{ab} = f_{ab}(r_{ab})\,\hat{r}_{ab} \tag{1.15}$$

where $\hat{r}_{ab} = \vec{r}_{ab}/|\vec{r}_{ab}|$ and $\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$. That is, the force is a scalar function of the magnitude of the position difference and is directed along $\hat{r}_{ab}$. The mathematical form may seem like a stronger statement than the verbal form. But such a dependence implies the force must be directed along $\hat{r}_{ab}$. The remaining dependence on $\vec{r}_{ab}$ must therefore be a scalar, and the only nonzero scalar that can be formed from $\vec{r}_{ab}$ is r_{ab}, so the scalar function f_{ab} must only be a function of r_{ab}.

Both forms are *assumptions* that must be checked observationally for different forces. For example, the Lorentz force on a charged particle, $\vec{F} = q\vec{v} \times \vec{B}$, satisfies the weak form but not the strong form. Forces that satisfy the strong form are called **central forces**; examples are gravitational and electrostatic forces.

Total Linear Momentum, Newton's Second Law, Center of Mass, Conservation of Linear Momentum

Now, let's consider the forces acting on a system of particles with masses m_a, positions $\vec{r}_a$, and momenta $\vec{p}_a$. Newton's second law for each particle is

$$\frac{d\vec{p}_a}{dt} = \vec{F}_a^{(e)} + \sum_{b\neq a} \vec{f}_{ab}$$

where the $^{(e)}$ superscript indicates *external* forces – forces exerted by something outside the system of particles – and the second term contains all the forces exerted on particle a by other particles of the system. Sum this over all particles to find

$$\frac{d}{dt}\sum_a m_a \frac{d}{dt}\vec{r}_a = \sum_a \vec{F}_a^{(e)} + \sum_{a,b,\,b\neq a} \vec{f}_{ab}$$

First, note that the second term vanishes due to the weak form of Newton's second law: every pairwise sum $\vec{f}_{ab} + \vec{f}_{ba} = 0$. Second, the total momentum of the system is

$$\vec{P} = \sum_a \vec{p}_a = \sum_a m_a \frac{d}{dt}\vec{r}_a \tag{1.16}$$

So we find that

$$\frac{d}{dt}\vec{P} = \sum_a \vec{F}_a^{(e)} \equiv \vec{F}^{(e)} \tag{1.17}$$

That is, the system can be treated as a point mass with momentum $\vec{P}$ acted upon by the total external force $\vec{F}^{(e)}$, and Newton's second law is obeyed for the equivalent point mass. Taking the analogy further, if the total mass of the system is

$$M = \sum_a m_a$$

and we define the **center of mass** as

$$\vec{R} \equiv \frac{1}{M}\sum_a m_a \vec{r}_a \tag{1.18}$$

and we assume the m_a are fixed in time, then Equation 1.17 becomes

$$M\frac{d^2}{dt^2}\vec{R} = \vec{F}^{(e)}$$

The analogy to a point mass continues to hold if we treat the center of mass as the position coordinate of the equivalent point mass.

If the total external force $\vec{F}^{(e)}$ vanishes along a direction $\vec{s}$, then the total linear momentum along that direction, $\vec{P}\cdot\vec{s}$, is constant. That is, *the total linear momentum of a system of particles subject to no external force is conserved.* It was not *a priori* obvious that this would occur; it is a result of the linearity of Newton's second law (linear dependence on position, mass, and force) and of the weak form of Newton's third law.

For a solid object, as opposed to a system of point particles, the natural extensions to the definitions are

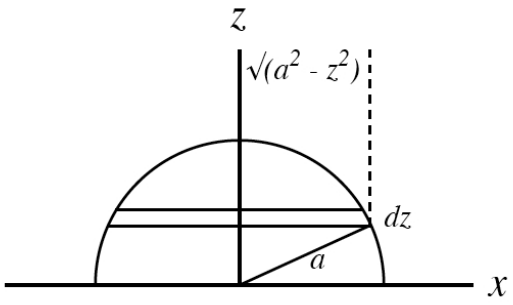
$$M = \int d^3r\, \rho(\vec{r}) \tag{1.19}$$

$$\vec{R} = \int d^3r\, \rho(r)\, \vec{r} \tag{1.20}$$

Notice that the integrand for $\vec{R}$ contains the vector $\vec{r}$, so the integral must be done component-by-component.

Example 1.11

(Thornton Example 9.1) Find the center of mass of a hemisphere of constant density ρ, and radius a.



The mass is obviously

$$M = \frac{2}{3}\,\pi\, a^3\, \rho$$

Choose a coordinate system wherein the azimuthal symmetry axis is aligned with the z axis. Consider a thin disk parallel to the xy plane at height z and having thickness dz. By symmetry, the center of mass of the disk must be at $z\hat{z}$. The radius of the thin disk is $\sqrt{a^2 - z^2}$. Therefore, the mass contribution of the disk is

$$dm = \pi \left(a^2 - z^2\right)\, \rho\, dz$$

Since each disk only contributes a component along z to the overall integral, clearly the center of mass must be along z. So we have

$$\begin{aligned} Z &= \frac{1}{M} \int z\, dm \\ &= \frac{1}{M} \int_0^a dz\, \pi \left(a^2 - z^2\right)\, \rho\, z \\ &= \frac{\pi\,\rho}{M} \left[a^2\, \frac{z^2}{2}\bigg|_0^a - \frac{z^4}{4}\bigg|_0^a \right] \\ &= \frac{\pi}{4\,M}\, \rho\, a^4 \\ &= \frac{3}{8}\, a \end{aligned}$$

Rocket Motion

The rocket problem is a classic application of conservation of momentum. First consider a rocket in the absence of gravity. It exhausts mass at a rate $\dot{m} = dm/dt$ where $\dot{m} < 0$. The exhaust gas is expelled at speed u (as measured in the rocket frame). We can find the motion of the rocket by requiring conservation of momentum:

$$\begin{aligned} p(t) &= p(t+dt) \\ mv &= (m+\dot{m}\,dt)(v+dv) + (-\dot{m}\,dt)(v-u) \end{aligned}$$

On the right side, the first term is the rocket momentum after losing mass $dm = \dot{m}\,dt < 0$ and gaining speed dv. The second term is the momentum of the expelled gas. The mass of the expelled gas is $-\dot{m}\,dt$ because $\dot{m} < 0$. The gas is expelled at speed u in the rocket frame, which is speed $v - u$ in fixed frame. We expand out the right side, dropping second order differentials like $dt\,dv$:

$$\begin{aligned} m\,v &= m\,v + \dot{m}\,v\,dt + m\,dv - \dot{m}\,v\,dt + \dot{m}\,u\,dt \\ m\,dv &= -\dot{m}\,u\,dt \end{aligned}$$

There are three unknowns in the above: m, v, and t. We can eliminate t by expanding $\dot{m}\,dt = (dm/dt)\,dt = dm$, which gives

$$\begin{aligned} \frac{dv}{u} &= -\frac{dm}{m} \\ v - v_0 &= -u\,[\log m - \log m_0] \\ v &= v_0 + u \log\left(\frac{m_0}{m}\right) \end{aligned}$$

The final speed depends on the exhaust speed and the ratio of initial to final mass. Clearly, the less mass left over, the larger the final speed. Note that $\dot{m}$ does not enter; it does not matter how quickly you expel the mass. This is sensible, as the thing that sets the final momentum is the total momentum of the gas that has been expelled. The rate at which it was expelled is irrelevant.

The above differential equation could also have been written

$$m\,\frac{dv}{dt} = -u\,\frac{dm}{dt} = u\left|\frac{dm}{dt}\right|$$

The right side is referred to as the **thrust** and has units of force. The left side looks more like mass $\times$ acceleration. The thrust is the effective force on the rocket due to expulsion of gas at rate $\dot{m}$ with speed u.

Since the final speed of the rocket depends only logarithmically on m_0/m, gaining final speed by simply trying to increase this ratio is a losing battle. A better way to do it is to use a multistage rocket. Let

$$\begin{aligned} m_0 &= \text{initial mass} \\ m_a &= \text{mass of first stage payload} \\ m_b &= \text{mass of first stage fuel container (empty)} \\ m_1 &= m_a + m_b \\ v_1 &= \text{final speed of first stage} \\ m_c &= \text{mass of second stage payload} \\ m_d &= \text{mass of second stage fuel container (empty)} \\ m_2 &= m_c + m_d \\ v_2 &= \text{final speed of second stage} \end{aligned}$$

Then the speeds are all related by

$$\begin{aligned} v_1 &= v_0 + u \log\left(\frac{m_0}{m_1}\right) \\ v_2 &= v_1 + u \log\left(\frac{m_a}{m_2}\right) \\ &= v_0 + u\left[\log\left(\frac{m_0}{m_1}\right) + \log\left(\frac{m_1}{m_2}\right)\right] \\ &= v_0 + u \log\left(\frac{m_0\, m_a}{m_1\, m_2}\right) \end{aligned}$$

Since $m_a < m_1$, the advantage is not gained directly by jettisoning the empty fuel container and engine m_b. Rather the advantage is gained because now m_2 can be much smaller than it would have been possible to make m_1, compensating for the lost factor m_a/m_1.

Now, let's repeat the problem in the presence of simple uniform gravitational field g. In the presence of an external force $-m\,g$, our equation of motion for the system is

$$\begin{aligned} \frac{dp}{dt} &= F^{(e)} = -m\,g \\ p(t+dt) - p(t) &= -m\,g\,dt \\ m\,dv + \dot{m}\,u\,dt &= -m\,g\,dt \\ dv &= -\left[\frac{\dot{m}}{m}\,u + g\right] dt \\ dv &= -\left[\frac{u}{m} + \frac{g}{\dot{m}}\right] dm \end{aligned}$$

where we have eliminated t in favor of m again. We integrate to find (remembering $\dot{m} < 0$):

$$\begin{aligned} v &= v_0 + \frac{g}{|\dot{m}|}\int_{m_0}^{m} dm' - u\int_{m_0}^{m}\frac{dm'}{m'} \\ &= v_0 - \frac{g}{|\dot{m}|}\,(m_0 - m) + u\log\left(\frac{m_0}{m}\right) \\ &= v_0 - g\,t + u\log\left(\frac{m_0}{m}\right) \end{aligned}$$

where in the last step we made use of the fact that the mass loss is constant so $m_0 - m = |\dot{m}\,t|$. So, the solution is pretty straightforward – same as the one in the absence of gravity, but there is an additional deceleration term due to gravity.

Angular Momentum, Conservation of Angular Momentum, External and Internal Torques

We consider the angular momentum of a system of particles. Let the position of particle a be

$$\vec{r}_a = \vec{R} + \vec{s}_a \tag{1.21}$$

where $\vec{R}$ is the position of the center of mass and $\vec{s}_a$ is the position relative to the center of mass. Since $\vec{R}$ may experience acceleration, $\vec{s}_a$ can be in a noninertial reference system.

We have to be careful which coordinates Newton's second law is applied to. The angular momentum of particle a in the inertial system is

$$\vec{L}_a = \vec{r}_a \times \vec{p}_a$$

Let us of course write out the total angular momentum:

$$\begin{aligned}
\vec{L} = \sum_a \vec{L}_a &= \sum_a \vec{r}_a \times \vec{p}_a \\
&= \sum_a \vec{r}_a \times m_a \dot{\vec{r}}_a \\
&= \sum_a \left(\vec{R} + \vec{s}_a\right) \times m_a \left(\dot{\vec{R}} + \dot{\vec{s}}_a\right) \\
&= \sum_a m_a \left[\left(\vec{R} \times \dot{\vec{R}}\right) + \left(\vec{R} \times \dot{\vec{s}}_a\right) + \left(\vec{s}_a \times \dot{\vec{R}}\right) + \left(\vec{s}_a \times \dot{\vec{s}}_a\right)\right]
\end{aligned}$$

Consider the middle two terms:

$$\sum_a m_a \left[\left(\vec{R} \times \dot{\vec{s}}_a\right) + \left(\vec{s}_a \times \dot{\vec{R}}\right)\right] = \vec{R} \times \left[\sum_a m_a \dot{\vec{s}}_a\right] + \left[\sum_a m_a \vec{s}_a\right] \times \dot{\vec{R}}$$

Since $\vec{s}_a$ is referenced to the center of mass, by definition the quantity $\sum_a m_a \vec{s}_a$ vanishes.

$$\sum_a m_a \vec{s}_a = \sum_a m_a \left(\vec{r}_a - \vec{R}\right) = M\vec{R} - M\vec{R} = 0$$

So our expression for $\vec{L}$ simplifies to

$$\begin{aligned}
\vec{L} &= \sum_a m_a \left[\left(\vec{R} \times \dot{\vec{R}}\right) + \left(\vec{s}_a \times \dot{\vec{s}}_a\right)\right] \\
&= \vec{R} \times M\dot{\vec{R}} + \sum_a \vec{s}_a \times m_a \dot{\vec{s}}_a \\
\vec{L} &= \vec{R} \times \vec{P} + \sum_a \vec{s}_a \times m_a \dot{\vec{s}}_a
\end{aligned} \tag{1.22}$$

Thus, the total angular momentum is the sum of the angular momentum of the center of mass and the angular momentum of the system relative to its center of mass. Remember that the center of mass system is not necessarily inertial, so $m_a \dot{\vec{s}}_a$, which looks like a linear momentum, may not behave reasonably. It is best to not call it a linear momentum.

The next obvious question is – what does Newton's second law tell us about $\dot{\vec{L}}$? We know

$$\begin{aligned}
\dot{\vec{L}} = \sum_a \dot{\vec{L}}_a &= \sum_a \vec{r}_a \times \dot{\vec{p}}_a \\
&= \sum_a \vec{r}_a \times \left[\vec{F}_a^{(e)} + \sum_{b \neq a} \vec{f}_{ab}\right] \\
&= \sum_a \vec{r}_a \times \vec{F}_a^{(e)} + \sum_{a,b,\, b<a} \left[\vec{r}_a \times \vec{f}_{ab} + \vec{r}_b \times \vec{f}_{ba}\right]
\end{aligned}$$

where in going from the second to the third line we have regrouped the terms in the sum over particle pairs. The first term is obviously due to external torques while the second term corresponds to internal torques. The weak form of Newton's third law gives $\vec{f}_{ba} = -\vec{f}_{ab}$, so we can rewrite

$$\begin{aligned}\dot{\vec{L}} &= \sum_a \vec{r}_a \times \vec{F}_a^{(e)} + \sum_{a,b,\, b<a} (\vec{r}_a - \vec{r}_b) \times \vec{f}_{ab} \\ &= \sum_a \vec{r}_a \times \vec{F}_a^{(e)} + \sum_{a,b,\, b<a} \vec{r}_{ab} \times \vec{f}_{ab}\end{aligned}$$

Now, if we use the **strong form** of Newton's third law, the second term vanishes because $\vec{f}_{ab}$ is directed along $\vec{r}_{ab}$. So we have a version of Newton's second law:

$$\begin{aligned}\dot{\vec{L}} &= \sum_a \vec{r}_a \times \vec{F}_a^{(e)} \\ &= \sum_a \vec{N}_a^{(e)} \equiv \vec{N}^{(e)}\end{aligned} \tag{1.23}$$

So, when the forces are central (and the strong form of Newton's third law holds), we have that the change in the total angular momentum of the system is determined by the total *external* torque and that the internal torque vanishes. When the external torque vanishes, the angular momentum is conserved.

Important Note: Note that the angular momentum and forces are defined in the original inertial reference frame, **not the center of mass frame**. This is because the angular momentum of the center of mass would vanish in the center-of-mass frame ($\vec{R} = 0$ in center-of-mass frame).

Kinetic Energy

Let us now consider the concepts of work and energy for a system of particles. Consider the work done a system in moving from configuration 1 with coordinate positions $\vec{r}_{a,1}$ to configuration 2 with coordinate positions $\vec{r}_{a,2}$. The work done is

$$\begin{aligned}W_{12} &= \sum_a \int_1^2 \vec{F}_a \cdot d\vec{r}_a \\ &= \sum_a \left(\frac{p_{a,2}^2}{2\, m_a} - \frac{p_{a,1}^2}{2\, m_a} \right) \\ &= \sum_a \frac{p_{a,2}^2}{2\, m_a} - \sum_a \frac{p_{a,1}^2}{2\, m_a} \equiv T_2 - T_1\end{aligned}$$

where the kinetic energy has been defined in the obvious fashion. Let's write this out in terms of the center of mass coordinates, and assuming the mass of the particles do not change:

$$\begin{aligned}\vec{p}_a &= m_a\, \dot{\vec{r}}_a = m_a \left(\dot{\vec{R}} + \dot{\vec{s}}_a \right) \\ p_a^2 &= m_a^2\, \dot{R}^2 + m_a^2\, \dot{s}_a^2 + 2\, m_a^2\, \dot{\vec{R}} \cdot \dot{\vec{s}}_a\end{aligned}$$

So then

$$\begin{aligned} T &= \sum_a \frac{p_a^2}{2\,m_a} \\ &= \sum_a \frac{1}{2}\,m_a\,\dot{R}^2 + \sum_a \frac{1}{2}\,m_a\,\dot{s}_a^2 + \dot{\vec{R}}\cdot\sum_a m_a\,\dot{\vec{s}}_a \\ &= \frac{1}{2}\,M\,\dot{R}^2 + \sum_a \frac{1}{2}\,m_a\,\dot{s}_a^2 + \dot{\vec{R}}\cdot\frac{d}{dt}\sum_a m_a\,\vec{s}_a \end{aligned}$$

The next-to-last step **assumes the m_a are fixed in time**. The last term in the last line vanishes because the $\vec{s}_a$ are defined relative to the center of mass. So we have

$$T = \frac{1}{2}\,M\,\dot{R}^2 + \sum_a \frac{1}{2}\,m_a\,\dot{s}_a^2 \tag{1.24}$$

$$= \frac{P^2}{2\,M} + \sum_a \frac{1}{2}\,m_a\,\dot{s}_a^2 \tag{1.25}$$

The total kinetic energy is the sum of the kinetic energy of motion of the center of mass of the entire body and the "internal" kinetic energy of the body. Note that we can write the first term in terms of the center of mass momentum, but we do not write the second term in terms of momenta relative to the center of mass because the center of mass reference frame may not be inertial. Momentum relative to the center of mass can be a useless (and misleading) quantity.

Potential Energy and Conservation of Energy

Returning to the work equation given above, let's split into terms for the external and internal forces and assume they are both conservative:

$$\begin{aligned} W_{12} &= \sum_a \int_1^2 \vec{F}_a^{(e)}\cdot d\vec{r}_a + \sum_{a,b,\,b\neq a}\int_1^2 \vec{f}_{ab}\cdot d\vec{r}_a \\ &= -\sum_a \int_1^2 \vec{\nabla}_a U_a\cdot d\vec{r}_a + \sum_{a,b,\,b\neq a}\int_1^2 \vec{f}_{ab}\cdot d\vec{r}_a \end{aligned}$$

Note that the gradient is with respect to $\vec{r}_a$. U_a and $\tilde{U}_{ab}$ need not be related in any way. The first term is easy to integrate, giving

$$-\sum_a \int_1^2 \vec{\nabla}_a U_a\cdot d\vec{r}_a = \sum_a \left[U_a(\vec{r}_{a,1}) - U_a(\vec{r}_{a,2})\right]$$

The second term is more difficult because it is not obviously a perfect differential. We can

reorganize the sum and simplify by grouping terms for a given particle pair:

$$\begin{aligned}
\sum_{a,b,\,b\neq a}\int_1^2 \vec{f}_{ab}\cdot d\vec{r}_a &= \sum_{a,b,\,b<a}\int_1^2 \left[\vec{f}_{ab}\cdot d\vec{r}_a \;+\vec{f}_{ba}\cdot d\vec{r}_b\right] \\
&= \sum_{a,b,\,b<a}\int_1^2 \vec{f}_{ab}\cdot [d\vec{r}_a \;- d\vec{r}_b] \\
&= \sum_{a,b,\,b<a}\int_1^2 \vec{f}_{ab}\cdot d\vec{r}_{ab} \\
&= \sum_{a,b,\,b<a}\left[\tilde{U}_{ab}(r_{ab,1}) - \tilde{U}_{ab}(r_{ab,2})\right]
\end{aligned}$$

where in the second step we have made form of the weak form of Newton's third law and in the last line we have used the strong form to rewrite the line integral as the pairwise potential energy $\tilde{U}_{ab}$. It is not obvious at this point that this new pairwise potential energy we have just defined as the line integral of the central force over the difference coordinate $\vec{r}_{ab}$ is the same thing as our original single-particle definition of potential energy. But we can see the equivalency by working from the original definition:

$$\begin{aligned}
\tilde{U}_{ab}(\vec{r}_{a,2},\vec{r}_{b,2}) - \tilde{U}_{ab}(\vec{r}_{a,1},\vec{r}_{b,1}) &= \tilde{U}_{ab}(\vec{r}_{a,2},\vec{r}_{b,2}) - \tilde{U}_{ab}(\vec{r}_{a,2},\vec{r}_{b,1}) \\
&\quad +\tilde{U}_{ab}(\vec{r}_{a,2},\vec{r}_{b,1}) - \tilde{U}_{ab}(\vec{r}_{a,1},\vec{r}_{b,1}) \\
&= -\int_{\vec{r}_{b,1}}^{\vec{r}_{b,2}} \vec{f}_{ba}(\vec{r}_b,\vec{r}_{a,2})\cdot d\vec{r}_b - \int_{\vec{r}_{a,1}}^{\vec{r}_{a,2}} \vec{f}_{ab}(\vec{r}_a,\vec{r}_{b,1})\cdot d\vec{r}_a \\
&= \int_{\vec{r}_{b,1}}^{\vec{r}_{b,2}} \vec{f}_{ab}(\vec{r}_{a,2},\vec{r}_b)\cdot d\vec{r}_b - \int_{\vec{r}_{a,1}}^{\vec{r}_{a,2}} \vec{f}_{ab}(\vec{r}_a,\vec{r}_{b,1})\cdot d\vec{r}_a \\
&= -\int_{\vec{r}_{a,2}-\vec{r}_{b,2}}^{\vec{r}_{a,2}-\vec{r}_{b,2}} f_{ab}(r_{ab})\,\hat{r}_{ab}\cdot d\vec{r}_{ab} - \int_{\vec{r}_{a,1}-\vec{r}_{b,1}}^{\vec{r}_{a,2}-\vec{r}_{b,1}} f_{ab}(r_{ab})\,\hat{r}_{ab}\cdot d\vec{r}_{ab} \\
&= -\int_{\vec{r}_{ab,1}}^{\vec{r}_{ab,2}} f_{ab}(r_{ab})\,\hat{r}_{ab}\cdot d\vec{r}_{ab} \\
&= \tilde{U}_{ab}(r_{ab,2}) - \tilde{U}_{ab}(r_{ab,1})
\end{aligned}$$

In going from the second line to the third line, we make use of the weak form of Newton's third law. In going from the third line to the fourth line, we make use of the strong form $\vec{f}_{ab}(\vec{r}_a,\vec{r}_b) = f_{ab}(|\vec{r}_{ab}|)\,\hat{r}_{ab}$ and also change variables to $\vec{r}_{ab}$. Going from the fourth line to the fifth line simply makes use of the fact that the lower limit of integration on the first term and the upper limit of integration on the second term are equal, and also we use $\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$. The final step makes use of the fact that the integral only depend on $\vec{r}_{ab}$. The point made by this derivation is that the pairwise the expression for the potential energy $\tilde{U}(\vec{r}_{ab,2}) - \tilde{U}(\vec{r}_{ab,1})$ is indeed the same as the expression one would expect from the single-particle definition of potential energy as long as the strong form of Newton's third law holds.

So, the total work done is

$$W_{12} = \sum_a [U_a(\vec{r}_{a,1}) - U_a(\vec{r}_{a,2})] + \sum_{a,b,\,b<a}\left[\tilde{U}_{ab}(r_{ab,1}) - \tilde{U}_{ab}(r_{ab,2})\right] \tag{1.26}$$

$$\equiv U_1 - U_2 \tag{1.27}$$

and so we have

$$\begin{aligned} T_2 - T_1 &= U_1 - U_2 \\ E_2 = T_2 + U_2 &= T_1 + U_1 = E_1 \end{aligned}$$

i.e., we have total energy conservation. We have assumed only that all the forces are conservative and that the internal forces are conservative and central.

We separate the potential energy into two terms:

$$U^{(e)} = \sum_a U_a(\vec{r}_a) \tag{1.28}$$

$$U^{(i)} = \sum_{a,b,\, b<a} \tilde{U}_{ab}(r_{ab}) = \frac{1}{2} \sum_{a,b,\, b\neq a} \tilde{U}_{ab}(r_{ab}) \tag{1.29}$$

In general, $U^{(i)}$ need not be constant in time. We define a **rigid body** as one for which the distances r_{ab} are constant; displacements $d\vec{r}_{ab}$ during motion are always perpendicular to $\vec{r}_{ab}$. Since we have earlier assumed the strong form of the third law – *i.e.*, central forces – then this immediately implies $\vec{f}_{ab} \cdot d\vec{r}_{ab} = 0$ for all displacements $d\vec{r}_{ab}$. Hence, no work is done by the internal forces and therefore $U^{(i)}$ remains constant. Since constant offsets in potential energy have no effect on motion, the internal potential for such systems can be ignored: all motion is due only to external forces.

Example 1.12

A projectile of mass M explodes in flight into three pieces. The first mass $m_1 = M/2$ continues to travel in the same direction as the original projectile. The second mass $m_2 = M/6$ travels in the opposite direction and $m_3 = M/3$ comes to rest. The energy E converted from chemical energy of the explosive to final state mechanical energy is five times the initial kinetic energy of the projectile. What are the velocities of the three pieces?

We begin by writing the final velocities in terms of the initial one:

$$\begin{aligned} \vec{v}_1 &= k_1\, \vec{v} \\ \vec{v}_2 &= -k_2\, \vec{v} \\ \vec{v}_3 &= 0 \end{aligned}$$

Conservation of linear momentum gives us

$$\begin{aligned} M\, \vec{v} &= m_1\, \vec{v}_1 + m_2\, \vec{v}_2 + m_3\, \vec{v}_3 \\ M &= \frac{M}{2}\, k_1 - \frac{M}{6}\, k_2 \\ k_2 &= 3\, k_1 - 6 \end{aligned}$$

Conservation of energy gives

$$\begin{aligned} 6\, \frac{1}{2}\, M\, v^2 &= \frac{1}{2}\, m_1\, v_1^2 + \frac{1}{2}\, m_2\, v_2^2 + \frac{1}{2}\, m_3\, v_3^2 \\ 6 &= \frac{1}{2}\, k_1^2 + \frac{1}{6}\, k_2^2 \end{aligned}$$

Inserting the result for k_2 into the conservation of energy equation gives

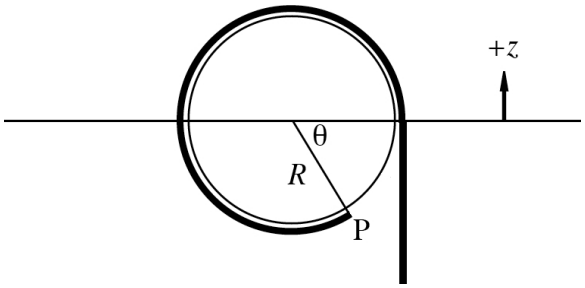
$$\begin{aligned} 36 &= 3\, k_1^2 + (3\, k_1 - 6)^2 \\ 36 &= 3\, k_1^2 + 9\, k_1^2 - 36\, k_1 + 36 \\ 0 &= k_1^2 - 3\, k_1 \end{aligned}$$

Clearly, the solution is $k_1 = 3, k_2 = 3$, so

$$\begin{aligned} \vec{v}_1 &= 3\,\vec{v} \\ \vec{v}_2 &= -3\,\vec{v} \\ \vec{v}_3 &= 0 \end{aligned}$$

Example 1.13

A rope of uniform linear density λ and mass m is wrapped one complete turn around a hollow cylinder of mass M and radius R. The cylinder rotates freely about its axis the rope unwraps. The rope ends are at $z = 0$ (one fixed, one loose) when point P is at $\theta = 0$. The system is slightly displaced from equilibrium (so motion will occur). Find the angular velocity as a function of the rotation angle θ of the cylinder.



We are trading the potential energy of the rope for the kinetic energy of the rope and cylinder. Let α be a parameter that describes the position along the rope, $0 \le \alpha \le 2\,\pi\,R$ where $2\,\pi\,R$ is the length of the rope. $\alpha = 0$ at the end that is fixed to the cylinder. The z position of the rope as a function of the angle θ and the parameter α is

$$z(\theta,\alpha) = \begin{cases} R\sin\left[2\pi - \left(\theta + \frac{\alpha}{R}\right)\right] & \theta + \frac{\alpha}{R} < 2\pi \\ -R\,\theta + (2\,\pi\,R - \alpha) & \theta + \frac{\alpha}{R} \ge 2\pi \end{cases}$$

Some explanation of the above form is necessary. The angle θ gives the angle of the start of the rope. The angle α/R is the angle between the start of the rope and the position α. Therefore, $\theta + \frac{\alpha}{R}$ gives the angle of the the position α relative to the θ origin, for the part of the rope that is on the cylinder.

- The cutoff point between the two forms is where the rope begins to unwind off the cylinder. This occurs for α such that the angle of the position α is 2π. As explained above, the angle of the position α is $\theta + \frac{\alpha}{R}$, so the cutoff is at $\theta + \frac{\alpha}{R} = 2\pi$.
- Before the cutoff point, the z coordinate of position α is just the z coordinate of a point on a circle of radius R and at angle $\theta + \frac{\alpha}{R}$, with an appropriate change of coordinate to a counterclockwise angle. The counterclockwise angle is $2\pi - (\theta + \frac{\alpha}{R})$. Hence, the z coordinate is $R\sin\left[2\pi - (\theta + \frac{\alpha}{R})\right]$.
- After the cutoff point, the rope just hangs straight down. The position of the end of the rope is the amount of the rope that has unwound. The amount of rope that has unwound is $R\,\theta$ because θ is the angle of the start of the rope on the cylinder. Therefore, the z coordinate of the end of the rope is $-R\,\theta$. A position α along the rope is at a distance $2\,\pi\,R - \alpha$ from the end of the rope (recall, $2\,\pi\,R$ is the length of the rope). So the z coordinate of the position α is $-R\,\theta + (2\,\pi\,R - \alpha)$.

We do not need to calculate the potential energy completely explicitly. We know that when the rope is fully wound, the potential energy vanishes because there is as much rope at $+z$ as at $-z$. So we only need to correct for the part that unwinds and hangs vertically. The potential energy is thus

$$U(\theta) = \int_{2\pi R - R\theta}^{2\pi R} d\alpha\, \lambda\, g\, [-R\,\theta + (2\,\pi\,R - \alpha)] - \int_0^{R\theta} d\beta\, \lambda\, g\, R \sin\left(2\pi - \frac{\beta}{R}\right)$$

In each term, the integrand is the product of the z position, the mass density, and the acceleration g, which gives the potential energy of the differential element of length $d\alpha$. The first term is for the hanging rope. The second term is the potential energy that would be present if the rope remained wound around the cylinder between angles 0 and θ (0 and $R\,\theta$ in rope length). We use a different variable β to distinguish from α. Thus, instead of calculating the full potential energy of the rope, we calculate the part that is due to the unwound portion of the rope.

We can simplify the first term by changing variables to $u = -R\,\theta + (2\,\pi\,R - \alpha)$; the limits of integration change to 0 and $-R\,\theta$. Continuing onward:

$$\begin{aligned} U(\theta) &= \lambda\, g \left\{ -\int_0^{-R\theta} du\, u - R^2 \cos(2\pi - \frac{\beta}{R}) \Big|_0^{R\theta} \right\} \\ &= \lambda\, g \left\{ -\frac{1}{2}\,(R\,\theta)^2 - R^2 \cos(2\pi - \theta) + R^2 \cos(2\pi) \right\} \\ &= -\lambda\, g\, R^2 \left\{ \frac{\theta^2}{2} + \cos\theta - 1 \right\} \end{aligned}$$

The kinetic energy is the sum of the kinetic energies of the rotating cylinder, the rotating rope, and the falling rope:

$$K(\theta) = \frac{1}{2}\, M \left(R\,\dot{\theta}\right)^2 + \frac{1}{2}\,\lambda\, R\,(2\,\pi - \theta)\, \left(R\,\dot{\theta}\right)^2 + \frac{1}{2}\,\lambda\, R\,\theta\, \left(R\dot{\theta}\right)^2$$

where, for the last term, we could simply take the rope velocity to be the velocity of the cylinder where the rope leaves the cylinder. Simplifying:

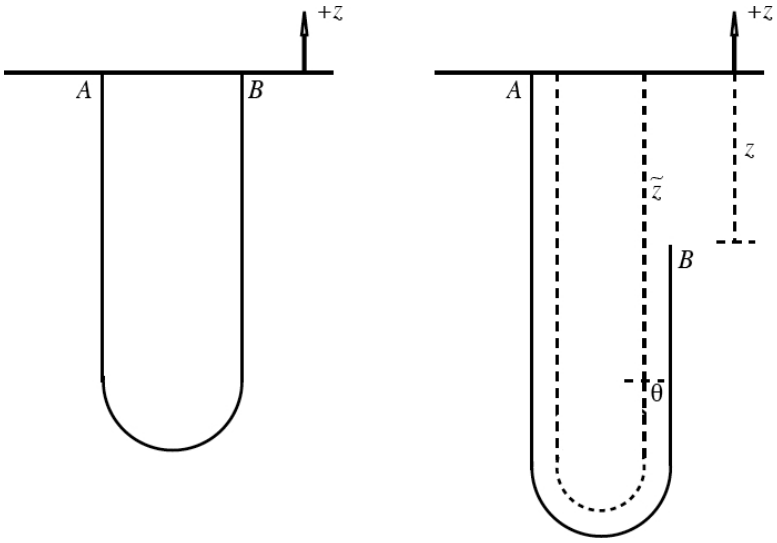
$$K(\theta) = \frac{1}{2}\, M \left(R\,\dot{\theta}\right)^2 + \frac{1}{2}\, m \left(R\,\dot{\theta}\right)^2$$

Initially, the total potential energy vanishes (because as much of the rope is at $+z$ as at $-z$) and the kinetic energy vanishes (because the angular velocity vanishes). So conservation of energy gives

$$\begin{aligned} 0 &= U(\theta) + K(\theta) \\ \lambda\, g\, R^2 \left\{ \frac{\theta^2}{2} + \cos\theta - 1 \right\} &= \frac{1}{2}\,(M + m) \left(R\,\dot{\theta}\right)^2 \\ \dot{\theta}^2 &= \frac{2\,\lambda\, g}{M + m} \left\{ \frac{\theta^2}{2} + \cos\theta - 1 \right\} \\ \dot{\theta}^2 &= \frac{m\, g}{2\,\pi\, R\,(M + m)} \left\{ \theta^2 + 2\cos\theta - 2 \right\} \end{aligned}$$

Example 1.14

(Thornton Example 9.2) A chain of uniform linear mass density λ, length b, and mass $M = b\,\lambda$ hangs from two points A and B, adjacent to each other. The end B is released. Find the tension in the chain at point A after the end B has fallen a distance z by (a) assuming free fall and (b) using energy conservation. You should treat the chain as follows: the portion above the bend point on the side toward end A is stationary. The portion above the bend point on the side toward end B falls all with the same velocity. The time-dependence is in the length and velocity of the falling portion. Neglect the bend region – assume the bend is perfect and instantaneous.



For both solutions, first note that the length of the portion of the chain that is fixed and the length that is falling is

$$\begin{aligned} l_{fixed} &= \frac{b-z}{2} \\ l_{fall} &= b - l_{fixed} = \frac{b+z}{2} \end{aligned}$$

That the above formulae are correct can be seen as follows. First, $b + |z| = b - z$ is the total length from A to the initial position of B including the gap z. (Remember, $z < 0$.) So $(b-z)/2$ is half that length, which gives the length from A to the bend point, which is the portion of the chain that is fixed.

Also, in order to find the tension, we will need to use the equation of motion for the center of mass of the chain:

$$\dot{P} = -M\,g + T$$

The momentum of the chain is

$$P = \lambda\, l_{fall}\, \dot{z} = \lambda \left(\frac{b+z}{2}\right) \dot{z}$$

where $\dot{z}$ is the speed of the falling end (and therefore of the entire falling section). So we have

$$\begin{aligned} T &= M\,g + \dot{P} \\ &= M\,g + \lambda\left[\left(\frac{b+z}{2}\right)\,\ddot{z} + \frac{1}{2}\,\dot{z}^2\right] \end{aligned}$$

(a) Assume free fall. Since we assume that the falling part of the chain is in free fall, it holds that

$$\begin{aligned} z &= -\frac{1}{2}\,g\,t^2 \\ \dot{z} &= -g\,t = \sqrt{-2\,g\,z} \\ \ddot{z} &= -g \end{aligned}$$

which gives

$$\begin{aligned} T &= M\,g + \lambda\left[-\left(\frac{b+z}{2}\right) - z\right] g \\ &= M\,g + \frac{\lambda\,g}{2}\,(-b - 3\,z) \\ &= \frac{M\,g}{2}\left(-\frac{3\,z}{b} + 1\right) \end{aligned}$$

When the chain is just released, $z = 0$ and $T = M\,g/2$. When the fall is finished and $z = -b$, we have $T = 2\,M\,g$. The results for the tension are somewhat nonsensical; as we will see, this is due to the assumption of free fall being incorrect.

(b) Now we solve by energy methods. Finding the kinetic energy as a function of position gives us $\dot{z}$, we can then differentiate to find $\ddot{z}$ and insert into the above equation for the tension. We find the kinetic energy by requiring conservation of energy.

The potential energy of the chain at any given z coordinate is found by integrating the potential energy of small mass elements dm over the chain. Let θ be a parameter that runs from 0 to b, starting at A, that indicates where the mass element is along the chain; θ is independent of how far end B has fallen. Let $\tilde{z}(\theta)$ be the z coordinate of the element θ. We know that

$$\tilde{z}(\theta) = \begin{cases} -\theta & \theta < l_{fixed} \\ (\theta - l_{fixed}) - l_{fixed} & \theta > l_{fixed} \end{cases}$$

The contribution of an element $d\theta$ at θ to the mass and potential energy are

$$\begin{aligned} dm &= \lambda\,d\theta \\ dU &= dm\,g\,\tilde{z}(\theta) \end{aligned}$$

so we then have

$$\begin{aligned}
U(z) &= \int_0^b d\theta\, \lambda\, g\, \tilde{z}(\theta) \\
&= \lambda\, g \left[-\int_0^{l_{fixed}} d\theta\, \theta + \int_{l_{fixed}}^b d\theta\, (\theta - 2\, l_{fixed}) \right] \\
&= \lambda\, g \left[-\frac{l_{fixed}}{2} + \frac{b^2 - l_{fixed}^2}{2} - 2\, l_{fixed}\, (b - l_{fixed}) \right] \\
&= \lambda\, g \left[l_{fixed}^2 - 2\, l_{fixed}\, b + \frac{b^2}{2} \right] \\
&= \lambda\, g \left[\left(\frac{b^2}{4} - \frac{b\, z}{2} + \frac{z^2}{4} \right) - \left(b^2 - b\, z \right) + \frac{b^2}{2} \right] \\
&= \lambda\, g \left[-\frac{b^2}{4} + \frac{b\, z}{2} + \frac{z^2}{4} \right]
\end{aligned}$$

The kinetic energy is given by the mass of the part of the chain that is falling and the speed at which it falls:

$$\begin{aligned}
K &= \frac{1}{2}\, \lambda\, l_{fall}\, \dot{z}^2 \\
&= \frac{1}{4}\, \lambda\, (b + z)\, \dot{z}^2
\end{aligned}$$

Now, we use conservation of energy, noting that $K = 0$ when $z = 0$:

$$\begin{aligned}
U(0) &= K + U(z) \\
-\lambda\, g\, \frac{b^2}{4} &= \frac{1}{4}\, \lambda\, (b + z)\, \dot{z}^2 + \lambda\, g \left[-\frac{b^2}{4} + \frac{b\, z}{2} + \frac{z^2}{4} \right] \\
-g \left[2\, b\, z + z^2 \right] &= (b + z)\, \dot{z}^2 \\
\dot{z}^2 &= -g\, \frac{2\, b\, z + z^2}{b + z}
\end{aligned}$$

Differentiating yields

$$\begin{aligned}
2\, \dot{z}\, \ddot{z} &= -g \left[\frac{2\, b + 2\, z}{b + z} - \frac{2\, b\, z + z^2}{(b + z)^2} \right] \dot{z} \\
&= -g \left[2 - \frac{2\, b\, z + z^2}{(b + z)^2} \right] \dot{z} \\
\ddot{z} &= -g \left[1 - \frac{1}{2}\, \frac{2\, b\, z + z^2}{(b + z)^2} \right]
\end{aligned}$$

Now, insert into our previous equation for the tension in terms of z, $\dot{z}$, and $\ddot{z}$ to find

$$\begin{aligned}
T &= M\,g + \lambda\left[\left(\frac{b+z}{2}\right)(-g)\left[1-\frac{1}{2}\frac{2\,b\,z+z^2}{(b+z)^2}\right]+\frac{1}{2}(-g)\,\frac{2\,b\,z+z^2}{b+z}\right] \\
&= M\,g - \lambda\,g\left[\frac{1}{2}\frac{(b+z)^2}{b+z}-\frac{1}{4}\frac{2\,b\,z+z^2}{b+z}+\frac{1}{2}\frac{2\,b\,z+z^2}{b+z}\right] \\
&= M\,g - \frac{1}{4}\frac{\lambda\,g}{b+z}\left[2\,(b+z)^2-(2\,b\,z+z^2)\right] \\
&= \lambda\,g\,\frac{b\,(b+z)}{b+z}-\frac{1}{4}\frac{\lambda\,g}{b+z}\left[2\,b^2+6\,b\,z+3\,z^2\right] \\
&= \frac{1}{4}\frac{\lambda\,g}{b+z}\left[2\,b^2-2\,b\,z-3\,z^2\right] \\
&= \frac{M\,g}{4}\frac{1}{b(b+z)}\left[2\,b^2-2\,b\,z-3\,z^2\right]
\end{aligned}$$

The two results for the tension are not the same! In the free-fall solution, the tension increases linearly as the chain falls, simply reflecting the fact that the amount of mass under tension increases linearly with the how far the end B has fallen. In the energy solution, the tension becomes infinite as $z \to -b$. Experimentally, it has been determined that the latter solution is closer to reality (Calkin and March, *Am. J. Phys*, **57**: 154 (1989)), though of course the tension does not become infinite (just really large, 25 times the chain weight).

The solutions differ because the free-fall solution makes an assumption about the motion of the chain while the energy method does not need to. This is seen by the fact that the relation between $\dot{z}$, $\dot{z}$ and $\ddot{z}$ is more complicated in the energy solution. Experimentally, it is seen that the latter relation is closer to reality and that the chain falls faster than free-fall because some tension is communicated through the bend and exerts an additional downward force on the falling part of the chain. In the energy solution, we see this additional force as the second term in $\ddot{z}$, which always makes $|\ddot{z}|$ larger (because $z < 0$).

1.3.2 The Virial Theorem

Here we prove the Virial Theorem, which relates the time-averaged kinetic energy for a bounded system to a quantity called the **virial**, which is just a time-averaged dot product of the force and position of the various particles in the system. In its basic form, the virial theorem does not have a clear intuitive interpretation, though it is certainly useful. When one considers the specific case of conservative forces that depend on particle radius, the virial becomes simply related to the potential energy of the system. Thus, we obtain a time-averaged relation between kinetic and potential energy. This is an incredibly powerful statement because it doesn't require specific knowledge of the particle orbits.

Generic Version

Consider an ensemble of particles, whose positions $\vec{r}_a$ and momenta $\vec{p}_a$ are bounded, meaning that there are upper limits on both. This means that the particles are both confined to a particular region of space and also that they never approach a force center that might impart to them infinite momentum. Define the quantity

$$S = \sum_a \vec{p}_a \cdot \vec{r}_a$$

Calculate the time-averaged rate of change of S:

$$\left\langle \frac{dS}{dt} \right\rangle = \frac{1}{\tau} \int_0^\tau \frac{dS}{dt}\, dt$$

The integrand is a total derivative, so the integral is done trivially:

$$\left\langle \frac{dS}{dt} \right\rangle = \frac{S(\tau) - S(0)}{\tau}$$

Since we assumed $\vec{r}_a$ and $\vec{p}_a$ are bounded, it also holds that S is bounded. Thus, by letting $\tau \to \infty$ – *i.e.*, by taking the average over an arbitrarily long time – we can make $\left\langle \frac{dS}{dt} \right\rangle \to 0$. Let us explicitly calculate $\left\langle \frac{dS}{dt} \right\rangle$ and use the fact that it vanishes:

$$0 = \left\langle \frac{dS}{dt} \right\rangle = \left\langle \sum_a \left[\vec{p}_a \cdot \dot{\vec{r}}_a + \dot{\vec{p}}_a \cdot \vec{r}_a \right] \right\rangle = \left\langle \sum_a \vec{p}_a \cdot \dot{\vec{r}}_a \right\rangle + \left\langle \sum_a \dot{\vec{p}}_a \cdot \vec{r}_a \right\rangle$$

The two terms can be time-averaged separately because time averaging is a linear operation. The first term is just $2\,T$, twice the kinetic energy.[1] The second term may be rewritten using the force on particle a, $\vec{F}_a$, using Newton's second law. We may thus write the above as

$$\langle T \rangle = -\frac{1}{2} \left\langle \sum_a \vec{F}_a \cdot \vec{r}_a \right\rangle \tag{1.30}$$

The quantity on the right side is known as the **virial**. The key result is that the time-averaged kinetic energy is related to a time-average of a quantity involving the forces and positions. The virial theorem is similar to the work-energy theorem, which relates the work done by a force on particle to the particle's kinetic energy and which is also derived using Newtons' second law, but the virial theorem pertains to time-averaged, summed quantities rather than to individual particle instantaneous quantities. What good does this do for us? The key is the time-averaging and summing over particles, which lets the virial theorem be used in unexpected ways.

Example 1.15: Ideal Gas Law

We can, for example, use the virial theorem to prove the ideal gas law! Consider a gas of temperature Θ confined to a box of volume V. The temperature is defined in terms of the average (over particles) kinetic energy of the gas particles, so we can relate the total time-averaged kinetic energy of the gas to the temperature:

$$\langle T \rangle = \frac{3}{2}\, N\, k\, \Theta$$

where N is the number of gas particles. To calculate the virial, we need to evlauate the time average of $\vec{F}_a \cdot \vec{r}_a$. The gas particles move freely except when they hit a wall, when an instantaneous force is exerted to reflect them from the wall. Let us write the sum for virial,

[1] Note: there is no ambiguity here about how to calculate T. $\vec{p}_a$ and $\vec{r}_a$ are not generalized coordinates, they are the Cartesian vectors describing the particles (think back to elementary mechanics). It always holds that $\vec{p}_a = m_a\, \dot{\vec{r}}_a$ and that $T = \frac{1}{2}\, m_a\, \dot{\vec{r}}_a^{\,2}$, hence $\vec{p}_a \cdot \dot{\vec{r}}_a = 2\,T$.

$\sum_a \vec{F}_a \cdot \vec{r}_a$, as an integral over the walls of the box. The average contribution to the force exerted on an area element dA of the wall by the gas at any instant in time is

$$d\vec{F} = \hat{n}\, P\, dA$$

where $\hat{n}$ is the outward normal at the wall. By Newton's third law, the force exerted on the gas by the wall is the same modulo a sign. The sum for the virial is then just an integral over the walls:

$$-\frac{1}{2}\left\langle \sum_a \vec{F}_a \cdot \vec{r}_a \right\rangle = \frac{1}{2}\int_S P\, \hat{n} \cdot \vec{r}\, dA$$

where S indicates the closed surface defining the box walls. We may do the surface integral using Gauss' theorem:

$$\int_S \hat{n} \cdot \vec{r}\, dA = \int_V \vec{\nabla} \cdot \vec{r}\, dV = 3\, V$$

Thus, we obtain

$$\begin{aligned} \langle T \rangle &= \frac{3}{2}\, P\, V \\ \implies \quad N\, k\, \Theta &= P\, V \end{aligned}$$

which is the ideal gas law. Note especially how we did the derivation using only information about the time-averaged force: we didn't need to know any details about the interaction of the particles with the walls except the average force per unit area, P, due to that interaction.

Conservative Power Law Potentials

If we now consider the specific case of particles being acted upon by a conservative force field that is derived from a potential energy that is a power law in particle radius from the center of force, we can evaluate the virial more explicitly. That is, we assume

$$\vec{F}_a = -\vec{\nabla}_a V(\vec{r}_a)$$

where $V(\vec{r})$ is the potential energy and where $\vec{\nabla}_a$ is the gradient with respect to particle a's position vector, $\vec{r}_a$. Note that we are assuming that all the particles move in a single potential energy that is a function of the particle position.[2] This assumption allows us to write the virial as

$$-\frac{1}{2}\left\langle \sum_a \vec{F}_a \cdot \vec{r}_a \right\rangle = \frac{1}{2}\left\langle \sum_a \vec{r}_a \cdot \vec{\nabla}_a V(\vec{r}_a) \right\rangle$$

Now, assume $V(\vec{r}_a) = k\, r_a^n$. Then, $\vec{\nabla}_a V(\vec{r}_a) = n\, k\, r_a^{n-1}\, \hat{r}_a$ and the virial becomes

$$\begin{aligned} -\frac{1}{2}\left\langle \sum_a \vec{F}_a \cdot \vec{r}_a \right\rangle &= \frac{1}{2}\left\langle \sum_a r_a\, n\, k\, r_a^{n-1} \right\rangle = \frac{n}{2}\left\langle \sum_a k\, r_a^n \right\rangle \\ &= \frac{n}{2}\left\langle \sum_a V(r_a) \right\rangle = \frac{n}{2}\,\langle U \rangle \end{aligned}$$

[2] Strictly speaking, pairwise central forces do not satisfy this form. But, for an ensemble of many particles, it is a very good approximation to say that each particle moves in a potential generated by the whole ensemble that looks like a potential fixed to the center of mass of the ensemble, which we take to be at rest. The ensemble potential is quite close to independent of the position of any single particle.

where $U = \sum_a V(r_a)$ is the total potential energy of the system. Thus, the virial theorem reduces to

$$\langle T \rangle = \frac{n}{2} \langle U \rangle \tag{1.31}$$

That is, we obtain a very simple relation between the time-averaged kinetic and potential energies of the system.

Example 1.16: The Virial Theorem in Astrophysics

The virial theorem is used widely in astrophysics because of the dominance of gravity and because it relates directly observable quantities – kinetic energy and temperature – to unobservable quantities – potential energy and mass. We assume gravitational forces, so $n = -1$. If we divide the virial theorem by the number of particles, we have

$$\frac{1}{N} \langle T \rangle = \frac{1}{2} \frac{1}{N} |\langle U \rangle|$$

(The sign in n has been canceled by the use of the absolute value sign.) That is, the kinetic energy per particle is half the potential energy per particle. We can use this in different ways to measure total masses of systems:

- If we are looking at a gas cloud, we can measure the gas temperature Θ by its free-free photon emission.[3] That gives us $\langle T \rangle$. The potential energy can be rewritten in terms of the cloud mass M, the typical gas particle mass μ, and the cloud-averaged particle radius. We denote this latter averaged radius as, somewhat uninformatively, the virial radius, R_v. The virial theorem then tells us

$$\frac{3}{2} k \Theta = \frac{1}{N} \langle T \rangle = \frac{1}{N} \frac{1}{2} |\langle U \rangle| = \frac{1}{2} G M \mu \left\langle \frac{1}{r} \right\rangle \equiv \frac{1}{2} G M \mu \frac{1}{R_v}$$
$$3 k \Theta = \frac{G M \mu}{R_v}$$

  Note that the averaging is done on $1/r$, not on r. A typical application would be to use the virial theorem to measure the cloud mass. One has to assume that the cloud is spherically symmetric and optically transparent to its own free-free emission; one can then infer from the observed photon radial distribution the shape (but not the normalization!) of the cloud's density profile. From the shape of unnormalized profile, one can calculate the virial radius. The gas is almost always mostly ionized hydrogen, so μ is known. That leaves the cloud mass as the only unknown. Thus, one can infer from only the photon emission and the virial theorem the cloud mass without any absolute knowledge of normalization of the photon emission in terms of the density. That's rather remarkable!

- If we are looking at a galaxy, we can measure the line-of-sight velocity of a subset of stars by redshift of known spectral lines. The same technique works for galaxies orbiting in a galaxy clusters. Assuming isotropy of the object, the line-of-sight velocity

[3] Free-free emission is just the process of electrons scattering via the Coulomb force off ions in a plasma, a gas that is hot enough that the bulk of the atoms are ionized. Since the electrons are accelerated in these scattering events, they emit light in the form of a photon. The typical photon energy depends on the plasma temperature; for the very hot plasma in galaxy clusters, which is at millions of degrees K, the photons are keV-energy X-rays. In our own galaxy, the emission is usually in the radio, with wavelength of 1 cm and longer.

and the velocity transverse to the line of sight will be equal on average (up to a $\sqrt{2}$). Assuming all the orbiting objects in the larger object are of roughly equal mass, the kinetic energy per particle is simply related to the rms of the measured line-of-sight velocity:

$$\frac{1}{N}\langle T\rangle = \frac{1}{2}\,m\,v_{3d,rms}^2 = \frac{3}{2}\,m\,v_{1d,rms}^2$$

where we relate the full 3-dimensional rms velocity to the measured one-dimension rms velocity assuming isotropy. We can do the same kind of thing as we did for the gas cloud, except now m will drop out:

$$\frac{3}{2}\,m\,v_{1d,rms}^2 = \frac{1}{N}\langle T\rangle = \frac{1}{N}\,\frac{1}{2}\,|\langle U\rangle| = \frac{1}{2}\,G\,M\,m\,\langle 1r\rangle \equiv \frac{1}{2}\,G\,M\,m\frac{1}{R_v}$$
$$3\,v_{1d,rms}^2 = \frac{G\,M}{R_v}$$

Since the test particles whose velocities we measure are discrete objects, we can just make a plot of their number density as a function of radius and from that calculate the virial radius. We can thus determine M only from our observations of the test particle positions and line-of-sight velocities!

1.3.3 Collisions of Particles

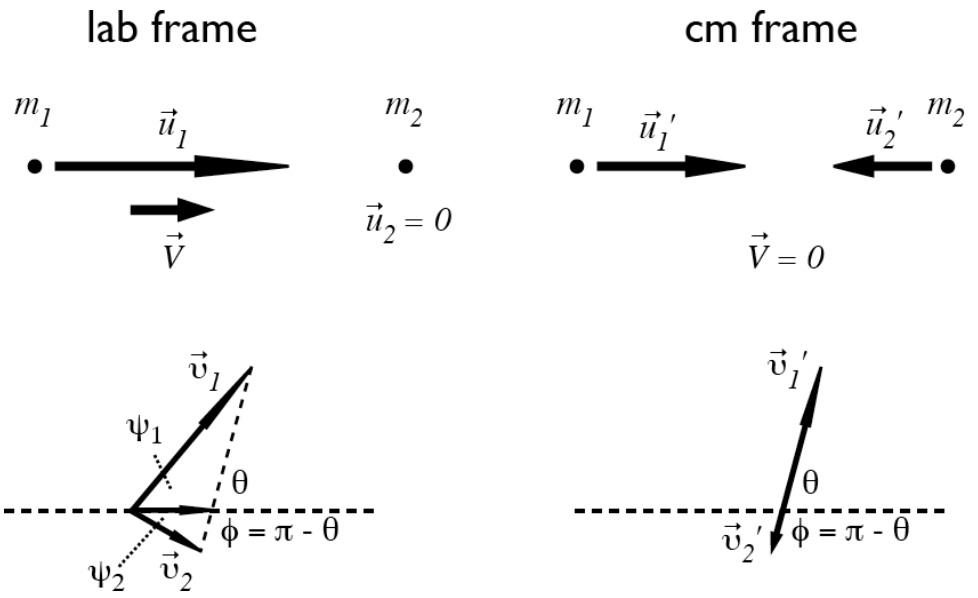
One useful application of the concepts we have described above for systems of particles is in the study of collisions of particles. For an isolated system of particles with no external forces acting on it, the total linear momentum is conserved. If the masses of the particles are fixed, the velocity of the center of mass is constant. The reference frame that moves with the center of mass is therefore inertial, and so we can use Newtonian mechanics in this reference frame. Frequently, working in this frame is much easier than in other frames because the algebra is reduced (no need to carry around an extra velocity) and symmetries are more easily apparent.

Transforming between Lab and Center-of-Mass Reference Frames

Start with some notation:

m_1, m_2	=	masses of particles
$\vec{u}_i, \vec{v}_i$	=	initial and final velocities of particle i in lab system
$\vec{u}'_i, \vec{v}'_i$	=	initial and final velocities of particle i in cm system
T_0, T'_0	=	total kinetic energy in lab and cm systems
T_i, T'_i	=	kinetic energy of particle i in lab and cm systems
$\vec{V}$	=	velocity of cm system with respect to lab system
ψ_i	=	deflection angle of particle i in lab system (angle between initial velocity of particle 1 and the final velocity of particle i, $\cos(\psi_i) = \vec{v}_i \cdot \vec{u}_1$)
θ	=	deflection angle in cm system (same for both particles by definition of cm system)

Qualitatively, scattering of two particles in the lab and center-of-mass systems looks as follows (we choose m_2 to be initially at rest in the lab system to provide the most extreme difference between lab and cm systems):



Because the total momentum vanishes in the cm system, the two particles must exit the collision with collinear velocity vectors. Thus, the final state is parameterized by only one angle θ. When transformed back to the lab system, there are now two angles ψ_1 and ψ_2 reflecting the degree of freedom of the value of the total velocity $\vec{V}$. Note that the vector $\vec{V}$ fully describes the components of $\vec{v}_1$ and $\vec{v}_2$ that are not along the line between the two particles since $\vec{v}_i = \vec{V} + \vec{v}_i'$ and we know the $\vec{v}_i'$ are collinear.

Let's consider the problem quantitatively. The center of mass satisfies

$$\begin{aligned} m_1\,\vec{r}_1 + m_2\,\vec{r}_2 &= M\,\vec{R} \\ m_1\,\vec{u}_1 + m_2\,\vec{u}_2 &= M\,\vec{V} \end{aligned}$$

In the case $\vec{u}_2 = 0$, which we can always obtain by working in the rest frame of m_2, we have

$$\vec{V} = \frac{m_1\,\vec{u}_1}{m_1 + m_2}$$

This is the velocity at which the center of mass moves toward m_2. Since the center-of-mass is stationary in the center-of-mass frame, we have $\vec{u}_2' = -\vec{V}$. We also have

$$\vec{u}_1' = \vec{u}_1 - \vec{V} = \frac{m_2\,\vec{u}_1}{m_1 + m_2}$$

Elastic Collisions: Kinematics

We now specialize to **elastic collisions**, wherein the internal kinetic and potential energies of the colliding bodies are unchanged. We also assume there are no external potential energies, so that the only energies are the "external" kinetic energies of the problem, Because we assume the internal energies are unchanged, conservation of energy implies conservation of *mechanical energy*. Because we assume no external potentials, conservation of mechanical energy implies conservation of "external" kinetic energy. For our above two-particle collision problem, conservation of linear momentum and energy in the center-of mass frame yield:

$$\begin{aligned} 0 = m_1\,\vec{u}_1' + m_2\,\vec{u}_2' &= m_1\,\vec{v}_1' + m_2\,\vec{v}_2' \\ \frac{1}{2}\,m_1\,u_1'^2 + \frac{1}{2}\,m_2\,u_2'^2 &= \frac{1}{2}\,m_1\,v_1'^2 + \frac{1}{2}\,m_2\,v_2'^2 \end{aligned}$$

Solving the first equation for $\vec{u}_2'$ and $\vec{v}_2'$ and inserting into the second equation gives:

$$\frac{1}{2}\left[m_1 + \frac{m_1^2}{m_2}\right] u_1'^2 = \frac{1}{2}\left[m_1 + \frac{m_1^2}{m_2}\right] v_1'^2$$

So, $v_1' = u_1'$ and therefore $v_2' = u_2'$. The kinetic energy of each particle is individually conserved *in the cm frame!* In the cm frame, we have

$$\begin{aligned}\vec{v}_1' &= u_1'\,(\hat{x}\,\cos\theta + \hat{y}\,\sin\theta) = \frac{m_2\,u_1}{m_1+m_2}\,(\hat{x}\,\cos\theta + \hat{y}\,\sin\theta)\\ \vec{v}_2' &= -u_2'\,(\hat{x}\,\cos\theta + \hat{y}\,\sin\theta) = -\frac{m_1\,u_1}{m_1+m_2}\,(\hat{x}\,\cos\theta + \hat{y}\,\sin\theta)\end{aligned}$$

We transform back to the lab frame by adding $\vec{V}$:

$$\begin{aligned}\vec{v}_1 = \vec{v}_1' + \vec{V} &= \hat{x}\left[\frac{m_2\,u_1}{m_1+m_2}\,\cos\theta + \frac{m_1\,u_1}{m_1+m_2}\right] + \hat{y}\,\frac{m_2\,u_1}{m_1+m_2}\,\sin\theta\\ \vec{v}_2 = \vec{v}_2' + \vec{V} &= \hat{x}\left[-\frac{m_1\,u_1}{m_1+m_2}\,\cos\theta + \frac{m_1\,u_1}{m_1+m_2}\right] - \hat{y}\,\frac{m_1\,u_1}{m_1+m_2}\,\sin\theta\end{aligned}$$

We can find the angles ψ_1 and ψ_2:

$$\begin{aligned}\tan\psi_1 &= \frac{\sin\theta}{\frac{m_1}{m_2}+\cos\theta}\\ \tan\psi_2 &= \frac{\sin\theta}{1-\cos\theta}\end{aligned}$$

For all cases, $\tan\psi_2 \geq 0$ so $0 \leq \psi_2 \leq \pi/2$. Particle 2 always forward scatters in the lab frame, as one would expect since it starts at rest. The behavior of particle 1 depends on whether $m_1 > m_2$, $m_1 = m_2$, or $m_1 < m_2$. Let's consider the cases separately:

- $m_1 = m_2$: The solution is quite simple in this case:

$$\begin{aligned}\tan\psi_1 &= \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2}\\ \tan\psi_2 &= \frac{\sin\theta}{1-\cos\theta} = \cot\frac{\theta}{2}\end{aligned}$$

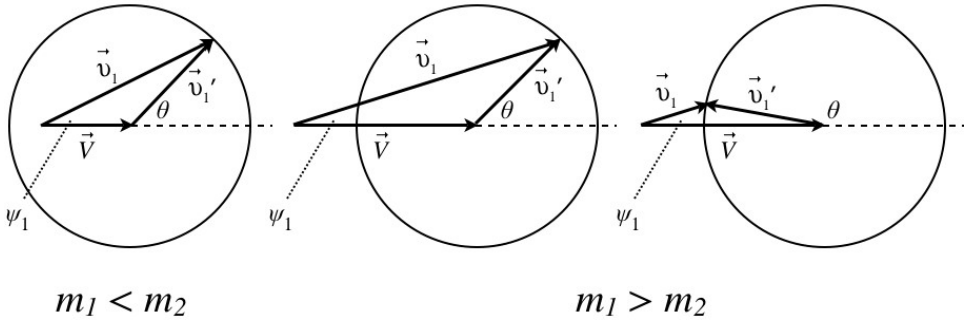
  For this particular case, $\psi_1 + \psi_2 = \frac{\pi}{2}$. The two particles emerge in the lab frame at right angles.

- $m_1 < m_2$: in this case, the denominator of $\tan\psi_1$ can be both positive and negative, while the numerator is always positive (by convention). Thus, there can be both forward and backward scattering of particle 1.

- $m_1 > m_2$: In this case, the denominator of $\tan\psi_1$ can only be positive, so for any cm scattering angle $0 \leq \theta \leq \pi$, we have $0 \leq \psi \leq \pi/2$. There can be only forward scattering of particle 1.

We can interpret the above in terms of the relative size of the cm speed V and the scattered speed of the first particle in the cm, v_1'. The ratio $m_1/m_2 = V/v_1'$. Since $\vec{v}_1 = \vec{V} + \vec{v}_1'$, the size of the ratio V/v_1' relative to 1 determines whether or not $\vec{v}_1$ can be negative.

We can derive another fact about the scattering by considering the solution geometrically. Since $\vec{v}_1 = \vec{V} + \vec{v}_1'$, the three vectors form a triangle. The possible shapes of the triangle depend on the value of V/v_1' relative to 1:



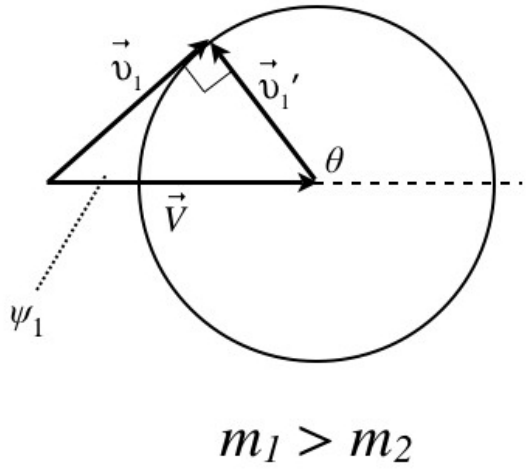
When $V/v_1' = m_1/m_2 < 1$, there is only one solution for $\vec{v}_1$ for a given value of ψ_1. For $V/v_1' > 1$, we see that a single value of ψ_1 can be derived from two different values of θ. It is not necessarily obvious from the formula for $\tan\psi_1$ that this would be true.

If $m_1 \ll m_2$, then $\tan\psi_1 \approx \tan\theta$ or $\psi_1 \approx \theta$; the lab and cm frames approximately coincide. For $m_1 \gg m_2$, we have $\tan\psi_1 \ll 1$ for all θ, so m_2 is always scattered almost along the incoming particle velocity vector regardless of the size of the incoming particle speed.

Let's calculate the maximum value of ψ_1. Consider the three cases separately:

- $m_1 = m_2$: Recall that $\psi_1 = \theta/2$ for this case, so clearly $\psi_{1,max} = \pi/2$.
- $m_1 < m_2$: Backward scattering in the lab frame is allowed in this case, so $\psi_{1,max} = \pi$.
- $m_1 > m_2$: This is the most difficult case because backward scattering in the cm frame will still be forward scattering in the lab frame. We can figure it out in two ways:
  - Geometrically: there is a simpler geometric derivation: There are two center-of-mass-frame scattering angles θ that result in the same lab frame angle ψ_1. These occur at the two intersection points with the circle of radius v_1' of a line that makes angle ψ_1 with $\vec{V}$. As ψ_1 increases, these two intersection points move together until they become identical. At this point, the vector $\vec{v}_1$ is tangent to the circle. A tangent to a circle is normal to the radius vector to that point, which is $\vec{v}_1'$. So, we have a right triangle with sides v_1' (subtending ψ_1) and v_1 and hypotenuse V. So, clearly,

$$\sin\psi_{1,max} = \frac{v_1'}{V} = \frac{m_2}{m_1}$$



– By calculus: we saw before that ψ_1 stays in the first quadrant for $m_1 > m_2$, so maximizing ψ_1 corresponds to maximizing $\tan\psi_1$. Let's take the derivative and set to zero:

$$\begin{aligned}
\frac{d}{d\theta}\tan\psi_1 &= \frac{\cos\theta}{\frac{m_1}{m_2}+\cos\theta} + \frac{\sin^2\theta}{\left(\frac{m_1}{m_2}+\cos\theta\right)^2} \\
&= \frac{\frac{m_1}{m_2}\cos\theta + \cos^2\theta + \sin^2\theta}{\frac{m_1}{m_2}+\cos\theta} \\
&= \frac{1+\frac{m_1}{m_2}\cos\theta}{\frac{m_1}{m_2}\cos\theta}
\end{aligned}$$

Requiring the above to vanish implies $\cos\theta = -m_2/m_1$. We then have

$$\begin{aligned}
\tan^2\psi_{1,max} &= \frac{\sin^2\theta}{\left(\frac{m_1}{m_2}+\cos\theta\right)^2} \\
&= \frac{1-\frac{m_2^2}{m_1^2}}{(\frac{m_1}{m_2}-\frac{m_2}{m_1})^2} \\
&= \frac{m_1^2\,m_2^2 - m_2^4}{(m_1^2-m_2^2)^2} \\
&= \frac{m_2^2}{m_1^2-m_2^2}
\end{aligned}$$

Now we use some trigonometric identities to find $\sin\psi_{1,max}$:

$$\begin{aligned}
\sin^2\psi_{1,max} &= \frac{1}{1+\cot^2\psi_{1,max}} \\
&= \frac{1}{1+\frac{m_1^2-m_2^2}{m_2^2}} \\
&= \frac{m_2^2}{m_1^2} \\
\sin\psi_{1,max} &= \frac{m_2}{m_1}
\end{aligned}$$

The interpretation of this relation is: the larger the mismatch in masses, the more forward-concentrated the scattering is.

Elastic Collisions: Energy

Recall earlier that we determined that the kinetic energy of each particle is conserved individually in the cm frame, which is convenient. What happens to the kinetic energies in the lab frame?

First, we know that the initial kinetic energy in the lab and cm frames for $\vec{u}_2 = 0$ are

$$\begin{aligned} T_0 &= \frac{1}{2}\, m_1\, u_1^2 \\ T_0' &= \frac{1}{2}\left[m_1\, u_1'^2 + m_2\, u_2'^2\right] \\ &= \frac{1}{2}\left[m_1\, \frac{m_2^2}{(m_1+m_2)^2}\, u_1^2 + m_2\, \frac{m_1^2}{(m_1+m_2)^2}\, u_1^2\right] \\ &= \frac{1}{2}\frac{m_1\, m_2}{m_1+m_2}\, u_1^2 \\ &= \frac{m_2}{m_1+m_2}\, T_0 \end{aligned}$$

Note that this implies $T_0' < T_0$. The final cm energies are

$$\begin{aligned} T_1' &= \frac{1}{2}\, m_1\, v_1'^2 = \frac{1}{2}\, m_1\, u_1'^2 \\ &= \frac{1}{2}\, m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 u_1^2 \\ &= \left(\frac{m_2}{m_1+m_2}\right)^2 T_0 \\ T_2' &= \frac{1}{2}\, m_2\, v_2'^2 = \frac{1}{2}\, m_2\, u_2'^2 \\ &= \frac{1}{2}\, m_2 \left(\frac{m_1}{m_1+m_2}\right)^2 u_1^2 \\ &= \frac{m_1\, m_2}{(m_1+m_2)^2}\, T_0 \\ T_1' + T_2' &= \frac{m_2^2 + m_1\, m_2}{(m_1+m_2)^2}\, T_0 \\ &= \frac{m_2}{m_1+m_2}\, T_0 = T_0' \end{aligned}$$

Obviously, we will want to find the final lab frame energies in terms of the initial lab frame energy. We can write

$$\frac{T_1}{T_0} = \frac{\frac{1}{2}\, m_1\, v_1^2}{\frac{1}{2}\, m_1\, u_1^2} = \frac{v_1^2}{u_1^2}$$

The law of cosines applied to the figure above relating $\vec{v}_1$, $\vec{v}_1'$, and $\vec{V}$ gives

$$v_1'^2 = v_1^2 + V^2 - 2\, v_1\, V \cos\psi_1$$

which then implies

$$\begin{aligned} \frac{T_1}{T_0} &= \frac{v_1^2}{u_1^2} = \frac{v_1'^2}{u_1^2} - \frac{V^2}{u_1^2} + 2\, \frac{v_1\, V}{u_1^2}\, \cos\psi_1 \\ &= \left(\frac{m_2}{m_1+m_2}\right)^2 - \left(\frac{m_1}{m_1+m_2}\right)^2 + 2\, \frac{v_1\, V}{u_1^2}\, \cos\psi_1 \\ &= \frac{m_2 - m_1}{m_1+m_2} + 2\, \frac{v_1\, V}{u_1^2}\, \cos\psi_1 \end{aligned}$$

We can rewrite the third term easily. Recall that $\vec{v}_1 = \vec{v}_1' + \vec{V}$. Since $\vec{V}$ has no y component, it therefore holds that $v_1 \sin\psi_1 = v_1' \sin\theta$. So we get

$$\begin{aligned}
\frac{T_1}{T_0} &= \frac{m_2 - m_1}{m_1 + m_2} + 2\,\frac{v_1' V}{u_1^2}\,\frac{\sin\theta}{\sin\psi_1}\,\cos\psi_1 \\
&= \frac{m_2 - m_1}{m_1 + m_2} + 2\,\frac{m_2}{m_1 + m_2}\,\frac{m_1}{m_1 + m_2}\,\frac{\sin\theta}{\tan\psi_1} \\
&= \frac{m_2 - m_1}{m_1 + m_2} + 2\,\frac{m_2}{m_1 + m_2}\,\frac{m_1}{m_1 + m_2}\,\sin\theta\,\frac{\frac{m_1}{m_2} + \cos\theta}{\sin\theta} \\
&= \frac{m_2 - m_1}{m_1 + m_2} + 2\,\frac{m_1\,m_2}{(m_1 + m_2)^2}\left(\frac{m_1}{m_2} + \cos\theta\right) \\
&= \frac{m_2^2 - m_1^2 + 2\,m_1^2}{(m_1 + m_2)^2} + 2\,\frac{m_1\,m_2}{(m_1 + m_2)^2}\,\cos\theta \\
&= 1 - \frac{2\,m_1\,m_2}{(m_1 + m_2)^2} + 2\,\frac{m_1\,m_2}{(m_1 + m_2)^2}\,\cos\theta \\
\frac{T_1}{T_0} &= 1 - \frac{2\,m_1\,m_2}{(m_1 + m_2)^2}\,(1 - \cos\theta)
\end{aligned}$$

We can write T_1/T_0 in terms of the lab-frame angle also. To do this, we need to express $\cos\theta$ in terms of ψ_1. Recall from earlier that

$$\tan\psi_1 = \frac{\sin\theta}{\frac{m_1}{m_2} + \cos\theta}$$

For notational convenience, let $r = m_1/m_2$. Let's manipulate the above:

$$\begin{aligned}
\tan^2\psi_1 &= \frac{\sin^2\theta}{(r + \cos\theta)^2} \\
(r + \cos\theta)^2 \tan^2\psi_1 &= 1 - \cos^2\theta \\
\cos^2\theta\,(\tan^2\psi_1 + 1) + 2\,r\,\tan^2\psi_1\,\cos\theta + r^2\tan^2\psi_1 - 1 &= 0 \\
\cos^2\theta\cos^{-2}\psi_1 + 2\,r\,\tan^2\psi_1\,\cos\theta + r^2\tan^2\psi_1 - 1 &= 0 \\
\cos^2\theta + 2\,r\,\sin^2\psi_1\,\cos\theta + r^2\sin^2\psi_1 - \cos^2\psi_1 &= 0
\end{aligned}$$

Apply the quadratic formula to find

$$\begin{aligned}
\cos\theta &= \frac{1}{2}\left[-2\,r\,\sin^2\psi_1 \pm \sqrt{4\,r^2\,\sin^4\psi_1 - 4\left(r^2\sin^2\psi_1 - \cos^2\psi_1\right)}\right] \\
&= -r\,\sin^2\psi_1 \pm \sqrt{r^2\,\sin^4\psi_1 - r^2\sin^2\psi_1 + 1 - \sin^2\psi_1} \\
&= -r\,\sin^2\psi_1 \pm \sqrt{(r^2\,\sin^2\psi_1 - 1)(\sin^2\psi_1 - 1)} \\
&= -r\,\sin^2\psi_1 \pm \cos\psi_1\sqrt{1 - r^2\,\sin^2\psi_1} \\
1 - \cos\theta &= 1 + r\left[\sin^2\psi_1 \pm \cos\psi_1\sqrt{r^{-2} - \sin^2\psi_1}\right]
\end{aligned}$$

Inserting this into our expression for T_1/T_0 in terms of θ, we find

$$\begin{aligned}
\frac{T_1}{T_0} &= 1 - \frac{2\,m_1\,m_2}{(m_1+m_2)^2}\left\{1 + \frac{m_1}{m_2}\left[\sin^2\psi_1 \pm \cos\psi_1\sqrt{\frac{m_2^2}{m_1^2} - \sin^2\psi_1}\right]\right\} \\
&= \frac{m_1^2+m_2^2}{(m_1+m_2)^2} - \frac{2\,m_1^2}{(m_1+m_2)^2}\left[\sin^2\psi_1 \pm \cos\psi_1\sqrt{\frac{m_2^2}{m_1^2} - \sin^2\psi_1}\right] \\
&= \frac{m_1^2+m_2^2}{(m_1+m_2)^2} + \frac{2\,m_1^2}{(m_1+m_2)^2}\left[-\sin^2\psi_1 \mp \cos\psi_1\sqrt{\frac{m_2^2}{m_1^2} - \sin^2\psi_1}\right] \\
&= \frac{m_1^2+m_2^2}{(m_1+m_2)^2} \\
&\quad + \frac{2\,m_1^2}{(m_1+m_2)^2}\left[-\frac{1}{2}\frac{m_2^2}{m_1^2} - \frac{1}{2}\,\sin^2\psi_1 + \frac{1}{2}\left(\frac{m_2^2}{m_1^2} - \sin^2\psi_1\right)\right. \\
&\quad \left. \mp \cos\psi_1\sqrt{\frac{m_2^2}{m_1^2} - \sin^2\psi_1} + \frac{1}{2}\,\cos^2\psi_1 - \frac{1}{2}\,\cos^2\psi_1\right] \\
&= \frac{m_1^2+m_2^2}{(m_1+m_2)^2} \\
&\quad + \frac{2\,m_1^2}{(m_1+m_2)^2}\left[-\frac{1}{2}\frac{m_2^2}{m_1^2} - \frac{1}{2}\left(\sin^2\,\psi_1 + \cos^2\psi_1\right) + \frac{1}{2}\left(\sqrt{\frac{m_2^2}{m_1^2} - \sin^2\psi_1} \mp \cos\psi_1\right)^2\right] \\
\frac{T_1}{T_0} &= \frac{m_1^2}{(m_1+m_2)^2}\left[\cos\psi_1 \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\psi_1}\right]^2
\end{aligned}$$

Notes on signs:

- We can see that the quantity under the square root sign is nonnegative and so always defined. Recall earlier we proved that for $m_1 > m_2$, the maximum value of $\sin\psi_1$ is m_2/m_1. So the quantity under the square root is nonnegative for this case. When $m_2 \geq m_1$, there will also be no problem because the maximum value of $\sin\psi_1$ is 1.
- We need to specify whether to use one or both of the possible solutions indicated by the $\pm$ sign. In the case $m_1 < m_2$, we know based on our geometric arguments that there can only be one solution. To understand which one to pick, let's determine the size of the square root quantity:

$$\begin{aligned}
\frac{m_2}{m_1} &> 1 \\
\Longrightarrow \frac{m_2^2}{m_1^2} - \sin^2\psi &> 1 - \sin^2\psi_1 = \cos^2\psi_1 \\
\Longrightarrow \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\psi} &> |\cos\psi_1|
\end{aligned}$$

  Now, we expect the incoming particle to lose more and more energy as scattering goes from forward to backward in the lab frame. Because $m_1 < m_2$, we know that backscattering is possible in the lab frame, and so ψ_1 may take on values that yield

$\cos\psi_1 < 0$. These two considerations lead us to choose the + sign: for $\psi_1 = 0$, both terms are positive and take on their maximum values (when $\cos\psi_1 = 1$ and $\sin\psi_1 = 0$). For backscattering, we have $\cos\psi_1 < 0$ and so the two terms have opposite sign, making T_1 smallest for backscattering.

If $m_1 > m_2$, then we know there are two solutions from our geometric arguments and so we should take both the $\pm$ solutions; there is no choice to make here. But, of course, it is interesting to relate these two solutions to the center-of-mass picture. As one can see from the earlier diagram, the outgoing vector $\vec{v}_1$ is longest (and hence T_1 is largest) when the scattering is forward in the center-of-mass frame. Thus, the + solution corresponds to forward scattering in the center-of-mass frame and the − solution to backward scattering in that frame.

If $m_1 = m_2$, then the square-root quantity becomes $|\cos\psi_1|$. Taking the − solution would give $T_1 = 0$ for all ψ_1, which is clearly nonsense. So we should take the + solution, giving

$$\frac{T_1}{T_0} = \frac{4\,m_1^2}{(m_1+m_2)^2}\cos^2\psi_1 = \cos^2\psi_1$$

when $m_1 = m_2$.

We can derive the kinetic energy of the recoiling particle in the lab frame in terms of the cm and lab frame angles also:

$$\begin{aligned}\frac{T_2}{T_0} &= 1 - \frac{T_1}{T_0} \\ &= 1 - \left[1 - \frac{2\,m_1\,m_2}{(m_1+m_2)^2}\,(1-\cos\theta)\right] \\ &= \frac{2\,m_1\,m_2}{(m_1+m_2)^2}\,(1-\cos\theta)\end{aligned}$$

To convert to the lab frame angle, we make use of the relation derived earlier:

$$\begin{aligned}\tan\psi_2 &= \frac{\sin\theta}{1-\cos\theta} \\ \tan^2\psi_2 &= \frac{\sin^2\theta}{(1-\cos\theta)^2} \\ &= \frac{1-\cos^2\theta}{(1-\cos\theta)^2} \\ &= \frac{1+\cos\theta}{1-\cos\theta} \\ \tan^2\psi_2\,[1-\cos\theta] &= 1+\cos\theta \\ \tan^2\psi_2 - 1 &= \cos\theta\left[1+\tan^2\psi_2\right] \\ \tan^2\psi_2 - 1 &= \cos\theta\cos^{-2}\psi_2 \\ \cos\theta &= \sin^2\psi_2 - \cos^2\psi_2 \\ 1-\cos\theta &= 2\cos^2\psi_2\end{aligned}$$

so then we find

$$\frac{T_2}{T_0} = \frac{4\,m_1\,m_2}{(m_1+m_2)^2}\cos^2\psi_2$$

Notes:

- In the special case $m_1 = m_2$, we have $T_1/T_0 = \cos^2\psi_1$ and $T_2/T_0 = \cos^2\psi_2 = \sin^2\psi_1$ because $\psi_1 + \psi_2 = \pi/2$ for this case.
- The center-of-mass scattering angle θ and the ratio of input to output kinetic energies are very simply related.

Example 1.17

A particle of mass m_1 elastically scatters from a particle of mass m_2 at rest. At what lab-frame angle should one place a detector to detect m_1 if it loses one-third of its momentum? Over what range m_1/m_2 is this possible? Calculate the scattering angle for $m_1/m_2 = 1$.

Our condition on the final state speed is

$$m_1\, v_1 = \frac{2}{3}\, m_1\, u_1$$

The energy ratio between output and input kinetic energy is

$$\begin{aligned}\frac{T_1}{T_0} &= \frac{\frac{1}{2}\, m_1\, v_1^2}{\frac{1}{2}\, m_1\, u_1^2} \\ &= \frac{v_1^2}{u_1^2} \\ &= \frac{4}{9}\end{aligned}$$

We equate this to our formula for the lab-frame scattering angle

$$\begin{aligned}\frac{4}{9} &= 1 - \frac{2\, m_1\, m_2}{(m_1 + m_2)^2}\,(1 - \cos\theta) \\ 1 - \cos\theta &= -\frac{5}{18}\frac{(m_1 + m_2)^2}{m_1\, m_2} \\ \cos\theta &= 1 - \frac{5}{18}\frac{(m_1 + m_2)^2}{m_1\, m_2} \equiv y\end{aligned}$$

We want the lab angle, though, so let's use the relation between cm and lab angle:

$$\begin{aligned}\tan\psi &= \frac{\sin\theta}{\frac{m_1}{m_2} + \cos\theta} \\ &= \frac{\sqrt{1 - y^2}}{\frac{m_1}{m_2} + y}\end{aligned}$$

Because $\tan\psi$ must be real, we require $1 - y^2 \geq 0$. Since $y \leq 1$ by definition, this corresponds to requiring $1 + y \geq 0$, or

$$\begin{aligned}1 + \left[1 - \frac{5}{18}\frac{(m_1 + m_2)^2}{m_1\, m_2}\right] &\geq 0 \\ -5\, x^2 + 26\, x - 5 &\geq 0 \\ (-5\, x + 1)(x - 5) &\geq 0\end{aligned}$$

The two roots are $x = 1/5$ and $x = 5$ and the inequality is satisfied in the range $1/5 < x < 5$; *i.e.*,

$$\frac{1}{5} \leq \frac{m_1}{m_2} \leq 5$$

That is, we can find the solution $\tan\psi$ given above when the mass ratio is in this range. For $m_1 = m_2$, we find

$$\begin{aligned} y &= -\frac{1}{9} \\ \tan\psi &= \frac{\sqrt{1-(-1/9)^2}}{1-1/9} = \sqrt{80/64} = \sqrt{5/4} \\ \psi &\approx 48.2^\circ \end{aligned}$$

Inelastic Collisions

We conclude with a very brief discussion of inelastic collisions. These are collisions in which linear momentum is conserved but mechanical energy is not because some of the energy goes into internal motions of the colliding objects. Recall that conservation of total linear momentum for a system required only that internal forces obey the weak form of the third law, whereas conservation of mechanical energy required the strong form and that the internal kinetic and potential energies be fixed. We must therefore adjust our law of conservation of energy to be

$$Q + T_1 + T_2 = T_1' + T_2'$$

where Q represents the amount of energy that will be liberated ($Q < 0$) or absorbed ($Q > 0$). A classic inelastic collision is the following one: A ball of putty is thrown at another ball of putty and they stick together, continuing on as one body. Clearly, energy was put into internal motion (heat, etc.) by the collision; the mechanical energy is not conserved.

One useful concept to consider when looking at inelastic collisions is that of **impulse**

$$\vec{P} = \int_{t_1}^{t_2} \vec{F} dt$$

The reason impulse is useful is that, though we usually do not know the details of $\vec{f}(t)$, we can determine the total impulse, which gives the total momentum change. Or, vice versa, we can find the force from the total impulse.

Example 1.18

A rope of linear density λ is dropped onto a table. Let part of the rope already have come to rest on the table. As the free end falls a distance z, find the force exerted on the table.

During a time dt, a portion of the rope $v\,dt$ comes to rest, where $v = |\dot{z}|$ is the fall speed. The momentum of the rope changes by an amount

$$dp = (\lambda\, v\, dt)\, v = \lambda\, v^2\, dt$$

This is equal but opposite to the change of momentum of the table. The force on the table due to this impulse is therefore

$$F_{impulse} = \frac{dp}{dt} = \lambda\, v^2 = 2\,\lambda\, z\, g$$

where the last step is made by making use of the kinematics of a freely falling object. If the table remains stationary as the rope falls on it, the table must be exerting an equal but opposite force back on the rope. The table thus exerts a total force on the rope

$$F = F_{impulse} + F_g = 3\,\lambda\,g\,z$$

The first part of the force brings the portion of the rope z to rest, while the second term keeps it from falling through the table under the influence of gravity.

Chapter 2

Lagrangian and Hamiltonian Dynamics

This chapter presents Lagrangian and Hamiltonian dynamics, an advanced formalism for studying various problems in mechanics. Lagrangian techniques can provide a much cleaner way of solving some physical systems than Newtonian mechanics, in particular the inclusion of constraints on the motion. Lagrangian techniques allow postulation of Hamilton's Principle of Least Action, which can be considered an alternative to Newton's second law as the basis of mechanics. Symmetry under transformations is investigated and seen to lead to useful conserved quantities. The Hamiltonian formalism is introduced, which is useful for proving various important formal theorems in mechanics and, historically, was the starting point for quantum mechanics.

2.1 The Lagrangian Approach to Mechanics

The fundamental idea of the Lagrangian approach to mechanics is to reformulate the equations of motion in terms of the dynamical variables that describe the degrees of freedom, and thereby to incorporate constraint forces into the definition of the degrees of freedom rather than explicitly including them as forces in Newton's second law.

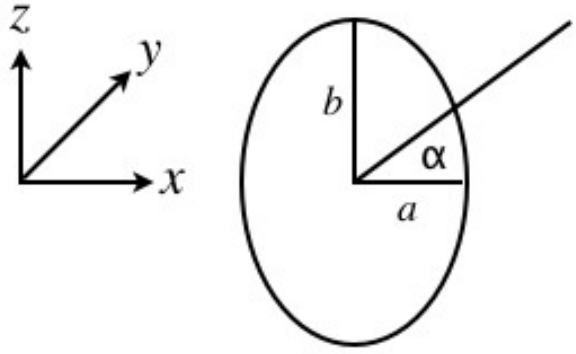
We discuss the basics of Lagrangian mechanics – degrees of freedom, constraints, generalized coordinates, virtual work, generalized forces, the Lagrangian and Hamiltonian functions, and various methods to derive equations of motion using these concepts. This section follows Hand and Finch Chapter 1, though in somewhat different order.

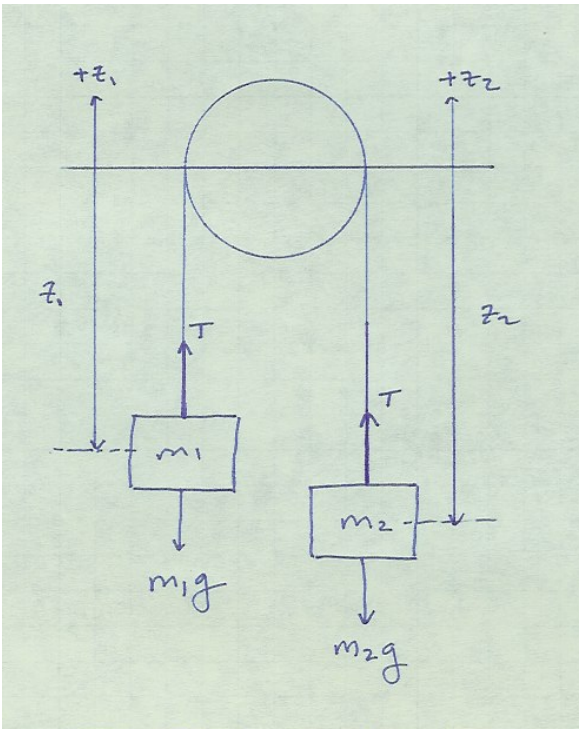
Be careful to realize that the Lagrangian approach is not independent of Newton's second law; the derivation of d'Alembert's principle, the critical step in developing Lagrangian mechanics, relies directly on Newton's second law. We will come to a new formulation of mechanics in the following section.

Throughout this section, we will work two examples alongside the theory. The first consists of a point particle sliding on an elliptical wire in the presence of gravity. The Cartesian coordinates of the particle satisfy

$$\left(\frac{x}{a(t)}\right)^2 + \left(\frac{z}{b(t)}\right)^2 = 1$$

We will at various points consider a and b to be time dependent or constant. The origin of the coordinate system is the stationary center of the ellipse. The second consists of an Atwood's machine, as in Example 1.4, except we now allow the rope length to be a function of time, $l = l(t)$.





Examples to be worked alongside theory.

2.1.1 Degrees of Freedom, Constraints, and Generalized Coordinates

Degrees of Freedom

Obviously, a system of M point particles that are unconstrained in any way has $3\,M$ degrees of freedom.

There is freedom, of course, in how we specify the degrees of freedom; *e.g.*:

- choice of origin
- coordinate system: cartesian, cylindrical, spherical, etc.
- center-of-mass vs. individual particles: $\{\vec{r}_k\}$ or $\{\vec{R}, \vec{s}_k = \vec{r}_k - \vec{R}\}$

But, the number of degrees of freedom is always the same; *e.g.*, in the center-of-mass system, the constraint $\sum_k \vec{s}_k = 0$ applies, ensuring that $\{\vec{r}_k\}$ and $\{\vec{R}, \vec{s}_k\}$ have same number of degrees of freedom.

The motion of such a system is completely specified by knowing the dependence of the available degrees of freedom on time.

Example 2.1:

In the elliptical wire example, there are *a priori* 3 degrees of freedom, the 3 spatial coordinates of the point particle. The constraints reduce this to one degree of freedom, as no motion in y is allowed and the motions in z and x are related. The loss of the y degree of freedom is easily accounted for in our Cartesian coordinate system; effectively, a 2D Cartesian sytem in x and z will suffice. But the relation between x and z is a constraint that cannot be trivially accomodated by dropping another Cartesian coordinate.

Example 2.2:

In the Atwood's machine example, there are *a priori* 2 degrees of freedom, the z coordinates of the two blocks. (We ignore the x and y degrees of freedom because the problem is inherently 1-dimensional.) The inextensible rope connecting the two masses reduces this to one degree of freedom because, when one mass moves by a certain amount, the other one must move by the opposite amount.

Constraints

Constraints may reduce the number of degrees of freedom; *e.g.*, particle moving on a table, rigid body, etc.

Holonomic constraints are those that can be expressed in the form

$$f(\vec{r}_1, \vec{r}_2, \ldots, t) \;\; = \;\; 0$$

For example, restricting a point particle to move on the surface of a table is the holonomic constraint $z - z_0 = 0$ where z_0 is a constant. A rigid body satisfies the holonomic set of constraints

$$|\vec{r}_i - \vec{r}_j| - c_{ij} \;\; = \;\; 0$$

where c_{ij} is a set of constants satisfying $c_{ij} = c_{ji} > 0$ for all particle pairs i, j. For the curious: it is remarkably hard to find the etymology of holonomic (or holonomy) on the web. I found the following (thanks to John Conway of Princeton):

I believe it was first used by Poinsot in his analysis of the motion of a rigid body. In this theory, a system is called "holonomic" if, in a certain sense, one can recover global information from local information, so the meaning "entire-law" is quite appropriate. The rolling of a ball on a table is non-holonomic, because one rolling along different paths to the same point can put it into different orientations.

However, it is perhaps a bit too simplistic to say that "holonomy" means "entire-law". The "nom" root has many intertwined meanings in Greek, and perhaps more often refers to "counting". It comes from the same Indo-European root as our word "number."

Nonholonomic constraints are, obviously, constraints that are not holonomic. Hand and Finch Chapter 1 Appendix A has a nice discussion. We present the highlights here. There are three kinds of nonholonomic constraints:

1. **Nonintegrable** or history-dependent constraints. These are constraints that are not fully defined until the full solution of the equations of motion is known. Equivalently, they are certain types of constraints involving velocities.

   The classic case of this type is a vertical disk rolling on a horizontal plane. If x and y define the position of the point of contact between the disk and the plane, ϕ defines the angle of rotation of the disk about its axis, and θ defines the angle between the rotation axis of the disk and the x-axis, then one can find the constraints

   $$\begin{aligned} \dot{x} &= -r\,\dot{\phi}\,\cos\theta \\ \dot{y} &= -r\,\dot{\phi}\,\sin\theta \end{aligned}$$

   The differential version of these constraints is

   $$\begin{aligned} dx &= -r\,d\phi\,\cos\theta \\ dy &= -r\,d\phi\,\sin\theta \end{aligned}$$

   These differential equations are not integrable; one cannot generate from the relations two equations $f_1(x,\theta,\phi) = 0$ and $f_2(y,\theta,\phi) = 0$. The reason is that, if one assumes the functions f_1 and f_2 exist, the above differential equations imply that their second derivatives would have to satisfy

   $$\frac{\partial^2 f}{\partial\theta\,\partial\phi} \neq \frac{\partial^2 f}{\partial\phi\,\partial\theta}$$

   which is a very unpleasant mathematical condition. Explicitly, suppose f_1 existed. Then we would be able to write

   $$f_1(x,\theta,\phi) = 0$$

   Let us obtain the differential version of the constraint by differentiating:

   $$\frac{\partial f_1}{\partial x}\,dx + \frac{\partial f_1}{\partial\theta}\,d\theta + \frac{\partial f_1}{\partial\phi}\,d\phi = 0$$

   This differential constraint should match the original differential constraint $dx = -r\,d\phi\,\cos\theta$. Identifying the coefficients of the differentials yields

   $$\frac{\partial f_1}{\partial x} = 1 \qquad \frac{\partial f_1}{\partial\theta} = 0 \qquad \frac{\partial f_1}{\partial\phi} = r\,\cos\phi$$

Taking the mixed second partial derivatives gives

$$\frac{\partial^2 f_1}{\partial\phi\,\partial\theta} = 0 \qquad \frac{\partial^2 f_1}{\partial\theta\,\partial\phi} = -r\,\sin\phi$$

which, clearly, do not match.

Such constraints are also called nonintegrable because one cannot integrate the differential equation to find a constraint on the coordinates. Nonintegrability is at the root of the etymology indicated in the quotation above: a differential relation such as the one above is a local one; if the differential relation is integrable, you can obtain the constraint at all points in space, *i.e.*, you can find the "entire law". Clearly, nonintegrability is also related to the fact that the constraint is velocity-dependent: a velocity-dependent constraint is a local constraint, and it may not always be possible to determine a global constraint from it.

2. inequality constraints; *e.g.*, particles required to stay inside a box, particle sitting on a sphere but allowed to roll off
3. problems involving frictional forces

Holonomic constraints may be divided into **rheonomic** ("running law") and **scleronomic** ("rigid law") depending on whether time appears explicitly in the constraints:

$$\begin{aligned} \text{rheonomic:} &\quad f(\{\vec{r}_k\}, t) = 0 \\ \text{scleronomic:} &\quad f(\{\vec{r}_k\}) = 0 \end{aligned}$$

At a technical level, the difference is whether $\frac{\partial f}{\partial t} = 0$ or not; the presence of this partial derivative affects some of the relations we will derive later.

Example 2.1:

For the elliptical wire example, the constraint equation is the one we specified initially:

$$\left(\frac{x}{a(t)}\right)^2 + \left(\frac{z}{b(t)}\right)^2 = 1$$

If a and/or b do indeed have time dependence, then the constraint is rheonomic. Otherwise, it is scleronomic.

Example 2.2:

For the Atwood's machine, the constraint equation is

$$z_1 + z_2 + l(t) = 0$$

where $l(t)$ is the length of the rope (we assume the pulley has zero radius). The signs on z_1 and z_2 are due to the choice of direction for positive z in the example. The constraint is again rheonomic if l is indeed time dependent, scleronomic if not.

Generalized Coordinates

In general, if one has j independent constraint equations for a system of M particles with $3\,M$ degrees of freedom, then the true number of degrees of freedom is $3\,M - j$. There

is dynamical behavior of the system in only these remaining degrees of freedom. One immediately asks the question – since there are fewer degrees of freedom than position coordinates, is there any way to eliminate those unnecessary degrees of freedom and thereby simplify analysis of the mechanical system? In our example, why keep both x and z if one of them would suffice? This question leads us to the idea of *generalized coordinates*, which are a set of $3\,M - j$ coordinates that have already taken into account the constraints and are independent, thereby reducing the complexity of the system.

For *holonomic* constraints, the constraint equations ensure that it will always be possible to define a new set of $3\,M - j$ **generalized coordinates** $\{q_k\}$ that fully specify the motion of the system subject to the constraints and that are *independent* of each other. The independence arises because the number of degrees of freedom must be conserved. The constraint equations yield (possibly implicit) functions

$$\vec{r}_i \;=\; \vec{r}_i(q_1, q_2, ..., q_{3M-j}, t) \tag{2.1}$$

that transform between the generalized coordinates and the original coordinates. It may not always be possible to write these functions analytically. Some of these coordinates may be the same as the original coordinates, some may not; it simply depends on the structure of the constraints. We will refer to the original coordinates as **position coordinates** to distinguish them from the reduced set of independent generalized coordinates.

Generalized coordinates are more than just a notational convenience. By incorporating the constraints into the definition of the generalized coordinates, we obtain two important simplifications: 1) the constraint forces are eliminated from the problem; and 2) the generalized coordinates are fully independent of each other. We shall see these simplifications put into effect when we discuss virtual work and generalized forces.

Just as the velocity corresponding to a coordinate $\vec{r}_j$ is $\dot{\vec{r}}_j = \frac{d}{dt}\vec{r}_j$, it is possible to define a **generalized velocity** $\dot{q}_k = \frac{d}{dt}q_k$. Note that in all cases, velocities are defined as **total time derivatives** of the particular coordinate. Remember that if you have a function $g = g(\{q_k\}, t)$, then the total time derivative $\frac{d}{dt}\,g$ is evaluated by the chain rule:

$$\frac{d}{dt}\,g(\{q_k\}, t) \;=\; \sum_k \frac{\partial g}{\partial q_k}\,\frac{dq_k}{dt} + \frac{\partial g}{\partial t} \tag{2.2}$$

It is very important to realize that, until a specific solution to the equation of motion is found, a generalized coordinate and its corresponding generalized velocity are *independent* variables. This can be seen by simply remembering that two initial conditions – $q_k(t = 0)$ and $\dot{q}_k(t = 0)$ are required to specify a solution of Newton's second law because it is a *second-order* differential equation. Higher-order derivatives are *not independent* variables because Newton's second law relates the higher-order derivatives to $\{q_k\}$ and $\{\dot{q}_k\}$. **The independence of $\{q_k\}$ and $\{\dot{q}_k\}$ is a reflection of the structure of Newton's second law, not just a mathematical theorem.**

Unless otherwise indicated, from here on we will assume all constraints are holonomic.

Example 2.1:

For the elliptical wire, the constraint equation

$$\left(\frac{x}{a(t)}\right)^2 + \left(\frac{z}{b(t)}\right)^2 = 1$$

can be used to define different generalized coordinate schemes. Two obvious ones are x and z; *i.e.*, let x be the generalized coordinate and drop the z degree of freedom, or vice versa. Another obvious one would be the azimuthal angle α,

$$\alpha = \tan^{-1}\left[\frac{z}{b(t)}\,\frac{a(t)}{x}\right]$$

The formal definitions $\vec{r}_i(\{q_k\},t)$ are then

$$x = a(t)\,\cos\alpha \qquad z = b(t)\,\sin\alpha$$

Here, we see the possibility of explicit time dependence in the relationship between the positions x and z and the generalized coordinate α.

Example 2.2:

For the Atwood's machine, either z_1 or z_2 could suffice as the generalized coordinate. Let's pick z_1, calling it Z to distinguish the generalized coordinate, giving

$$z_1 = Z \qquad z_2 = -l(t) - Z$$

This case is pretty trivial.

"Dot Cancellation"

For holonomic constraints, there is a very important statement that we will make much use of later:

$$\frac{\partial\vec{r}_i}{\partial q_k} \;=\; \frac{\partial\dot{\vec{r}}_i}{\partial\dot{q}_k} \tag{2.3}$$

Conceptually, what this says is *The differential relationship between a given degree of freedom and a generalized coordinate is the same as the differential relationship between the corresponding velocities.* This statement relies on having holonomic constraints. Heuristically, one can understand the rule as simply arising from the fact that holonomic constraints use only the positions and time to define the generalized coordinates; as a result, any relationships between positional velocities and generalized velocities must be determined only by relationships between positions and generalized coordinates. The above relationship is then the simplest possible one.

We derive the result rigorously by starting with the total time derivative $\dot{\vec{r}}_i$:

$$\dot{\vec{r}}_i = \frac{d}{dt}\,\vec{r}_i(\{q_k\},t) \;=\; \sum_k \frac{\partial\vec{r}_i}{\partial q_k}\,\frac{dq_k}{dt} + \frac{\partial\vec{r}_i}{\partial t}$$

The last term does not exist if the constraints are scleronomic, but that is not really important here. Now, take the partial derivative with respect to $\dot{q}_l$; this selects out the term in the sum with $k = l$, and drops the t term:

$$\frac{\partial\dot{\vec{r}}_i}{\partial\dot{q}_l} \;=\; \sum_k \frac{\partial\vec{r}_i}{\partial q_k}\,\delta_{kl} = \frac{\partial\vec{r}_i}{\partial q_l}$$

and so the dot cancellation relation is proven.

Dot cancellation does not necessarily hold if the constraints are nonholonomic. Suppose $\vec{r}_i = \vec{r}_i(\{q_k\}, \{\dot{q}_k\}, t)$. Then our partial derivative would be

$$\dot{\vec{r}}_i = \frac{d}{dt}\,\vec{r}_i(\{q_k\}, \{\dot{q}_k\}, t) \quad = \quad \sum_k \left[\frac{\partial \vec{r}_i}{\partial q_k}\,\dot{q}_k + \frac{\partial \vec{r}_i}{\partial \dot{q}_k}\,\ddot{q}_k\right] + \frac{\partial \vec{r}_i}{\partial t}$$

Newton's second law relates $\ddot{q}_k$ to q_k and $\dot{q}_k$. By definition, if there is a velocity-dependent nonholonomic constraint, then there is a velocity-dependent force. Thus, $\ddot{q}_k$ will certainly depend directly on $\dot{q}_k$ and the second term in the sum will yield additional terms when one does the next step, taking the partial derivative with respect to $\dot{q}_k$. The first and last terms might also yield additional unwanted terms: if $\vec{r}_i$ depends on $\dot{q}_k$, then there may still be $\dot{q}_k$-dependent terms left in $\frac{\partial \vec{r}_i}{\partial t}$, which will survive when the partial derivative with respect to $\dot{q}_k$ is taken. Either way, the dot cancellation result would be invalidated.

Example 2.1:

For the elliptical wire, we have given the relations between the positions x, z and the generalized coordinate α

$$x = a(t)\,\cos\alpha \qquad z = b(t)\,\sin\alpha$$

We find the relations between the velocities by differentiating with respect to time:

$$\dot{x} = \dot{a}\,\cos\alpha - a\,\sin\alpha\,\dot{\alpha} \qquad \dot{z} = \dot{b}\,\sin\alpha + b\,\cos\alpha\,\dot{\alpha}$$

Note that we do have terms of the form $\partial\vec{r}_i/\partial t$, the terms with $\dot{a}$ and $\dot{b}$. Taking the partial derivatives, we find

$$\begin{aligned} \frac{\partial x}{\partial \alpha} &= -a\,\sin\alpha \qquad & \frac{\partial z}{\partial \alpha} &= b\,\cos\alpha \\ \frac{\partial \dot{x}}{\partial \dot{\alpha}} &= -a\,\sin\alpha \qquad & \frac{\partial \dot{z}}{\partial \dot{\alpha}} &= b\,\cos\alpha \end{aligned}$$

where the $\dot{a}$ and $\dot{b}$ terms have disappeared because $\dot{\alpha}$ does not appear in them. We see the dot cancellation works.

Example 2.2:

The Atwood's machine example is as follows; it is somewhat nontrivial if l is a function of time. Differentiating the relations between z_1, z_2, and Z gives

$$\dot{z}_1 = \dot{Z} \qquad \dot{z}_2 = -\dot{l} - \dot{Z}$$

So then

$$\begin{aligned} \frac{\partial z_1}{\partial Z} &= 1 \qquad & \frac{\partial z_2}{\partial Z} &= -1 \\ \frac{\partial \dot{z}_1}{\partial \dot{Z}} &= 1 \qquad & \frac{\partial \dot{z}_2}{\partial \dot{Z}} &= -1 \end{aligned}$$

Note that dot cancellation works even if l is time dependent.

2.1.2 Virtual Displacement, Virtual Work, and Generalized Forces

Virtual Displacement

We define a **virtual displacement** $\{\delta\vec{r}_i\}$ as an infinitesimal displacement of the system coordinates $\{\vec{r}_i\}$ that satisfies the following criteria (stated somewhat differently than in Hand and Finch, but no different in meaning):

1. The displacement satisfies the constraint equations, but may make use of any remaining unconstrained degrees of freedom.
2. The time is held fixed during the displacement.
3. The generalized velocities $\{\dot{q}_k\}$ are held fixed during the displacement.

A virtual displacement can be represented in terms of position coordinates or generalized coordinates. The advantage of generalized coordinates, of course, is that they automatically respect the constraints. An arbitrary set of displacements $\{\delta q_k\}$ can qualify as a virtual displacement if conditions (2) and (3) are additionally applied, but an arbitrary set of displacements $\{\delta\vec{r}_i\}$ may or may not qualify as a virtual displacement depending on whether the displacements obey the constraints. All three conditions will become clearer in the examples. Explicitly, the relation between infinitesimal displacements of generalized coordinates and virtual displacements of the position coordinates is

$$\delta\vec{r}_i = \sum_k \frac{\partial\vec{r}_i}{\partial q_k}\,\delta q_k \tag{2.4}$$

This expression has content: there are fewer $\{q_k\}$ than $\{\vec{r}_i\}$, so the fact that $\delta\vec{r}_i$ can be expressed only in terms of the $\{q_k\}$ reflects the fact that the virtual displacement respects the constraints. One can put in any values of the $\{\delta q_k\}$ and obtain a virtual displacement, but not every possible set of $\{\delta\vec{r}_i\}$ can be written in the above way.

Example 2.1:

For the elliptical wire, it is easy to see what kinds of displacements satisfy the first two requirements. Changes in x and z are related by the constraint equation; we obtain the relation by applying the displacements to the constraint equation. We do this by writing the constraint equation with and without the displacements and differencing the two:

$$\text{with displacements:} \qquad \left(\frac{x+\delta x}{a(t)}\right)^2 + \left(\frac{z+\delta z}{b(t)}\right)^2 = 1$$

$$\text{without displacements:} \qquad \left(\frac{x}{a(t)}\right)^2 + \left(\frac{z}{b(t)}\right)^2 = 1$$

$$\text{difference:} \qquad \left(\frac{x+\delta x}{a(t)}\right)^2 - \left(\frac{x}{a(t)}\right)^2 + \left(\frac{z+\delta z}{b(t)}\right)^2 - \left(\frac{z}{b(t)}\right)^2 = 0$$

$$\frac{x\,\delta x}{[a(t)]^2} + \frac{z\,\delta z}{[b(t)]^2} = 0$$

$$\Rightarrow \quad \frac{\delta x}{\delta z} = -\frac{a(t)}{b(t)}\frac{z}{x}$$

All terms of second order in δx or δz are dropped because the displacements are infinitesimal.[1] The result is that δx and δz cannot be arbitrary with respect to each other and are related by where the particle is in x and z and the current values of a and b; this clearly satisfies the first requirement. We have satisfied the second requirement, keeping time fixed, by treating a and b as constant: there has been no δt applied, which would have added derivatives of a and b to the expressions. If a and b were truly constant, then the second requirement would be irrelevant. The third requirement is not really relevant here because the generalized velocities do not enter the constraints in this holonomic case; but they will enter, for example, the kinetic energy, so it must be kept in mind.

The relation between the virtual displacements in the positions and in the generalized coordinate is easy to calculate:

$$\begin{aligned} x = a\,\cos\alpha \quad &\Rightarrow \quad \delta x = -a\,\sin\alpha\;\delta\alpha \\ z = b\,\sin\alpha \quad &\Rightarrow \quad \delta z = b\,\cos\alpha\;\delta\alpha \end{aligned}$$

$$\Longrightarrow \quad \frac{\delta x}{\delta z} = -\frac{a}{b}\tan\alpha$$

We see that there is a one-to-one correspondence between all infinitesimal displacements $\delta\alpha$ of the generalized coordinate and *virtual* displacements of the positional coordinates $(\delta x, \delta z)$, as stated above. The displacements of the positional coordinates that cannot be generated from $\delta\alpha$ by the above expressions are those that do not satisfy the constraints and are disallowed.

Example 2.2:

For our Atwood's machine example, the constraint equation

$$z_1 + z_2 + l(t) = 0$$

is easily converted to differential form, giving

$$\delta z_1 + \delta z_2 = 0$$

Again, remember that we do not let time vary, so $l(t)$ contributes nothing to the differential. This equation is what we would have arrived at if we had started with an infinitesimal displacement δZ of the generalized coordinate Z (holding time fixed according to condition (2)):

$$\delta z_1 = \delta Z \qquad \delta z_2 = -\delta Z \quad \Rightarrow \quad \delta z_1 + \delta z_2 = 0$$

Fixing time prevents appearance of derivatives of $l(t)$. Again, we also see the one-to-one correspondence between infinitesimal displacements of the generalized coordinate and virtual displacements of the position coordinates.

[1] This is one of the first applications of Taylor expansions in this course. This kind of thing will be done regularly, so get used to the technique!

Virtual Work

Using the virtual displacement, we may define **virtual work** as the work that would be done on the system by the forces acting on the system as the system undergoes the virtual displacement $\{\delta\vec{r}_i\}$:

$$\delta W \equiv \sum_{ij} \vec{F}_{ij} \cdot \delta\vec{r}_i$$

where $\vec{F}_{ij}$ is the jth force acting on the coordinate of the ith particle $\vec{r}_i$.[2]

Example 2.1:

In our elliptical wire example, $\vec{F}_{11}$ would be the gravitational force acting on the point mass and $\vec{F}_{12}$ would be the force exerted by the wire to keep the point mass on the wire ($i = 1$ only because there is only one object involved). The virtual work is

$$\delta W = \sum_{i=1}^{1} \left[\vec{F}_{i1} + \vec{F}_{i2}\right] \cdot \delta\vec{r}_i$$

Example 2.2:

In our Atwood's machine example, the two masses feel gravitational forces $\vec{F}_{11} = -m_1\, g\, \hat{z}$ and $\vec{F}_{21} = -m_2\, g\, \hat{z}$. The tension in the rope is the force that enforces the constraint that the length of rope between the two blocks is fixed, $\vec{F}_{12} = T\, \hat{z}$ and $\vec{F}_{22} = T\, \hat{z}$. T may be a function of time if l varies with time, but it is certainly the same at the two ends of the rope at any instant.

At this point, we specialize to constraints that do no net work when a virtual displacement is applied. This assumption is critical. Making this assumption implies that only the *non-constraint* forces need be included in the sum over j because the terms due to constraints yield no contribution.

The assumption deserves some detailed discussion. It is not clear whether it is possible to state general rules about which kinds of constraints satisfy the assumption. In fact, Schaum's Outline on Lagrangian Dynamics (D. A. Wells) says "While the truth of this statement is easily demonstrated with simple examples, a general proof is usually not attempted. It may be regarded as a postulate." Goldstein simply states that "We now restrict ourselves to systems for which *the net virtual work of the forces of constraint is zero*" and makes no statement about the general applicability of the assumption. Note that Hand and Finch completely gloss over this subtlety; they simply state "Recall that since constraint forces always act to maintain the constraint, they point in a direction perpendicular to the movement of the parts of the system. This means that the constraint forces do not contribute anything to the virtual work." The first sentence is patently false, as our Atwood's machine example shows!

Let us try to at least get an intuitive idea of how different kinds of constraints satisfy the assumption. There are clearly three kinds:

[2] We deviate from Hand and Finch's notation here by adding the j index; for our examples, it is useful to use the j index to distinguish the different forces acting on a given particle. Hand and Finch's $\vec{F}_i$ is simply $\sum_j \vec{F}_{ij}$; it gives the total force acting on particle i.

1. "normal forces": If $\vec{F}_{ij} \cdot \delta\vec{r}_i$ vanishes for a single particle i and a single constraint j, then, the constraint force must act on only one particle and must act normal to the motion. Our elliptical wire constraint is of this form. The constraint defining rigid-body motion, $|\vec{r}_a - \vec{r}_b| = c_{ab}$ for all particles a, b in the body, is similar in form: an allowed virtual displacement keeps the length of the vector separation of the two particles fixed but allows its orientation to change, while the force that maintains the constraint is the central force between the two, which acts along the separation vector and thus perpendicular to the virtual displacement. It is not quite the same as the single-particle version, but it still can be considered a normal force because the constraint force and virtual displacement are perpendicular.
2. "single-constraint satisfaction": Not all constraints are "normal forces"; see our Atwood's machine example below, where the constraint force acts along the virtual displacement so that $\vec{F}_{ij} \cdot \delta\vec{r}_i \neq 0$ but $\sum_i \vec{F}_{ij} \cdot \delta\vec{r}_i$ does vanish due to summation over i. In this case, once one sums over the particles that are affected by a particular constraint, then the sum vanishes. For this type of constraint, each constraint j satisfies the assumption $\sum_i \vec{F}_{ij} \cdot \delta\vec{r}_i = 0$ independently. Of course, normal forces are a special subset of this class, but it is instructive to consider them separately.
3. "interlocking constraint satisfaction": I admittedly cannot think of an example, but one can imagine in a general sense that some set of interlocking constraints, where multiple coordinates appear in multiple constraints, might require the summation over both i and j for the assumption to hold.

Because of the possibility that there exist situations of the third type, we use the most generic assumption we need to proceed with our derivation, which is the third one. We write that down as

$$\sum_{ij} \vec{F}_{ij}^{(c)} \cdot \delta\vec{r}_i = 0$$

where the $^{(c)}$ superscript restricts the sum to constraint forces but the sum is over *all* constraint forces and *all* particles. Mathematically, the assumption lets us drop the part of the virtual work sum containing constraint forces, leaving

$$\delta W = \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \delta\vec{r}_i$$

where the $^{(nc)}$ superscript indicates that the sum is only over non-constraint forces.

Example 2.1:

In our elliptical wire example, the force exerted by the wire, $\vec{F}_{12}$, acts to keep the point mass i on the wire; the force is therefore always normal to the wire. The virtual displacement $\delta\vec{r}_1$ must always be tangential to the wire to satisfy the constraint. So, $\sum_{i=1}^{1} \vec{F}_{i2} \cdot \delta\vec{r}_i = 0$. The only non-constraint force is gravity, $\vec{F}_{11}$, so we are left with

$$\delta W = \sum_{j=1}^{1} \sum_{i=1}^{1} \vec{F}_{ij}^{(nc)} \cdot \delta\vec{r}_i = \sum_{i=1}^{1} \vec{F}_{i1} \cdot \delta\vec{r}_i = \vec{F}_{11} \cdot \delta\vec{r}_1 = -m\,g\,\delta z$$

δW will in general not vanish; it gives rise to the dynamics of the problem.

Example 2.2:

In the Atwood's machine example, the constraint forces $\vec{F}_{21}$ and $\vec{F}_{22}$ act along the rope. The virtual displacements are also along the rope. Clearly, $\vec{F}_{21} \cdot \delta\vec{r}_1 = F_{21}\,\delta z_1$ and $\vec{F}_{22} \cdot \delta\vec{r}_2 = F_{22}\,\delta z_2$ do not vanish. But the sum does:

$$\sum_{i=1}^{2} \vec{F}_{i2} \cdot \delta\vec{r}_i = F_{21}\delta z_1 + F_{22}\delta z_2 = T(\delta z_1 + \delta z_2) = 0$$

Notice that, in this case, all the terms pertaining to the particular constraint force have to be summed in order for the result to hold. The virtual work and non-constraint force sum is then

$$\delta W = \sum_{i=1}^{2}\sum_{j=1}^{1} \vec{F}_{ij}^{(nc)} \cdot \delta\vec{r}_i = \sum_{i=1}^{2} \vec{F}_{i1} \cdot \delta\vec{r}_i = -g\,(m_1\,\delta z_1 + m_2\,\delta z_2)$$

Note that, unless $m_1 = m_2$, δW will in general not vanish, again giving rise to the dynamics of the problem.

Generalized Force

Our discussion of generalized coordinates essentially was an effort to make use of the constraints to eliminate the degrees of freedom in our system that have no dynamics. Similarly, the constraint forces, once they have been taken account of by transforming to the generalized coordinates, would seem to be irrelevant. We will show how they can be eliminated in favor of *generalized forces* that contain only the non-constraint forces.

To define generalized forces, we combine Equation 2.4, the relation between virtual displacements of position coordinates and generalized coordinates, with Equation 2.5, the relation between virtual work and non-constraint forces:

$$\begin{aligned}
\delta W &= \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \delta\vec{r}_i \\
&= \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \left[\sum_k \frac{\partial\vec{r}_i}{\partial q_k}\,\delta q_k\right] \\
&= \sum_k \left[\sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \frac{\partial\vec{r}_i}{\partial q_k}\right] \delta q_k \\
&\equiv \sum_k \mathcal{F}_k\,\delta q_k
\end{aligned}$$

where

$$\mathcal{F}_k \equiv \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \frac{\partial\vec{r}_i}{\partial q_k} = \frac{\delta W}{\delta q_k} \tag{2.5}$$

is the **generalized force** along the kth generalized coordinate. The last form, $\mathcal{F}_k = \delta W/\delta q_k$, says that the force is simple the ratio of the work done to the displacement when a virtual displacement of only the kth generalized coordinate is performed; it is of course possible to displace *only* the kth generalized coordinate because the generalized coordinates

are mutually independent. It is important to remember that the generalized force is found by summing only over the non-constraint forces: the constraint forces have already been taken into account in defining generalized coordinates.

The above definition is very sensible. In Section 1.1.3, we defined work to be the line integral of force. Infinitesimally, a force $\vec{F}$ causing a displacement $\delta\vec{r}$ does work $\delta W = \vec{F}\cdot\delta\vec{r}$. The generalized force is the exact analogue: if work δW is done when the ensemble of forces act to produce a generalized coordinate displacement δq_k, then the generalized force $\mathcal{F}_k$ doing that work is $\mathcal{F}_k = \delta W/\delta q_k$. But the generalized force is a simplification because it is only composed of the non-constraint forces.

Example 2.1:

In the elliptical wire example, the generalized force for the α coordinate ($k = 1$) is

$$\begin{aligned}\mathcal{F}_\alpha &= \vec{F}_{11}\cdot\frac{\partial\vec{r}_1}{\partial\alpha} = -m\,g\,\hat{z}\cdot\left(\hat{x}\,\frac{\partial x}{\partial\alpha}+\hat{z}\,\frac{\partial z}{\partial\alpha}\right) = -m\,g\,\hat{z}\cdot(-\hat{x}\,a\sin\alpha+\hat{z}\,b\cos\alpha)\\ &= -m\,g\,b\,\cos\alpha\end{aligned}$$

The constraint force, which acts in both the x and z directions and is α-dependent, does not appear in the generalized force.

Example 2.2:

In the Atwood's machine example, the generalized force for the Z coordinate ($k = 1$ again) is

$$\begin{aligned}\mathcal{F}_Z &= \vec{F}_{11}\cdot\frac{\partial\vec{r}_1}{\partial Z}+\vec{F}_{21}\cdot\frac{\partial\vec{r}_2}{\partial Z} = -m_1\,g\,\frac{\partial z_1}{\partial Z}-m_2\,g\,\frac{\partial z_2}{\partial Z}\\ &= (m_2-m_1)\,g\end{aligned}$$

Again, the constraint force (the rope tension) is eliminated. Because Z is just z_1 in this case, the generalized force in Z is just the net force on m_1 acting in the z_1 direction.

2.1.3 d'Alembert's Principle and the Generalized Equation of Motion

The reader is at this point no doubt still wondering, so what? This section gets to the what.

d'Alembert's Principle

Our definition of virtual work was

$$\delta W \;=\; \sum_{ij}\vec{F}_{ij}\cdot\delta\vec{r}_i$$

where the sum includes *all* (constraint and non-constraint) forces. Assuming our position coordinates are in an inertial frame (but not necessarily our generalized coordinates), Newton's second law tells us $\sum_j\vec{F}_{ij} = \dot{\vec{p}}_i$: the sum of all the forces acting on a particle give the rate of change of its momentum. We may then rewrite δW:

$$\delta W \;=\; \sum_i\sum_j\vec{F}_{ij}\cdot\delta\vec{r}_i = \sum_i\dot{\vec{p}}_i\cdot\delta\vec{r}_i$$

But, we found earlier that we could write the virtual work as a sum over only non-constraint forces,

$$\delta W = \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \delta \vec{r}_i$$

Thus, we may derive the relation

$$\sum_i \left[\sum_j \vec{F}_{ij}^{(nc)} - \dot{\vec{p}}_i \right] \cdot \delta \vec{r}_i = 0 \tag{2.6}$$

The above equation is referred to as **d'Alembert's principle**. Its content is that the rate of change of momentum is determined **only by the non-constraint forces**. In this form, it is not much use, but the conclusion that the rate of change of momentum is determined only by *non-constraint* forces is an important physical statement. Note that the multiplication by the virtual displacement must be included in order for the statement to hold; the statement $\vec{F}_i^{(nc)} - \dot{\vec{p}}_i = 0$ is *not* in general true; just consider our elliptical wire and Atwood's machine examples.

We may use d'Alembert's principle to relate generalized forces to the rate of change of the momenta:

$$\sum_k \mathcal{F}_k \, \delta q_k = \delta W = \sum_k \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{i,k} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \, \delta q_k$$

Now, unlike the $\{\delta \vec{r}_i\}$, the $\{\delta q_k\}$ are mutually independent. Therefore, we may conclude that equality holds for each term of the sum separately (unlike for Equation 2.6), providing a different version of **d'Alembert's principle**:

$$\sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k = \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \tag{2.7}$$

This is now a very important statement: the generalized force for the kth generalized coordinate, which can be calculated from the non-constraint forces only, is related to a particular weighted sum of momentum time derivatives (the weights being the partial derivatives of the position coordinates with respect to the generalized coordinates). **Effectively, we have an analogue of Newton's second law, but including only the non-constraint forces.** This can be a major simplification in cases where the constraint forces are complicated or simply not known.

d'Alembert's principle is not yet useful in the current form because the left side contains generalized forces depending on the generalized coordinates but the right side has derivatives of the momenta associated with the position coordinates. We need time derivatives with respect to the generalized coordinates on the right side so all the dynamics can be calculated in the generalized coordinates.

Generalized Equation of Motion

Here we perform the manipulation needed to make d'Alembert's principle useful. We know from Newtonian mechanics that work is related to kinetic energy, so it is natural to expect

the virtual work due to a differential displacement $\{\delta\vec{r}_i\}$ to be related to some sort of small change in kinetic energy. We first begin with a formal definition of kinetic energy:

$$T \equiv \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i = T(\{q_k\}, \{\dot{q}_k\}, t) \tag{2.8}$$

T should be obtained by first writing T in terms of position velocities $\{\dot{\vec{r}}_i\}$ and then using the definition of the position coordinates in terms of generalized coordinates to rewrite T as a function of the generalized coordinates and velocities. T may depend on all the generalized coordinates and velocities and on time because the $\{\vec{r}_i\}$ depend on the generalized coordinates and time and a time derivative is being taken, which may introduce dependence on the generalized velocities (via the chain rule, as seen earlier). The partial derivatives of T are

$$\begin{aligned}
\frac{\partial T}{\partial q_k} &= \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} \\
\frac{\partial T}{\partial \dot{q}_k} &= \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \sum_i \vec{p}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}
\end{aligned}$$

where in the last step we have made use of dot cancellation because the constraints are assumed to be holonomic.

Now, we have $\vec{p}_i$ floating around but we need $\dot{\vec{p}}_i$. The natural thing is then to take a time derivative. We do this to $\partial T/\partial \dot{q}_k$ (instead of $\partial T/\partial q_k$) because we want to avoid second-order time derivatives if we are to obtain something algebraically similar to the right side of d'Alembert's principle. We find

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i \vec{p}_i \cdot \frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_k}$$

Referring back to the second form of d'Alembert's principle (Equation 2.7), we see that the first term in the expression is the generalized force $\mathcal{F}_k$ for the kth coordinate. Continuing onward, we need to evaluate the second term. We have

$$\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_k} = \sum_l \left[\frac{\partial^2 \vec{r}_i}{\partial q_l \partial q_k}\dot{q}_l + \frac{\partial^2 \vec{r}_i}{\partial \dot{q}_l \partial q_k}\ddot{q}_l\right] + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_k}$$

When we exchange the order of the derivatives in the second term, we see that the second term vanishes because our holonomic constraint assumption – that the generalized velocities do not enter the constraints, and thus do not enter the relation between position and generalized coordinates – implies $\partial \vec{r}_i/\partial \dot{q}_k = 0$. In the last term, we can trivially exchange the order of the partial derivatives. We can bring $\partial/\partial q_k$ outside the sum in the first term because $\partial \dot{q}_l/\partial q_k = 0$. Thus, we have

$$\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_k} = \frac{\partial}{\partial q_k}\left\{\sum_l \left[\frac{\partial \vec{r}_i}{\partial q_l}\dot{q}_l\right] + \frac{\partial \vec{r}_i}{\partial t}\right\} = \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$$

where the last step simply used the chain rule for evaluation of $\dot{\vec{r}}_i = d\vec{r}_i/dt$. Essentially, we have demonstrated that the total time derivative d/dt and the partial derivative $\partial/\partial q_k$ commute when acting on $\vec{r}_i$ for holonomic constraints, which is a nontrivial statement because

q_k is time-dependent. We emphasize that it was the assumption of holonomic constraints that let us discard the second term above. Had that term remained, the dependence on $\ddot{q}_l$ would have made it impossible to bring $\partial/\partial q_k$ outside the sum because $\ddot{q}_l$ in general may depend on q_l (via Newton's second law). So we now have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) &= \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} \\ &= \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \frac{\partial T}{\partial q_k}\end{aligned}$$

or

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} \tag{2.9}$$

Recalling d'Alembert's principle (Equation 2.7), we may rewrite the above:

$$\sum_i \vec{F}_i^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} \tag{2.10}$$

This is the **generalized equation of motion**. The left side is completely determined by the non-constraint forces and the constraint equations. The right side is just derivatives of the kinetic energy with respect to the generalized coordinates and velocities. Thus, we obtain a differential equation for the motion in the generalized coordinates.

We note that no information has been lost; rather than explicitly solving for the effect of the constraint forces on the motion, we incorporate the constraint forces in the definition of the generalized coordinates, then we solve for the motion in the generalized coordinates. Once we have done the math, we can invert the constraints to give the motion in the original coordinates.

Example 2.1:

For the elliptical wire example, the kinetic energy in terms of position coordinate velocities is

$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{z}^2\right)$$

We have previously obtained formulae for $\dot{x}$ and $\dot{z}$ in terms of $\dot{\alpha}$:

$$\dot{x} = \dot{a}\,\cos\alpha - a\,\sin\alpha\,\dot{\alpha} \qquad \dot{z} = \dot{b}\,\sin\alpha + b\,\cos\alpha\,\dot{\alpha}$$

Let us specialize to the case $\dot{a} = 0$ and $\dot{b} = 0$ to avoid creating an unilluminating algebraic nightmare; so

$$\dot{x} = -a\,\sin\alpha\,\dot{\alpha} \qquad \dot{z} = b\,\cos\alpha\,\dot{\alpha}$$

We use these to rewrite the kinetic energy in terms of $\dot{\alpha}$:

$$T = \frac{m}{2}\left[a^2\dot{\alpha}^2\sin^2\alpha + b^2\dot{\alpha}^2\cos^2\alpha\right]$$

This is an important example of how to convert T from a function of the position velocities to a function of the generalized coordinates and velocities. Now take the prescribed derivatives:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} = \frac{d}{dt}\left[m\,\dot{\alpha}\left(a^2\,\sin^2\alpha + b^2\,\cos^2\alpha\right)\right] - 2\,m\,\dot{\alpha}^2\left(a^2 - b^2\right)\sin\alpha\,\cos\alpha$$
$$= m\,\ddot{\alpha}\left(a^2\,\sin^2\alpha + b^2\,\cos^2\alpha\right)$$

In taking the total derivative in the first term, we obtain two terms: the one displayed in the last line above, and one that exactly cancels the last term in the first line.[3] We have the generalized force $\mathcal{F}_\alpha$ from before, $\mathcal{F}_\alpha = -m\,g\,b\,\cos\alpha$, so the generalized equation of motion is

$$\mathcal{F}_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha}$$
$$-m\,g\,b\,\cos\alpha = m\,\ddot{\alpha}\left(a^2\,\sin^2\alpha + b^2\,\cos^2\alpha\right)$$
$$\ddot{\alpha} = -\,g\,\frac{b\cos\alpha}{a^2\sin^2\alpha + b^2\cos^2\alpha}$$

Specializing to $a = b = r$ (circular wire), this simplifies to

$$\ddot{\alpha} = -\,g\,\frac{\cos\alpha}{r}$$

One can schematically see what equations of motion one would have obtained if generalized coordinates had not been used. There would be two equations, one for x and one for z. Both would have a component of the constraint force and the z equation would have a gravity term. The constraint equation on x and z would be used to eliminate x, giving two equations for $\ddot{z}$ and the unknown constraint force. The constraint force could be eliminated using one of the equations, resulting in one, rather complicated, equation of motion in z. Clearly, the above equation of motion in α is much simpler!

Example 2.2:

For the Atwood's machine example, things are significantly simpler. The kinetic energy is

$$T = \frac{1}{2}\left(m_1\,\dot{z}_1^2 + m_2\,\dot{z}_2^2\right)$$

Rewriting using the generalized coordinate Z gives

$$T = \frac{1}{2}\left(m_1 + m_2\right)\dot{Z}^2$$

[3] This is our first encounter with an equation that has both total time derivatives and partial derivatives with respected to q_k and $\dot{q}_k$. One must realize that the total time derivative acts on all variables that have any time dependence. This is why two terms came out of the total time derivative term; one term arising from the total time derivative acting on $\dot{\alpha}$, which gives $\ddot{\alpha}$, and another arising from the total time derivative acting on $\sin^2\alpha$ and $\cos^2\alpha$, which gave the additional term that canceled the $\partial T/\partial\alpha$ term.

The kinetic energy derivatives term is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Z}}\right) - \frac{\partial T}{\partial Z} = (m_1 + m_2)\,\ddot{Z}$$

Using $\mathcal{F}_Z = (m_2 - m_1)\,g$ from earlier, the generalized equation of motion is

$$\begin{aligned} \mathcal{F}_Z &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Z}}\right) - \frac{\partial T}{\partial Z} \\ (m_2 - m_1)\,g &= (m_1 + m_2)\,\ddot{Z} \\ \ddot{Z} &= -\frac{m_1 - m_2}{m_1 + m_2}\,g \end{aligned}$$

which is the same equation of motion obtained for z_1 in Example 1.4 in Section 1.1. But, using the formalism of constraints and generalized coordinates, we have no mention of the rope tension in the equations of motion.

2.1.4 The Lagrangian and the Euler-Lagrange Equations

For conservative non-constraint forces, we can obtain a slightly more compact form of the generalized equation of motion, known as the **Euler-Lagrange equations**.

Generalized Conservative Forces

Now let us specialize to non-constraint forces that are conservative; *i.e.*,

$$\vec{F}_i^{(nc)} \;=\; -\vec{\nabla}_i U(\{\vec{r}_j\})$$

where $\vec{\nabla}_i$ indicates the gradient with respect to $\vec{r}_i$. Whether the constraint forces are conservative is irrelevant; we will only explicitly need the potential for the non-constraint forces. U is assumed to be a function only of the coordinate positions; there is no explicit dependence on time or on velocities, $\partial U/\partial t = 0$ and $\partial U/\partial \dot{\vec{r}}_i = 0$.[4] Let us use this expression in writing out the generalized force:

$$\mathcal{F}_k = \sum_i \vec{F}_i^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = -\sum_i \vec{\nabla}_i U(\{\vec{r}_j\}) \cdot \frac{\partial \vec{r}_i}{\partial q_k} = -\frac{\partial}{\partial q_k} U(\{q_l\}, t)$$

In the last step we make use of the holonomic constraints to rewrite U as a function of the $\{q_l\}$ and possibly t and realize that the previous line is just the partial derivative of U with respect to q_k.[5]

Thus, rather than determining the equation of motion by calculating the generalized force from the non-constraint forces and the coordinate transformation relations, we can rewrite the potential energy as a function of the generalized coordinates and calculate the generalized force by gradients thereof.

Example 2.1:

[4] It is possible to consider time-dependent potential energy functions in the following, but we hold off on that discussion until Section 2.1.9.

[5] Some care must be taken with the time dependence. U is not initially a function of t. But the constraints may be rheonomic, so some dependence of U on t may appear when the coordinate transformation is done, and $\partial U/\partial t$ may be nonzero. This should not be taken to imply that somehow the potential has become nonconservative – the time-dependence arises purely through the rheonomic constraint. If there is any such confusing circumstance, one should always transform U back to position coordinates to check for time-dependence. And note that U may remain time-independent in some rheonomic constraint cases; see Example 2.5.

For the elliptical wire example, the potential energy function is due to gravity,

$$U(z) = m\,g\,z$$

Rewriting in terms of α gives

$$U(\alpha;t) = m\,g\,b(t)\,\sin\,\alpha$$

The generalized force is then

$$\mathcal{F}_\alpha = -\frac{\partial U(\alpha;t)}{\partial \alpha} = -m\,g\,b(t)\,\cos\alpha$$

as obtained before. Note that we may allow b to be a function of time without ruining the conservative nature of the potential energy – U becomes a function of t through the definition of the generalized coordinate but, obviously, if it was initially a conservative potential, a transformation of coordinates cannot change that.

Example 2.2:

For the Atwood's machine, the potential energy function is

$$U(z_1, z_2) = g\,(m_1\,z_1 + m_2\,z_2)$$

Rewriting in terms of Z gives

$$U(Z;t) = g\,[(m_1 - m_2)\,Z - m_2\,l(t)]$$

The generalized force is

$$\mathcal{F}_Z = -\frac{\partial U(Z;t)}{\partial Z} = g\,(m_2 - m_1)$$

as found earlier. Again, l is allowed to be a function of time without ruining the conservative nature of the potential energy.

The Euler-Lagrange Equations

An even simpler method exists. We may rewrite the generalized equation of motion using the above relation between generalized force and gradient of the potential energy as

$$-\frac{\partial U}{\partial q_k} \;=\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}$$

Define the **Lagrangian**

$$L \;\equiv\; T - U \tag{2.11}$$

Since we have assumed holonomic constraints, we have that $\partial U/\partial \dot{q}_k = 0$. This lets us replace $d/dt\,(\partial T/\partial \dot{q}_k)$ with $d/dt\,(\partial L/\partial \dot{q}_k)$, giving

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} \;=\; 0 \tag{2.12}$$

This is the **Euler-Lagrange** equation. There is one for each generalized coordinate q_k. While we still end up taking partial derivatives of U, we no longer need to intepret these as generalized forces; we work directly from the Lagrangian to generate an equation of motion.[6]

Example 2.1:

For the elliptical wire, again taking a and b to be constant, the Lagrangian is

$$L = T - U = \frac{m}{2}\left[a^2\dot{\alpha}^2\sin^2\alpha + b^2\dot{\alpha}^2\cos^2\alpha\right] - m\,g\,b\,\sin\alpha$$

We have already calculated all the relevant partial derivatives to obtain the Euler-Lagrange equations, we simply restate them here:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\alpha}}\right) &= \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\alpha}}\right)\\ &= \frac{d}{dt}\left[m\,\dot{\alpha}\left(a^2\,\sin^2\alpha + b^2\,\cos^2\alpha\right)\right]\\ &= m\,\ddot{\alpha}\left(a^2\,\sin^2\alpha + b^2\,\cos^2\alpha\right) + 2\,m\,\dot{\alpha}^2\left(a^2 - b^2\right)\sin\alpha\,\cos\alpha\\ \frac{\partial L}{\partial\alpha} &= \frac{\partial T}{\partial\alpha} - \frac{\partial U}{\partial\alpha}\\ &= 2\,m\,\dot{\alpha}^2\left(a^2 - b^2\right)\sin\alpha\,\cos\alpha - m\,g\,b\,\cos\alpha\end{aligned}$$

The Euler-Lagrange equation then is

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\alpha}}\right) - \frac{\partial L}{\partial\alpha} &= 0\\ m\,\ddot{\alpha}\left(a^2\,\sin^2\alpha + b^2\,\cos^2\alpha\right) + m\,g\,b\,\cos\alpha &= 0\end{aligned}$$

which is equivalent to the generalized equation of motion found before using the generalized force and derivatives of the kinetic energy.

Example 2.2:

For the Atwood's machine, again taking l to be constant, the Lagrangian is

$$L = T - U = \frac{1}{2}\left(m_1 + m_2\right)\dot{Z}^2 + g\left[\left(m_2 - m_1\right)\,Z + m_2\,l\right]$$

Note that the $m_2\,l\,g$ term is constant and could be dropped without affecting the ensuing Euler-Lagrange equations. It is a constant offset to the potential energy, which we know cannot affect the dynamics. Again, all the necessary derivatives have already been calculated:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{Z}}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{Z}}\right) &= \left(m_1 + m_2\right)\ddot{Z}\\ \frac{\partial L}{\partial Z} = -\frac{\partial U}{\partial Z} &= g\left(m_2 - m_1\right)\end{aligned}$$

The Euler-Lagrange equation then is

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{Z}}\right) - \frac{\partial L}{\partial Z} &= 0\\ \left(m_1 + m_2\right)\ddot{Z} - g\left(m_2 - m_1\right) &= 0\end{aligned}$$

which is equivalent again to the generalized equation of motion found earlier.

[6] Note that explicit time-dependence in U would not have ruined the derivation – we only needed $\partial U/\partial\dot{q}_k = 0$ to move U inside the $d/dt\,(\partial/\partial\dot{q}_k)$ operator. We will return to this in Section 2.1.9.

2.1.5 The Hamiltonian

We seek a conserved quantity, one whose total time derivative vanishes. We can construct one from the Lagrangian; it is called the **Hamiltonian** and has the form

$$H \equiv \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \tag{2.13}$$

The total time derivative of the Hamiltonian is

$$\frac{d}{dt} H = \sum_k \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \sum_k \dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{d}{dt} L$$

We can use the Euler-Lagrange equation to rewrite the middle term, which gives

$$\frac{d}{dt} H = \sum_k \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \sum_k \dot{q}_k \frac{\partial L}{\partial q_k} - \frac{d}{dt} L$$

The first two terms are most of the total derivative of L; one is left only with

$$\frac{d}{dt} H = -\frac{\partial L}{\partial t}$$

Thus, if time does not explicitly appear in the Lagrangian, the Hamiltonian is completely conserved. If the constraints are scleronomic (the potential is implicitly assumed to be conservative because we are able to calculate a Lagrangian), then time will not appear explicitly and H will definitely be conserved. It can be shown that H is the total energy of the system, $H = T + U$. For rheonomic constraints, H may still be conserved, but may not be the energy. We will investigate these conservation laws in more detail later.

Example 2.1:

For the elliptical wire, H is (using $\partial L/\partial \dot{q}_k$ obtained earlier):

$$\begin{aligned} H &= \dot{\alpha} \frac{\partial L}{\partial \dot{\alpha}} - L = m\,\dot{\alpha}^2 \left(a^2 \sin^2\alpha + b^2 \cos^2\alpha\right) - T + U \\ &= 2\,T - T + U = T + U \\ &= \frac{m}{2}\,\dot{\alpha}^2 \left(a^2 \sin^2\alpha + b^2 \cos^2\alpha\right) + m\,g\,b\,\sin\alpha \end{aligned}$$

Rather than doing the algebra in all its explicit gore, we have made use of the fact that the $\dot{\alpha}\,(\partial L/\partial \dot{\alpha})$ term is in fact just $2\,T$. This simplification occurs in cases where the constraints are scleronomic because T becomes a simple quadratic form in the generalized velocities. This can be shown explicitly (Hand and Finch Problem 1.9). In this case, we also note that, since $\partial L/\partial t = 0$ for a and b fixed, it holds that $dH/dt = 0$: H is conserved. Note that proving this by explicitly taking the derivative of H would be rather painful and require use of the Euler-Lagrange equations (to replace the inevitable $\ddot{\alpha}$ terms that would arise).

Example 2.2:

For the Atwood's machine example, we have a similar situation: the constraints are scleronomic (when l is constant), so

$$H = T + U = \frac{1}{2}\,(m_1 + m_2)\,\dot{Z}^2 - g\,[(m_2 - m_1)\;Z + m_2\,l]$$

and H is conserved. Note again that the $m_2\,l\,g$ term is constant and could be dropped.

2.1.6 Cyclic Coordinates and Canonical Momenta

If the Lagrangian contains $\dot{q}_k$ but not q_k, we can easily see from the Euler-Lagrange equation that the behavior of that coordinate is trivial:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

which implies

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}$$

is constant or conserved and is termed the **canonical momentum conjugate to** q_k or the **canonically conjugate momentum**. The coordinate q_k is termed **ignorable** or **cyclic**. Once the value of p_k is specified by initial conditions, it does not change. In simple cases, the canonical momentum is simply a constant times the corresponding generalized velocity, indicating that the velocity in that coordinate is fixed and the coordinate evolves linearly in time. In more complicated cases, the dynamics are not so trivial, but one does still obtain very useful relations between coordinates and/or velocities that are not generally true. For example, for a particle moving in two dimensions under the influence of a central force like gravity, there is no dependence of L on the azimuthal angle ϕ, so the angular momentum $p_\phi = m\,\rho^2\,\dot{\phi}$ is constant. This tells us the useful fact $\dot{\phi} \propto \rho^{-2}$ – as the particle moves inward toward the origin, its angular velocity must increase in a specific way. In general, then, a cyclic coordinate results in a conserved momentum that simplifies the dynamics in the cyclic coordinate.

The above definition of canonical momentum holds even when the q_k coordinate is not cyclic; we will see its use in the future.

Example 2.1:

For the elliptical wire, the canonical momentum is

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = m\,\dot{\alpha}\left(a^2 \sin^2\alpha + b^2\cos^2\alpha\right)$$

The astute reader will notice that, when $a = b = r$ giving a circular wire, p_α is the angular momentum about the axis of the circle. α is not cyclic, so the Euler-Lagrange equation for it is not trivial and this momentum is not conserved. If gravity were eliminated, and $a = b = r$, then α would become cyclic because $\sin^2\alpha + \cos^2\alpha = 1$. Angular momentum, and hence angular velocity, would remain fixed at its initial value: the bead would simply circumnavigate the wire at constant speed. But if $a \neq b$, then we are left with α dependence in T even if $U = 0$ and so the Lagrangian is not cyclic in α.

Example 2.2:

For the Atwood's machine, the canonical momentum is

$$p_Z = \frac{\partial L}{\partial \dot{Z}} = (m_1 + m_2)\,\dot{Z}$$

In this case, if $U = 0$ and l is constant, Z does indeed become cyclic and p_Z is conserved. If the blocks are initially at rest, they stay at rest, and if they are initially moving with some speed $\dot{Z}$, that speed is preserved.

2.1.7 Summary

This section has followed some long logical paths, so we quickly summarize the logic here.

degrees of freedom + holonomic constraints
↓
generalized coordinates with dot cancellation

generalized coordinates + virtual work + Newton's second law
↓
d'Alembert's principle

d'Alembert's principle + generalized force
↓
generalized equation of motion

generalized equation of motion + conservative force
↓
Lagrangian, Euler-Lagrange equations

Lagrangian + scleronomic constraints
↓
conservation of Hamiltonian

Lagrangian + cyclic coordinates
↓
conserved conjugate momenta

2.1.8 More examples

In addition to the examples we did alongside the derivation, let us do a few more to further illustrate our results.

Example 2.3

Sliding block on sliding inclined plane. See Hand and Finch Sections 1.1 and 1.2 for details; we reproduce some of the more interesting parts here. See Figure 1.1 of Hand and Finch for a sketch.

Let $_b$ and $_p$ subscripts denote the block and the inclined plane. The constraints are that the block cannot move perpendicular to the plane and that the plane cannot move perpendicular to the flat surface it sits on. Therefore, the natural generalized coordinates are X, the horizontal position of the vertical edge of the plane, and d, the distance the block has slid down the plane. (Let h be the height of the plane and α be the angle). Note that d is defined relative to a noninertial reference frame! The constraints can be rewritten as the following transformation equations:

$$\begin{aligned}
\vec{r}_p(t) &= \hat{x}\, X \\
\vec{r}_b(t) &= \vec{r}_p(t) + \hat{y}\, h + d\,[\hat{x}\, \cos\alpha - \hat{y}\, \sin\alpha] \\
&= \hat{x}\,(d\, \cos\alpha + X) + \hat{y}\,(h - d\, \sin\alpha)
\end{aligned}$$

Note that, though d is a noninertial coordinate, the constraints are still *scleronomic* because time does not appear explicitly in the transformation relations; rheonomic and noninertial sometimes go hand-in-hand, but not always. The assorted partial derivatives are

$$\begin{aligned}
\frac{\partial \vec{r}_p}{\partial X} &= \hat{x} \\
\frac{\partial \vec{r}_p}{\partial d} &= 0 \\
\frac{\partial \vec{r}_b}{\partial X} &= \hat{x} \\
\frac{\partial \vec{r}_b}{\partial d} &= \hat{x}\,\cos\alpha - \hat{y}\,\sin\alpha
\end{aligned}$$

The non-constraint forces are

$$\begin{aligned}
\vec{F}_p^{(nc)} &= -M\,g\,\hat{y} \\
\vec{F}_b^{(nc)} &= -m\,g\,\hat{y}
\end{aligned}$$

This last step was very important – we did not have to put in any normal forces, etc., or take any projections. We just blindly put in the non-constraint forces – gravity only in this case – and then we will use the generalized equation of motion to do all the work. The generalized forces are

$$\begin{aligned}
\mathcal{F}_X &= \vec{F}_p^{(nc)} \cdot \frac{\partial \vec{r}_p}{\partial X} + \vec{F}_b^{(nc)} \cdot \frac{\partial \vec{r}_b}{\partial X} \\
&= 0 \\
\mathcal{F}_d &= \vec{F}_p^{(nc)} \cdot \frac{\partial \vec{r}_p}{\partial d} + \vec{F}_b^{(nc)} \cdot \frac{\partial \vec{r}_b}{\partial d} \\
&= m\,g\,\sin\alpha
\end{aligned}$$

Now, we calculate the kinetic energy and the relevant derivatives:

$$\begin{aligned}
T &= \frac{1}{2}\,M\,\dot{X}^2 + \frac{1}{2}\,m\,\left\{\frac{d}{dt}\left[\hat{x}\,(X + d\,\cos\alpha) + \hat{y}\,d\,\sin\alpha\right]^2\right\} \\
&= \frac{1}{2}\,(M+m)\,\dot{X}^2 + \frac{1}{2}\,m\left(\dot{d}^2 + 2\,\dot{d}\,\dot{X}\,\cos\alpha\right) \\
\frac{\partial T}{\partial X} = \frac{\partial T}{\partial d} &= 0 \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}}\right) &= \frac{d}{dt}\left([m+M]\,\dot{X} + m\,\dot{d}\,\cos\alpha\right) \\
&= [m+M]\,\ddot{X} + m\,\ddot{d}\,\cos\alpha \\
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{d}}\right) &= m\,\left[\ddot{d} + \ddot{X}\,\cos\alpha\right]
\end{aligned}$$

The generalized equations of motion are then

$$\begin{aligned}
X &: \; 0 = [m+M]\,\ddot{X} + m\,\ddot{d}\,\cos\alpha \\
d &: \; m\,g\,\sin\alpha = m\left[\ddot{d} + \ddot{X}\,\cos\alpha\right]
\end{aligned}$$

We can solve the X equation for $\ddot{X}$:

$$\ddot{X} = -\frac{m}{M+m}\,\ddot{d}\,\cos\alpha$$

and insert into the d equation:

$$\begin{aligned} m\,g\,\sin\alpha &= m\,\ddot{d}\left[1-\frac{m}{M+m}\,\cos^2\alpha\right] \\ \ddot{d} &= g\left(\frac{M+m}{m}\right)\frac{\sin\alpha}{\frac{M}{m}+\sin^2\alpha} \end{aligned}$$

which gives us a constant acceleration in d and thus lets us find the full motion for d. We can plug in to find

$$\ddot{X} = -g\,\frac{\sin\alpha\,\cos\alpha}{\frac{M}{m}+\sin^2\alpha}$$

Finally, we write out the accelerations in the original coordinates using the transformation relations:

$$\begin{aligned} \ddot{r}_p &= \hat{x}\,\ddot{X} \\ &= -\hat{x}\,g\,\frac{\sin\alpha\,\cos\alpha}{\frac{M}{m}+\sin^2\alpha} \\ \ddot{r}_b &= \hat{x}\left(\ddot{d}\,\cos\alpha+\ddot{X}\right)-\hat{y}\,\ddot{d}\,\sin\alpha \\ &= \hat{x}\,g\,\frac{M}{m}\,\frac{\sin\alpha\,\cos\alpha}{\frac{M}{m}+\sin^2\alpha}-\hat{y}\,g\,\left(\frac{M+m}{m}\right)\frac{\sin^2\alpha}{\frac{M}{m}+\sin^2\alpha} \\ &= \hat{x}\,g\,\frac{\sin\alpha\,\cos\alpha}{1+\frac{m}{M}\,\sin^2\alpha}-\hat{y}\,g\,\left(\frac{M+m}{M}\right)\frac{\sin^2\alpha}{1+\frac{m}{M}\,\sin^2\alpha} \\ &= \frac{g}{1+\frac{m}{M}\,\sin^2\alpha}\left[\hat{x}\,\sin\alpha\,\cos\alpha-\hat{y}\,\left(\frac{M+m}{M}\right)\sin^2\alpha\right] \end{aligned}$$

Clearly, the great advantage in doing this problem using the generalized equation of motion is that we could completely ignore the constraint forces *and* we could use the non-inertial coordinate d as if it were inertial, except insofar as we had to calculate the kinetic energy in the noninertial coordinate system.

We can do the problem even more easily using the Lagrangian formalism. The kinetic energy was given above; the potential energy is

$$U = m\,g\,(h-d\sin\alpha)$$

so

$$\begin{aligned} L &= T-U \\ &= \frac{1}{2}\,(M+m)\,\dot{X}^2+\frac{1}{2}\,m\left(\dot{d}^2+2\,\dot{d}\,\dot{X}\,\cos\alpha\right)-m\,g\,(h-d\sin\alpha) \end{aligned}$$

The partial derivatives are

$$\begin{aligned} \frac{\partial U}{\partial\dot{X}}=\frac{\partial U}{\partial\dot{d}} &= 0 \\ \frac{\partial U}{\partial X} &= 0 \\ \frac{\partial U}{\partial d} &= -m\,g\,\sin\alpha \end{aligned}$$

We may then write down Euler's equations (making use of the above derivatives of U and the derivatives of T calculated earlier):

$$\begin{aligned} X &: [m+M]\,\ddot{X} + m\,\ddot{d}\,\cos\alpha = 0 \\ d &: m\left[\ddot{d} + \ddot{X}\,\cos\alpha\right] - m\,g\,\sin\alpha = 0 \end{aligned}$$

which are the same equations we found earlier. The main advantages in using the Lagrangian technique when the non-constraint forces are conservative are

- it is usually easier to write down a potential energy than forces and one does not have to calculate $\sum_i \vec{F}_i^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$
- one directly calculates the partial derivatives with respect to the q_k

We may calculate the Hamiltonian

$$\begin{aligned} H &\equiv \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \\ &= \dot{X}\left\{[m+M]\,\dot{X} + m\,\dot{d}\,\cos\alpha\right\} + \dot{d}\left\{m\left[\dot{d} + \dot{X}\cos\alpha\right]\right\} \\ &\quad - \left\{\frac{1}{2}\,(M+m)\,\dot{X}^2 + \frac{1}{2}\,m\left(\dot{d}^2 + 2\,\dot{d}\,\dot{X}\cos\alpha\right) - m\,g\,(h - d\sin\alpha)\right\} \\ &= \frac{1}{2}\,(M+m)\,\dot{X}^2 + \frac{1}{2}\,m\left(\dot{d}^2 + 2\,\dot{d}\,\dot{X}\cos\alpha\right) + m\,g\,(h - d\sin\alpha) \\ &= T + U \end{aligned}$$

which is the total energy of the system. We can see that energy is conserved by recognizing that $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$ vanishes because there is no explicit dependence of the Lagrangian on time. The canonically conjugate momenta are:

$$\begin{aligned} p_X &= \frac{\partial L}{\partial \dot{X}} = [m+M]\,\dot{X} + m\,\dot{d}\,\cos\alpha \\ p_d &= \frac{\partial L}{\partial \dot{d}} = m\left[\dot{d} + \dot{X}\cos\alpha\right] \end{aligned}$$

Note that it is $\dot{d}$ that multiplies $\cos\alpha$ in p_X and $\dot{X}$ that multiples $\cos\alpha$ in p_d.

Example 2.4

Hand and Finch, Section 1.9

Remarkably enough, the Euler-Lagrange equation works to some extent in non-inertial frames. The kinetic and potential energies must first be defined in an inertial frame. But then transformation to an accelerating frame can simply be treated as a rheonomic constraint. This is one of those cases where the potential energy is time-independent when written in the position coordinates, but obtains time-dependence when transformed to generalized coordinates because of the rheonomic constraints. For example, consider throwing a ball upward in an upwardly accelerating elevator. The transformation equation is simply

$$z = q + v_0\,t + \frac{1}{2}\,a\,t^2$$

where we have assumed the elevator starts at $t = 0$ with velocity v_0 and accelerates with acceleration a. In the inertial frame, the ball has Lagrangian

$$L = \frac{1}{2} m \dot{z}^2 - m g z$$

Rewriting in terms of the generalized coordinate q, we have

$$\begin{aligned} L &= \frac{1}{2} m [\dot{q} + v_0 + a t]^2 - m g \left[q + v_0 t + \frac{1}{2} a t^2 \right] \\ &= \frac{1}{2} m \dot{q}^2 + m \dot{q} [v_0 + a t] + \frac{1}{2} m [v_0 + a t]^2 - m g q - m g \left[v_0 t + \frac{1}{2} a t^2 \right] \end{aligned}$$

The partial derivatives are

$$\begin{aligned} \frac{\partial L}{\partial \dot{q}} &= m [\dot{q} + v_0 + a t] \\ \frac{\partial L}{\partial q} &= -m g \end{aligned}$$

So the Euler-Lagrange equation is

$$\begin{aligned} m [\ddot{q} + a] + m g &= 0 \\ \ddot{q} &= -(g + a) \\ \dot{q} &= u_0 - (g + a) t \\ q &= u_0 t - \frac{1}{2}(g + a) t^2 \end{aligned}$$

If we then transform back to the inertial frame as a check, we find

$$z = (u_0 + v_0) t - \frac{1}{2} g t^2$$

which is as you would expect: if the ball is thrown upward with speed u_0 relative to the elevator, its initial velocity in the inertial frame is $u_0 + v_0$ and the ball decelerates under gravity as normal. In the noninertial frame, though, the ball decelerates faster, with deceleration $g + a$, because of the acceleration of the elevator.

It is interesting to look at the Hamiltonian because, due to the rheonomic constraints, it is neither the total energy nor is it conserved:

$$\begin{aligned} H = \dot{q} \frac{\partial L}{\partial \dot{q}} - L &= m \dot{q} [\dot{q} + v_0 + a t] - L \\ &= \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m [v_0 + a t]^2 + m g q + m g \left[v_0 t + \frac{1}{2} a t^2 \right] \end{aligned}$$

Clearly, H is time-dependent, and it has no apparent simple relationship to the energy as calculated in the inertial frame, $T + U$.

Example 2.5

Consider a heavy bead sliding on a stiff rotating wire held at a fixed angle, with the bead experiencing the force of gravity and any constraint forces; see Hand and Finch Figure 1.2 for a sketch. Let's apply our formalism to the problem.

The constraint is that the bead must remain on the wire, and that the wire rotates with angular velocity ω. The natural generalized coordinate is q, the distance that the bead is from the origin along the wire. Let α be the polar angle between the axis of rotation and the wire; it is fixed by assumption. We consider the case shown in Hand and Finch, where the wire is angled upward so that the bead does not fall off.

The coordinate transformation relations are

$$\vec{r}(t) = \hat{x}\, q\, \sin\alpha\, \cos\omega t + \hat{y}\, q\, \sin\alpha\, \sin\omega t + \hat{z}\, q\, \cos\alpha$$

Note that t appears explicitly. Though we have not written the constraint equations explicitly, the presence of t in the transformation from position coordinates to generalized coordinates implies that the constraint is rheonomic. Let's first go the generalized equation of motion route, so we need to find the partial derivatives of the transformation relation:

$$\frac{\partial \vec{r}}{\partial q} = \hat{x}\, \sin\alpha\, \cos\omega t + \hat{y}\, \sin\alpha\, \sin\omega t + \hat{z}\, \cos\alpha$$

The non-constraint force is

$$\vec{F}^{(nc)} = -m\, g\, \hat{z}$$

So the generalized force is

$$\mathcal{F}_q = \vec{F}^{(nc)} \cdot \frac{\partial \vec{r}}{\partial q} = -m\, g\, \cos\alpha$$

To calculate the kinetic energy, we will need the velocity:

$$\begin{aligned} \dot{\vec{r}} &= \frac{\partial \vec{r}}{\partial q}\,\dot{q} + \frac{\partial \vec{r}}{\partial t} \\ &= \hat{x}\,\dot{q}\,\sin\alpha\,\cos\omega t + \hat{y}\,\dot{q}\,\sin\alpha\,\sin\omega t + \hat{z}\,\dot{q}\,\cos\alpha \\ &\quad -\hat{x}\; q\,\sin\alpha\,\omega\,\sin\omega t + \hat{y}\, q\,\sin\alpha\,\omega\,\cos\omega t \end{aligned}$$

The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}\, m\, \dot{\vec{r}} \cdot \dot{\vec{r}} \\ &= \frac{1}{2}\, m\, \left(\dot{q}^2 + q^2\, \omega^2\, \sin^2\alpha\right) \end{aligned}$$

where a large amount of algebra has been suppressed. The partial derivatives of T are

$$\begin{aligned} \frac{\partial T}{\partial q} &= m\, q\, \omega^2\, \sin^2\alpha \\ \frac{\partial T}{\partial \dot{q}} &= m\,\dot{q} \end{aligned}$$

The generalized equation of motion therefore is

$$\begin{aligned} \mathcal{F}_q &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} \\ -m\, g\, \cos\alpha &= \frac{d}{dt}\,(m\dot{q}) - m\, q\, \omega^2\, \sin^2\alpha \\ \ddot{q} - q\, \omega^2\, \sin^2\alpha &= -g\, \cos\,\alpha \end{aligned}$$

We can also employ Lagrangian techniques. The potential energy and the Lagrangian are

$$\begin{aligned} U &= m\,g\,z = m\,g\,q\,\cos\alpha \\ L &= T - U = \frac{1}{2}\,m\,\left(\dot{q}^2 + q^2\,\omega^2\,\sin^2\alpha\right) - m\,g\,q\,\cos\alpha \end{aligned}$$

Note that, even though the constraints are rheonomic, U remains time-independent. This is a special case – while there was time dependence in the transformation relations, there was no time dependence in the relation between z and q, and it is only z that appears in the potential energy. The partial derivatives of U are

$$\begin{aligned} \frac{\partial U}{\partial q} &= m\,g\,\cos\alpha \\ \frac{\partial U}{\partial \dot{q}} &= 0 \end{aligned}$$

Therefore, the Euler-Lagrange equation is

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\,(T-U) - \frac{\partial}{\partial q}\,(T-U) &= 0 \\ m\,\ddot{q} - m\,q\,\omega^2\,\sin^2\alpha + m\,g\,\cos\alpha &= 0 \\ \ddot{q} - \omega^2\,q\,\sin^2\alpha &= -g\,\cos\alpha \end{aligned}$$

as seen before.

The canonical momentum is

$$p_q = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

The Hamiltonian is

$$\begin{aligned} H &= \dot{q}\,\frac{\partial L}{\partial \dot{q}} - L \\ &= m\,\dot{q}^2 - L \\ &= \frac{1}{2}\,m\,\dot{q}^2 - \frac{1}{2}\,m\,q^2\,\omega^2\,\sin^2\alpha + m\,g\,q\,\cos\alpha \end{aligned}$$

Note that this Hamiltonian is **not** the total energy of the system. H is conserved because $\partial L/\partial t$ vanishes. But, because of the negative sign on the second term, H is not the total energy. This occurs because the constraint was rheonomic.

Example 2.6

Another good example would be the ladder sliding against the wall problem that is used as an example in Hand and Finch Chapter 1. Try working it out yourself as we have worked out the examples above.

2.1.9 Special Nonconservative Cases

So far we have only considered the Lagrangian when there is a potential energy function that can be derived from conservative forces. There are some special nonconservative cases in which the Lagrangian formalism can continue to be used with some modifications. The discussions of velocity-dependent potentials and general nonconservative forces are based on Goldstein Section 1.5.

Time-Dependent Potentials

In defining the Lagrangian, we specialized to cases in which the potential energy was time-independent in position coordinates. However, one can imagine cases in which U has explicit time dependence but is instantaneously conservative – for example, the gravitational potential of a binary star system in which the stars orbit about the center of mass with constant angular velocity. At distances comparable to the stars' separation, the potential energy function will be strongly time dependent.[7] How can such cases be treated?

To first order, there is no change in the derivation or Euler-Lagrange equations. U can obviously be defined in a sensible fashion; the potential energy is path-independent if one holds time fixed as the line integral of the force is calculated. The generalized force can still be written as $\mathcal{F}_k = -\partial U(\{q_l\};t)/\partial q_k$. And the rewriting of the generalized equation of motion as the Euler-Lagrange equations continues to go through because $\partial U/\partial \dot{q}_k$ continues to vanish; this is what was required to move U inside the $d/dt(\partial/\partial \dot{q}_k)$ operator. So, the Euler-Lagrange equations continue to hold. The Hamiltonian may be defined, and instantaneously corresponds to the energy of the system if the constraints are scleronomic, but now it will not be conserved because $\partial L/\partial t \neq 0$.

More generally, if U is not at least instantaneously conservative, it does not seem likely that the Euler-Lagrange equations will hold. (It is not clear that the concept of U is even sensible in such cases.) Remember that the generalized equation of motion, using the generalized force, is the more fundamental relation. If it is not possible to define a potential energy function U such that $\mathcal{F}_k = -\partial U(\{q_l\};t)/\partial q_k$, then it is simply not possible to convert the generalized equation of motion to the Euler-Lagrange equations.

Velocity-Dependent Potentials

Suppose we have generalized force that can be written in terms of a velocity-dependent potential energy $U(\{q_k\},\{\dot{q}_k\})$ in the following manner:

$$\mathcal{F}_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{2.14}$$

If this is possible, then the Euler-Lagrange equation still holds for $L = T - U$ defined in this way because of the total time derivative term. U may be called a "generalized potential" or "velocity-dependent potential". It is not a potential energy in the conventional sense because it depends on more than just the particle position; it cannot be calculated from a line integral of the generalized force. But it does permit the use of a Lagrangian, which allows us to generalize many of the concepts we developed above. We may obtain a canonical momentum and a Hamiltonian from the Lagrangian, and we are assured that all of the properties we have studied continue to hold because we know the Lagrangian satisfies the Euler-Lagrange equation.

Why are velocity-dependent potentials of interest? There is one extremely important force that can be derived from a velocity-dependent potential, the Lorentz force of a magnetic field acting on a moving charged particle. A charged particle moving in electric and magnetic fields feels a force

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$$

[7] Note that we must consider the nonrelativistic limit so that we may assume the gravitational potential at all positions in space tracks the current positions of the stars; there is no retardation due to the finite propagation speed of gravity.

where q is the particle charge, $\vec{v}$ is the particle velocity, $\vec{E}$ is the electric field

$$\vec{E}(\vec{x}) = -\vec{\nabla}\phi(\vec{x}) - \frac{\partial \vec{A}(\vec{x})}{\partial t}$$

and $\vec{B}$ is the magnetic field

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})$$

with $\phi(\vec{x})$ being the electrostatic potential and $\vec{A}(\vec{x})$ the vector potential. The above force can be derived via the above formula from the generalized potential

$$U(\vec{x}, \vec{v}) = q\left[\phi(\vec{x}) - \vec{A}(\vec{x}) \cdot \vec{v}\right]$$

It is easy to see that this is the correct potential function by applying Equation 2.14:

$$\begin{aligned} \vec{\mathcal{F}} &= -\vec{\nabla} U + \frac{d}{dt}\vec{\nabla}_v U \\ &= -q\,\vec{\nabla}\,\phi + q\,\vec{\nabla}\left(\vec{A}\cdot\vec{v}\right) - q\,\frac{d}{dt}\left[\vec{\nabla}_v\left(\vec{A}\cdot\vec{v}\right)\right] \end{aligned}$$

where $\vec{\nabla}_v$ is the gradient with respect to the components of velocity rather than the components of position (just a notational shorthand). In the last term, $\vec{\nabla}_v\left(\vec{A}\cdot\vec{v}\right) = \vec{A}$ because $\vec{A}$ does not depend on $\vec{v}$ and $\vec{\nabla}_v$ acting on $\vec{v}$ picks out the components of $\vec{v}$. Next, we use the vector identity

$$\vec{a} \times (\vec{\nabla} \times \vec{b}) \;=\; \vec{\nabla}\,(\vec{a}\cdot\vec{b}) - (\vec{a}\cdot\vec{\nabla})\,\vec{b}$$

with $\vec{a} = \vec{v}$ and $\vec{b} = \vec{A}$. The use of this identity relies on the fact that $\vec{\nabla}$ does not act on $\vec{v}$ here.[8] With this identity, we can rewrite the above as

$$\begin{aligned} \vec{\mathcal{F}} &= -q\,\vec{\nabla}\,\phi + q\left[\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}\right] - q\,\frac{d}{dt}\vec{A} \\ &= -q\,\vec{\nabla}\,\phi + q\left[\vec{v}\times\vec{B}\right] - q\,\frac{\partial\vec{A}}{\partial t} \\ &= q\left[\vec{E} + \vec{v}\times\vec{B}\right] \end{aligned}$$

where we have used

$$\frac{d\,\vec{A}}{dt} = \frac{\partial A}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

(which can be proven using the chain rule).

[8] More specifically: the identity holds in the above form only when $\vec{a}$ is not acted on by $\vec{\nabla}$: we have moved $\vec{a}$ from the left side of $\vec{\nabla}$ (LHS of equation) to its right side (first term on RHS). More generally, one has to be a bit more careful. It is not possible to clearly write the general result using just vector notation, but it can be written using index notation:

$$\left[\vec{a}\times(\vec{\nabla}\times\vec{b})\right]_i = \sum_j a_j\nabla_i b_j - \sum_j a_j\nabla_j b_i$$

The key point is that in the first term, $\vec{a}$ is in a dot product with $\vec{b}$, but $\vec{\nabla}$ must be allowed to act on $\vec{b}$ first, and not as $\vec{\nabla}\cdot\vec{b}$.

With the above potential function, the Lagrangian is now

$$L = \frac{1}{2}\,m\,v^2 - q\,\left[\phi(\vec{x}) - \vec{A}(\vec{x})\cdot\vec{v}\right]$$

The canonical momentum is

$$\begin{aligned}\vec{p} = \vec{\nabla}_v L &= m\,\vec{v} + q\,\vec{\nabla}_v\left(\vec{A}\cdot\vec{v}\right)\\ &= m\,\vec{v} + q\,\vec{A}\end{aligned}$$

The Hamiltonian is:

$$\begin{aligned}H &= \vec{p}\cdot\vec{v} - L\\ &= m\,v^2 + q\,\vec{A}\cdot\vec{v} - \frac{1}{2}\,m\,v^2 + q\,\phi - q\,\vec{A}\cdot\vec{v}\\ &= \frac{1}{2}\,m\,v^2 + q\,\phi\\ &= \frac{1}{2\,m}\left(\vec{p} - q\,\vec{A}\right)^2 + q\,\phi\end{aligned}$$

The lack of any magnetic terms in the third line should not be too surprising – after all, since the magnetic field exerts a force perpendicular to velocity, it can do no work on a particle and thus does not contribute to its energy. But $\vec{A}$ becomes important dynamically when we consider the form in the last line because, in Hamiltonian mechanics (which we will discuss later), $\vec{p}$ and $\vec{x}$ are the physically significant variables; $\vec{A}$ appears when we write H in terms of $\vec{p}$ and $\vec{x}$ instead of in terms of $\vec{v}$ and $\vec{x}$. Quantum mechanics begins with the Hamiltonian formulation, so those of you who have taken quantum mechanics will no doubt recognize the form of the Hamiltonian in the last line.

Nonconservative Forces in General

While it is not in general possible to include other nonconservative forces in the Lagrangian, we can see how to write the Euler-Lagrange equations with nonconservative forces by returning to the generalized equation of motion. Recall Equation 2.10:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \mathcal{F}_k$$

We obtained the Euler-Lagrange equation by writing $\mathcal{F}_k$ as the gradient of a potential and moving it to the left side. If we have nonconservative forces, we can leave them on the right side, moving only conservative forces to the left side and including them in the Lagrangian. More generally, any force than can be included in the Lagrangian via some sort of potential moves to the left side. Thus, we can write a generalized Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \mathcal{F}_k^{no-L} \tag{2.15}$$

where $\mathcal{F}^{no-L}$ encompasses all forces that cannot be included in the Lagrangian.

2.1.10 Symmetry Transformations, Conserved Quantities, Cyclic Coordinates and Noether's Theorem

We discuss the relation between symmetry transformations of the Lagrangian, cyclic coordinates, and conserved quantities; the relationship is encoded in Noether's Theorem. Noether's Theorem brings to the fore the importance of these cyclic coordinates and canonical momenta.

This material is covered in Hand and Finch Sections 5.1 and 5.2.

Coordinate Transformations

Suppose we have a Lagrangian for a system, $L(q, \dot{q}, t)$. Let us define a new set of coordinates $q'(t)$. (Let q and q' be abbreviations for multi-dimensional, multi-particle sets of generalized coordinates $\{q_k\}$ and $\{q'_k\}$.) This is just a relabeling of the configuration of the system. For example, a point P in 3-dimensional space may be known as (x, y, z) in the original system and (x', y', z') in the new system. Without a doubt, we will have a new Lagrangian, $L'(q', \dot{q}', t)$, with the two Lagrangians related by

$$L'(q', \dot{q}', t) = L(q(q', \dot{q}', t), \dot{q}(q', \dot{q}', t), t) \tag{2.16}$$

There will likely be different equations of motion after relabeling. Though, because what we have done is **only** a relabeling, the actual physical solutions must be equivalent. That is, if one obtains physical solutions $q(t)$ and $q'(t)$ from the two different Lagrangians, the original relabeling procedure must show them to be equivalent physical paths.

Some further explanation of the transformation is in order. We wrote it as simply a relabeling. But it could have equally well been a movement of the system. These two interpretations are as follows:

- **passive** transformation: we have simply defined a new coordinate system, a relabeling of the points in space.
- **active** transformation: by changing coordinate systems, we have actally *moved* the system of particles and potentials relative the original coordinate system so that the system now has new coordinates q' in the original system. Note that *everything* must move; for example, if one has a nonuniform gravitational field, and one moves the particles but not the field, the dynamics will of course change!

To some extent, it doesn't matter which interpretation you like; the physical result will be the same.

To clarify these abstractions, consider the following examples:

1. Mirror transformation:

$$\begin{aligned} x' &= -x \\ y' &= y \\ z' &= z \end{aligned}$$

   Our transformation could be interpreted in two ways:

   - passive: the particle is not being moved; rather, we are defining a new coordinate system (x', y', z') that is a reflection of the original one and relabeling all points in space according to the transformation.

- active: the particle is actually reflected in x through the origin by the transformation.

2. Rotation transformation about the z axis:

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta \\ y' &= x\sin\theta + y\cos\theta \\ z' &= z \end{aligned}$$

or, equivalently,

$$\begin{aligned} x &= x'\cos\theta + y'\sin\theta \\ y &= -x'\sin\theta + y'\cos\theta \\ z &= z' \end{aligned}$$

Again, there are two interpretations:

- passive: the particle is not being moved; rather, we are defining a new coordinate system (x', y', z') that is rotated from the old one and relabeling all points in space according to the transformation.
- active: the particle is actually rotated about the origin by an angle θ by the transformation.

3. Translation:

$$\begin{aligned} x' &= x - a \\ y' &= y \\ z' &= z \end{aligned}$$

($a > 0$ for specificity). The two interpretations of the transformation are:

- passive: we are simply defining a new coordinate system whose origin is a distance a to the $+x$ direction of the old origin.
- active: the particle is actually moved a distance a to $-x$ of where it was originally.

Continuous Transformations

Let us now specialize to transformations that can be written as a function of a continuous parameter s (or set of parameters), such that $q'(t) = Q(s,t)$ with $Q(0,t) = q(t)$. The rotation and translation transformations above are examples of continuous transformations, with the parameters being the rotation angle θ or the translation distance a, respectively.

Symmetry Transformations

Now, let's consider transformations for which the Lagrangian satisfies a more sophisticated requirement:

$$L'(q', \dot{q}', t) = L(q', \dot{q}', t) \tag{2.17}$$

This statement has content that the original transformation relation Equation 2.16 did not have. Here, we are requiring that, in spite of relabeling of the coordinates, the original Lagrangian still holds. We say the Lagrangian is **invariant under the transformation** if the above holds, that the system is **symmetric under the transformation** or that the

transformation is a **symmetry transformation** of the system. We will see this invariance has physical content.

To clarify these abstractions, consider the following examples and counterexamples. For all, we consider a particle moving under the influence of gravitational field centered at the origin. The Lagrangian is

$$L(x,y,z) = \frac{1}{2}m\,\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - U\left(\sqrt{x^2+y^2+z^2}\right)$$

1. Example: Consider the reflection transformation. The new Lagrangian is

$$\begin{aligned} L'(x',y',z') &= L\left(x(x',y',z'),y(x',y',z'),z(x',y',z')\right) \\ &= \frac{1}{2}m\left(\left(-\dot{x}'\right)^2+\dot{y}'^2+\dot{z}'^2\right) - U\left(\sqrt{\left(-x'\right)^2+y'^2+z'^2}\right) \\ &= L(x',y',z') \end{aligned}$$

   The Lagrangian is clearly invariant under the transformation.

2. Example: Consider the rotation transformation about the z axis. We easily see

$$\begin{aligned} L'(x',y',z') &= L\left(x(x',y',z'),y(x',y',z'),z(x',y',z')\right) \\ &= \frac{1}{2}m\left[\left(\dot{x}'\,\cos\theta+\dot{y}'\,\sin\theta\right)^2+\left(-\dot{x}'\,\sin\theta+\dot{y}'\,\cos\theta\right)^2+\dot{z}'^2\right] \\ &\quad -U\left(\sqrt{\left(x'\,\cos\theta+y'\,\sin\theta\right)^2+\left(-x'\,\sin\theta+y'\,\cos\theta\right)^2+z'^2}\right) \\ &= \frac{1}{2}m\left(\dot{x}'^2+\dot{y}'^2+\dot{z}'^2\right) - U\left(\sqrt{x'^2+y'^2+z'^2}\right) \\ &= L(x',y',z') \end{aligned}$$

   The Lagrangian is invariant under the transformation.

3. Counterexample: Consider the simple translation transformation. The new Lagrangian is

$$\begin{aligned} L'(x',y',z') &= L\left(x(x',y',z'),y(x',y',z'),z(x',y',z')\right) \\ &= \frac{1}{2}m\left(\dot{x}'^2+\dot{y}'^2+\dot{z}'^2\right) - U\left(\sqrt{\left(x'+a\right)^2+y'^2+z'^2}\right) \\ &\neq L(x',y',z') \end{aligned}$$

   The Lagrangian is clearly not invariant under the transformation. This is an especially interesting case because it highlights the difference between a symmetry transformation and a garden-variety coordinate transformation. When we perform the coordinate transformation, we of course get a perfectly valid Lagrangian and, were we to solve for the dynamics using that Lagrangian, we would obtain the same physical path (accounting for the coordinate transformation) as with the original Lagrangian. *But the Lagrangians are different.* For a symmetry transformation, not only is the resulting physical path the same, but the Lagrangians themselves are the same.

Note how the above invariance or lack of invariance is independent of the interpretation of the transformation (passive or active).

Another way to express symmetry transformations

We have so far asked about $L'(q', \dot{q}', t)$ and whether it satisfies $L'(q', \dot{q}', t) = L(q', \dot{q}', t)$. We can ask a differently phrased but equivalent question: does it hold that $L(q', \dot{q}', t) = L(q, \dot{q}, t)$? Note that, in each of the above examples, this latter statement was the one we really tested. In every case, we have $L'(q', \dot{q}', t) = L(q(q', \dot{q}', t), \dot{q}(q', \dot{q}', t), t)$ (because that is what you get simply from coordinate relabeling), but the thing we had to then prove was $L(q(q', \dot{q}', t), \dot{q}(q', \dot{q}', t), t) = L(q', \dot{q}', t)$, and it was the latter step that did not always work. We will therefore say that a Lagrangian is invariant under the transformation $q \to q'$ if

$$L(q(q', \dot{q}', t), \dot{q}(q', \dot{q}', t), t) = L(q', \dot{q}', t) \tag{2.18}$$

and therefore

$$L(q'(q, \dot{q}, t), \dot{q}'(q, \dot{q}, t), t) = L(q, \dot{q}, t) \tag{2.19}$$

This may seem like a backwards way of looking at it – write the old Lagrangian using the new coordinates and ask if it looks like the old Lagrangian in the old coordinates – but we have proved above that it is a mathematically equivalent statement, and we will see this version is more convenient to work with. To be explicit, let's redo our examples using this form:

1. Example: Reflection:

$$\begin{aligned} L(x', y', z') &= \frac{1}{2} m \left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2\right) - U\left(\sqrt{x'^2 + y'^2 + z'^2}\right) \\ &= \frac{1}{2} m \left((-\dot{x})^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(\sqrt{(-x)^2 + y^2 + z^2}\right) \\ &= L(x, y, z) \end{aligned}$$

2. Example: Rotation:

$$\begin{aligned} L(x', y', z') &= \frac{1}{2} m \left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2\right) - U\left(\sqrt{x'^2 + y'^2 + z'^2}\right) \\ &= \frac{1}{2} m \left[(\dot{x}\cos\theta - \dot{y}\sin\theta)^2 + (\dot{x}\sin\theta + \dot{y}\cos\theta)^2 + \dot{z}^2\right] \\ &\quad -U\left(\sqrt{(x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 + z^2}\right) \\ &= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(\sqrt{x^2 + y^2 + z^2}\right) \\ &= L(x, y, z) \end{aligned}$$

3. Counterexample: Translation:

$$\begin{aligned} L(x', y', z') &= \frac{1}{2} m \left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2\right) - U\left(\sqrt{x'^2 + y'^2 + z'^2}\right) \\ &= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(\sqrt{(x-a)^2 + y^2 + z^2}\right) \\ &\neq L(x, y, z) \end{aligned}$$

The math is the same, though reversed, but the interpretation is somewhat different.

Noether's Theorem and Conserved Quantities

Suppose that we do indeed have a Lagrangian $L(q, \dot{q}, t)$ that is invariant under the continuous transformation $Q(s, t)$. Does this knowledge get us anything?

Begin by noting that, if $Q(s, t)$ is a symmetry transformation of L, then the transformed Lagrangian does not depend on s:

$$\frac{d}{ds} L(Q(s,t), \dot{Q}(s,t), t) = \frac{d}{ds} L(q(t), \dot{q}(t), t) = 0$$

Note that we have used the second manner of writing the invariance under transformation, Equation 2.19. The statement may not seem entirely obvious. Certainly, $L(Q(s,t), \dot{Q}(s,t), t)$ depends on s! Yes, there is explicit dependence on s, but the second form of the invariance under transformation requirement implies that this form that has explicit dependence on s can be rewritten in a form that has no dependence on s, which then allows us to claim the above. As a check, consider the rotation and translation transformations given earlier, with $s = \theta$ and $s = a$, where $L(x', y', z') = L(x, y, z)$ always held: $L(x, y, z)$ does not depend on s, so clearly then $L(x', y', z')$ cannot depend on s.

Let's derive the implications of the above. Applying the chain rule to obtain $\frac{dL}{ds}$ gives

$$\frac{\partial L}{\partial Q}\frac{dQ}{ds} + \frac{\partial L}{\partial \dot{Q}}\frac{d\dot{Q}}{ds} = 0$$

If we apply the Euler-Lagrange equation $\frac{\partial L}{\partial Q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = 0$ to replace $\frac{\partial L}{\partial Q}$, we find

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right)\frac{dQ}{ds} + \frac{\partial L}{\partial \dot{Q}}\frac{d\dot{Q}}{ds} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\frac{dQ}{ds}\right) &= 0 \end{aligned}$$

where in the last step we made use of the fact that t and s are completely independent of each other so the order of their derivatives can be exchanged. Recall that $p = \frac{\partial L}{\partial \dot{Q}}$, so we have

$$I(q, \dot{q}) \equiv p \left.\frac{dQ}{ds}\right|_{s=0}$$

is constant. We have evaluated $p = \frac{\partial L}{\partial \dot{Q}}$ and $\frac{dQ}{ds}$ at $s = 0$ because the above relation holds for all s and so we may choose $s = 0$ for convenience. I is written as a function of q and $\dot{q}$ because the right side is evaluated at $s = 0$. More generally, we have

$$I_j(\{q_k\}, \{\dot{q}_k\}, t) \equiv \sum_{k=1}^{N} p_k \left.\frac{dQ_k}{ds_j}\right|_{\{s_j=0\}} \tag{2.20}$$

is constant for any set of symmetry transformations $Q_k(\{s_j\})$ indexed by j.

The above equation is **Noether's Theorem**: If the Lagrangian possesses a set of P continuous symmetry transformations parameterized by P parameters $\{s_j\}$, then there are P conserved quantities associated with the transformations given by the above form.

Let's consider some examples.

1. Translation transformation on a translation-invariant Lagrangian. Consider the Lagrangian

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - m\, g\, z$$

and the transformation

$$\vec{r}' = \vec{r} + \vec{a}$$

where $\vec{a}$ is a constant. Let's apply our second method for checking for invariance:

$$\begin{aligned} L(x', y', z') &= \frac{1}{2} m \left(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2\right) - m\, g\, z' \\ &= \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - m\, g\, z - m\, g\, a_z \end{aligned}$$

where a_z is the z component of $\vec{a}$. The Lagrangian is clearly only invariant if $a_z = 0$, *i.e.*, if the translation is transverse to the gradient of the potential term. So the symmetry transformation is

$$\begin{aligned} x'(a_x, a_y) &= x + a_x \\ y'(a_x, a_y) &= y + a_y \end{aligned}$$

Since the transformation has two parameters a_x and a_y, there are two conserved quantities, which we can find using Equation 2.20:

$$\begin{aligned} a_x &: I_x = p_x \left.\frac{dx'}{da_x}\right|_{a_x=0} + p_y \left.\frac{dy'}{da_x}\right|_{a_x=0} = p_x = m\,\dot{x} \\ a_y &: I_y = p_x \left.\frac{dx'}{da_y}\right|_{a_y=0} + p_y \left.\frac{dy'}{da_y}\right|_{a_y=0} = p_y = m\,\dot{y} \end{aligned}$$

which are the conventional mechanical momenta along the x and y directions. Notice how the conserved quantities go with the parameters a_x and a_y, which in this case map to x and y but may not always!

2. Rotational transformation on a spherically symmetric (rotationally invariant) Lagrangian. We consider the same Lagrangian we considered earlier:

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - U\left(\sqrt{x^2 + y^2 + z^2}\right)$$

We have already demonstrated that this Lagrangian is invariant under the transformation

$$\begin{aligned} x'(\theta) &= x\,\cos\theta - y\,\sin\theta \\ y'(\theta) &= x\,\sin\theta + y\,\cos\theta \\ z'(\theta) &= z \end{aligned}$$

There is one parameter, θ, so there is one conserved quantity

$$\begin{aligned} I &= p_x \left.\frac{dx'}{d\theta}\right|_{\theta=0} + p_y \left.\frac{dy'}{d\theta}\right|_{\theta=0} \\ &= p_x\left(-x\sin 0 - y\cos 0\right) + p_y\left(x\cos 0 - y\sin 0\right) \\ &= x\,p_y - y\,p_x = l_z \end{aligned}$$

which you know as the angular momentum about the z axis. Rotations about the x and y axes would have yielded conserved quantities $l_x = y\,p_z - z\,p_y$ and $l_y = z\,p_x - x\,p_z$.

Noether's Theorem and Cyclic Coordinates

We conclude by returning to **cyclic coordinates**, which were first mentioned in Section 2.1. Recall that a cyclic coordinate is one that does not appear explicitly in the Lagrangian (though its time derivative may, and indeed must for the coordinate to be nontrivial). We have seen in Section 2.1 that the canonical momentum p_c corresponding to a cyclic coordinate q_c is conserved because the Euler-Lagrange equation for that coordinate becomes

$$\dot{p}_c = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_c} = \frac{\partial L}{\partial q_c} = 0$$

We can see that our discussion of symmetry transformations is consistent with this. Consider a translation in the coordinate q_c, $q_c' = q_c + a$. Clearly, the Lagrangian is invariant under the transformation because q_c does not appear explicitly in L. The conserved quantity is

$$I_c = p_c \left.\frac{dq_c'}{da}\right|_{a=0} = p_c$$

Thus, the conservation of canonical momentum for a cyclic coordinate is just a special case of Noether's theorem. Of course, this had to be true because we used the Euler-Lagrange equations to derive Noether's theorem.

Philosophical Epilogue

It is difficult to overstate the importance of symmetry transformations and Noether's theorem to the development of classical mechanics, classical field theory, quantum field theory, and much of the physics of the twentieth century. New theories, especially in quantum field theory, are frequently based on assumptions about what symmetry transformations nature should obey. Requirements of invariance under *global* symmetry transformations of the kind described here are frequently a starting point for new Lagrangians and provide conserved quantities ("charges"). For example, electric charge conservation can be demonstrated to be a consequence of the invariance of the quantum mechanics of a particle in an electrostatic field under phase transformations $\psi' = e^{i\phi}\psi$. Elevation of such global symmetries to local symmetries – that the Lagrangian must be invariant under symmetry transformations for which the transformation parameter s is an arbitrary function of position – require the existence of the gauge fields of particle physics (photons, W and Z bosons, gluons) and can be used to "derive" general relativity.

2.2 Variational Calculus and Dynamics

We present variational calculus and the variational approach to dynamics, which proceeds from the **Principle of Least Action**. The section will develop the variational calculus and derive the Euler-Lagrange equation from a variational principle rather than from Newton's second law. Holonomic constraints will be included via the technique of Lagrange multipliers, which is an alternative to defining a set of unconstrained generalized coordinates that have eliminated any constrained degrees of freedom. Some nonholonomic constraints may be dealt with via variational dynamics.

While it may seem that both this section and the previous one were new formulations of mechanics, one must realize that the Lagrangian mechanics introduced in the previous chapter derived directly from Newton's second law. The use of constraints, generalized variables, the Lagrangian, and the derivation of the Euler-Lagrange equation should be consider "technology," not new physics. This section, on the other hand, provides a new principle on which to base mechanics, an alternative to Newton's second law.

Another difference between the last section and this one is that, in the last section, we continued the Newtonian mechanical tendency to derive the equations of motion from a "differential principle" – *i.e.*, the equations of motion only care about what is happening locally. As Goldstein says, the Principle of Least Action is different in that it is an "integral principle," determining the equations of motion from a requirement on the entire motion of the system between a pair of start and end times.

The naming and history are no doubt becoming very confusing. The line of reasoning presented in the previous section was due to Lagrange and so is known as Lagrangian mechanics. But Lagrange was also responsible for the application of variational calculus to mechanics, and so the material derived in this section is also sometimes referred to as Lagrangian dynamics. But, in fact, the Principle of Least Action was formulated by Hamilton, not Lagrange. The confusion will continue when we embark on Hamiltonian dynamics in the next section. We will refer to the use of the calculus of variations in mechanics as variational dynamics or variational mechanics to distinguish from the Lagrangian mechanics studied in the previous section.

We following Hand and Finch Chapter 2, though with some pedagogical differences in how certain results are derived. Refer also Thornton Chapters 6 and 7.

2.2.1 The Variational Calculus and the Euler Equation

We begin by studying the variational calculus as a purely mathematical tool. A better name is perhaps "functional calculus" because we will be considering the concept of differentiation with respect to functions rather than numbers or sets of numbers. We will derive the Euler equation, an important result of variational calculus that we will apply to mechanics.

Functionals and Variations

- function = rule that maps an input set of numbers to an output set of numbers
- **functional** = rule that maps a function or set of functions to an output set of numbers. Non-generic example:

$$I[y] \equiv \int_{x_0}^{x_1} dx\, F\left(y, \frac{dy}{dx}, x\right) \tag{2.21}$$

where x is the variable of integration, y is a function of x, and F is a simple function that accepts three arguments. For example, $F = y^2 + (\frac{dy}{dx})^3 + x^4$. The functional

$I[y]$ is only one, specific example of a functional. Not every functional is based on an integration. Furthermore, the choice of the arguments of the function F – y, $\frac{dy}{dx}$, and x – is special.

- "Variation" in the function y: the functional equivalent of the differential in a variable dy; small changes in the function y at every point x, holding the endpoint values $y(x_0)$ and $y(x_1)$ fixed, and denoted by δy. There will also be a variation in $\frac{dy}{dx}$, denoted by $\delta\,\frac{dy}{dx}$, though it is entirely specified by the variation δy because $\delta\frac{dy}{dx} = \frac{d}{dx}\delta y$. (This statement does not contradict the argument in our discussion of Lagrangian mechanics that a generalized coordinate and its associated generalized velocity – here represented more abstractly by y and $\frac{dy}{dx}$ – are independent.[9]) To visualize the variation δy, think of flexible, stretchable string held fixed at its endpoints.
- "Variation" in the functional I: The change in I due to the variations δy and $\delta\frac{dy}{dx}$ is found in the way familiar from elementary calculus:

$$\begin{aligned}\delta I &\equiv I[y+\delta y] - I[y] \\ &= \int_{x_0}^{x_1} dx\, F\left(y+\delta y, \frac{dy}{dx} + \delta\frac{dy}{dx}, x\right) - \int_{x_0}^{x_1} dx\, F\left(y, \frac{dy}{dx}, x\right)\end{aligned} \tag{2.22}$$

δI is the "variation". We can further evaluate it using the chain rule:

$$F\left(y+\delta y, \frac{dy}{dx} + \delta\frac{dy}{dx}, x\right) - F\left(y, \frac{dy}{dx}, x\right) \;=\; \frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial \frac{dy}{dx}}\,\delta\frac{dy}{dx} + \mathcal{O}(\delta^2)$$

where $\mathcal{O}(\delta^2)$ denotes the higher-order terms being ignored in this linear expansion. So we have

$$\begin{aligned}\delta I &= \int_{x_0}^{x_1} dx \left[\frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial \frac{dy}{dx}}\,\delta\frac{dy}{dx}\right] \\ &= \int_{x_0}^{x_1} dx \left[\frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial \frac{dy}{dx}}\,\frac{d}{dx}\,\delta y\right] \\ &= \int_{x_0}^{x_1} dx \left[\frac{\partial F}{\partial y}\,\delta y - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right)\,\delta y\right] + \left[\frac{\partial F}{\partial \frac{dy}{dx}}\,\delta y\right]\Bigg|_{x_0}^{x_1} \\ &= \int_{x_0}^{x_1} dx \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right)\right]\delta y\end{aligned}$$

where in the next-to-last step we integrated by parts. The "surface term" vanished due to the boundary conditions on δy.

[9] It is true that, for a given "candidate" path $y(x)$, the function $\frac{dy}{dx}$ is completely specified. However, we do not know at this stage – *i.e.*, before writing and solving Euler's equation – which candidate path $y(x)$ is the correct one. There are a whole family of candidate paths that pass through a given y value at a given x, but whose first and higher-order derivatives all differ, and we allow any of them to be candidate paths. One might still worry that, if y and $\frac{dy}{dx}$ at a given x are known, that the entire path $y(x)$ is specified. That is simply not true because one has infinite freedom in the higher order derivatives of $y(x)$ at this stage. Another way of stating it is that, if y and $\frac{dy}{dx}$ at a given x are known, is it not possible to predict y for nearby x values? No, because the Taylor expansion connecting y at two values of x is an infinite power series with an infinite number of derivatives to be specified. Thus, until a physical path $y(x)$ is found via Euler's equation, y and $\frac{dy}{dx}$ are independent variables; but, for a particular candidate path, they are not.

Extremum Condition and the Euler Equation

A next obvious question to consider in carrying along the analogy from elementary calculus to variational calculus is: can we find a path $y^*(x)$ such that δI at the path $y^*(x)$ vanishes for all choices of δy? This is the analogy to asking, in elementary calculus, whether there exists a point where the derivative vanishes. We can certainly set δI as given above to zero. This is not particularly illuminating until one realizes that the condition that δI vanish for all δy implies that the quantity inside the brackets must vanish because δy is arbitrary. Recall that δy is a function of x and has arbitrary value at each point x, so δy cannot be pulled outside the integral; the bracketed quantity must indeed vanish for δI to vanish. Thus, we are left with Euler's equation:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right) = 0 \tag{2.23}$$

The reader will no doubt consider the structure of the above equation familiar.

Let's consider some examples of how one might use the above result.

Example 2.5

A classic example of functional minimization is to minimize the length of a path between two points in a plane. Let us find the path that minimizes the distance from the point (x_0, y_0) to the point (x_1, y_1), where the path is specified by a function $y(x)$ that indicates how to get from the start point to the end point.

The functional we will calculate is

$$I \equiv \int_{x_0}^{x_1} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

That this is the appropriate formula can be seen by considering a small element of the path dx in going from x to $x + dx$. There will be a corresponding vertical travel dy. The two distances form a right triangle whose hypotenuse $\sqrt{dx^2 + dy^2}$ is the distance actually traveled in going from $(x, y(x))$ to $(x + dx, y(x + dx))$. Playing fast and loose with differentials, we see the contribution to the length is just as indicated above. It is of course more rigorously justified by Taylor expanding $(x + dx, y(x + dx))$ in terms of (x, y). In any case, we have $F(y, \frac{dy}{dx}, x) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, so Euler's equation is

$$\begin{aligned} 0 - \frac{d}{dx}\left(\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{-\frac{1}{2}} \frac{dy}{dx}\right) &= 0 \\ \Longrightarrow \frac{dy}{dx} &= m \\ y &= mx + b \end{aligned}$$

where m and b are constants. So, clearly, the shortest path is a straight line. The constants are specified by requiring that the path start and end on the desired endpoints, giving

$$\begin{aligned} \frac{dy}{dx} &= \frac{y_1 - y_0}{x_1 - x_0} \\ y &= \frac{y_1 - y_0}{x_1 - x_0}\, x + \left(y_0 - \frac{y_1 - y_0}{x_1 - x_0}\, x_0\right) \end{aligned}$$

Example 2.6

The brachistochrone problem, the problem that led to the foundation of the variational calculus. Find the curve joining two points (x_0, y_0) and (x_1, y_1) that describes the path that a particle beginning from rest follows when it falls under the influence of gravity between the two points in the least time.

The quantity we want to minimize is the time taken. The time is the path length divided by the velocity. One might think that the natural independent variable is time. But, we don't know the end time; that is in fact the quantity we are trying to determine! So clearly t can't be the independent variable. The other obvious choice is x. Clearly, the quantity we want to minimize is

$$T = \int_{x_0}^{x_1} dt = \int_{x_0}^{x_1} \frac{ds}{v} = \int_{x_0}^{x_1} dx \, \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{v}$$

The problem now is: what to do with v? It involves derivatives with respect to time, not x. Here, we make use of the initial condition: at $t = 0$, $v = 0$. Let us also set $x_0, y_0 = 0$ without loss of generality in order to simplify the algebra. Therefore, conservation of energy implies

$$\frac{1}{2} m v^2 = m g y(x)$$

So we may rewrite

$$T = \int_{x_0}^{x_1} dx \, \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{-2 g y(x)}}$$

The integrand is clearly now a function that can be treated by the variational technique we have developed; the function F is

$$F\left(y, \frac{dy}{dx}, x\right) = \sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{-2 g y(x)}}$$

Euler's equation is

$$-\frac{F}{2y} - \frac{d}{dx}\left[\frac{F}{1 + \left(\frac{dy}{dx}\right)^2} \frac{dy}{dx}\right] = 0$$

$$-\frac{F}{2y} - \frac{dF}{dx} \frac{1}{1 + \left(\frac{dy}{dx}\right)^2} \frac{dy}{dx} + \frac{F}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2} 2 \left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2} - \frac{F}{1 + \left(\frac{dy}{dx}\right)^2} \frac{d^2y}{dx^2} = 0$$

$$-\frac{F}{2y} - \frac{dF}{dx} \frac{1}{1 + \left(\frac{dy}{dx}\right)^2} \frac{dy}{dx} - F \frac{1 - \left(\frac{dy}{dx}\right)^2}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2} \frac{d^2y}{dx^2} = 0$$

Now,

$$\begin{aligned}\frac{dF}{dx} &= \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial \frac{dy}{dx}}\frac{d^2y}{dx^2} \\ &= -\frac{F}{2\,y}\frac{dy}{dx} + \frac{F}{1+\left(\frac{dy}{dx}\right)^2}\frac{dy}{dx}\frac{d^2y}{dx^2}\end{aligned}$$

So we then have for our Euler equation

$$\begin{aligned}-\frac{F}{2\,y}\left[1-\frac{1}{1+\left(\frac{dy}{dx}\right)^2}\left(\frac{dy}{dx}\right)^2\right] - \frac{F}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}\left[\left(\frac{dy}{dx}\right)^2+\left(1-\left(\frac{dy}{dx}\right)^2\right)\right]\frac{d^2y}{dx^2} &= 0 \\ -\frac{F}{2\,y}\frac{1}{1+\left(\frac{dy}{dx}\right)^2} - \frac{F}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}\frac{d^2y}{dx^2} &= 0\end{aligned}$$

F never vanishes, so it can be eliminated. If we multiply across by some factors, we find

$$1+\left(\frac{dy}{dx}\right)^2 + 2\,y\,\frac{d^2y}{dx^2} = 0$$

This differential equation can be solved by making some substitutions. First, define $u = \sqrt{y}\,\frac{dy}{dx}$. Then we may rewrite the above equation as

$$\begin{aligned}1+2\sqrt{y}\,\frac{du}{dx} &= 0 \\ 2\,\frac{du}{dx} &= -\frac{1}{\sqrt{y}} = -\frac{1}{u}\frac{dy}{dx} \\ -\frac{d}{dx}\,u^2 = -2\,u\,\frac{du}{dx} &= \frac{dy}{dx}\end{aligned}$$

which is now a perfect differential. Integrating, we find (where we choose the form of the constant of integration for later convenience)

$$\begin{aligned}-u^2 &= y+2\,a \\ y\left(\frac{dy}{dx}\right)^2 &= -(y+2\,a) \\ \frac{dy}{dx} &= \pm\sqrt{\frac{-(y+2\,a)}{y}} \\ \frac{y\,dy}{\sqrt{-y\,(y+2\,a)}} &= dx\end{aligned}$$

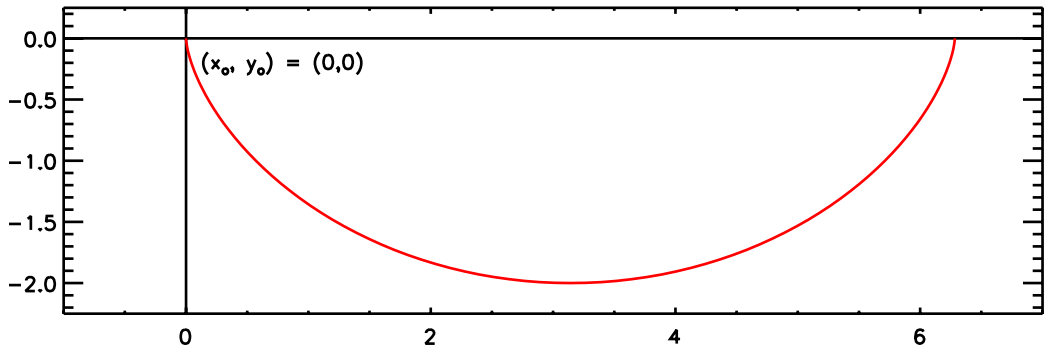
This kind of differential is not that unusual and can be solved by trigonometric substitution. Let $y = -a\,(1-\cos\theta)$. Then we have $dy = -a\,\sin\theta\,d\theta$ and

$$\begin{aligned}\frac{a\,(1-\cos\theta)\;a\,\sin\theta\,d\theta}{\left[a^2\,(1-\cos\theta)\,(1+\cos\theta)\right]^{1/2}} &= \pm dx \\ \frac{a\,(1-\cos\theta)\;a\,\sin\theta\,d\theta}{\left[a^2\,(1-\cos^2\theta)\right]^{1/2}} &= \pm dx \\ a\,(1-\cos\theta)\;d\theta &= \pm dx\end{aligned}$$

which is easily integrated. Our solution is therefore

$$\begin{aligned} x &= \pm a\,(\theta - \sin\theta) \\ y &= -a\,(1 - \cos\theta) \end{aligned}$$

which is the final parametric solution for the curve that minimizes the travel time. The $\pm$ sign is chosen based on whether $x_1 > 0$ or $x_1 < 0$. The results describe a **cycloid**, the curve that is traced out by a single point on the edge of a rolling disk. The constant a is determined by the position of the endpoint. The relation between θ and time can be found implicitly by using the relation between velocity and potential energy that we began with.



Cycloid solution to brachistochrone problem, for $x_1 > 0$ and $a = 1$.

Multiple Dependent Variables

What if we have multiple dependent variables y_k on which our function F depends? The obvious example is a function F that depends on the three-dimensional spatial position of a particle. What does requiring $\delta I = 0$ yield?

If we return to the derivation of the variation δI, we easily see that y could have been generalized to be an array of dependent variables y_k. We would have ended up with the relation

$$\delta I = \int_{x_0}^{x_1} dx \sum_k \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy_k}{dx}} \right) \right] \delta y_k$$

Now, if the y_k are all independent (*i.e.*, there are no constraint equations connecting them), then the variations δy_k are independent, in which case the quantity in square brackets must vanish separately for each k. That is, we have an Euler equation for each k,

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy_k}{dx}} \right) = 0 \tag{2.24}$$

2.2.2 The Principle of Least Action and the Euler-Lagrange Equation

To make use of the above mathematics, we need an integral quantity that will give rise to the Euler-Lagrange equations that we derived in the Section 2.1 via d'Alembert's principle. The form of the Euler equation suggests that if we take $F = L$, we will recover the Euler-Lagrange equations. Thus, we define the **action**,

$$S = \int_{t_0}^{t_1} dt\, L\,(q(t), \dot{q}(t), t) \tag{2.25}$$

and we require that the physical path satisfy the **Principle of Least Action**, that

$$\delta S = 0 \tag{2.26}$$

for the physical path. This is also known as **Hamilton's Principle** because he was the first one to suggest it as a general physical principle. Plugging in for $F = L$ in the Euler equation gives us the Euler-Lagrange equation,

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

Thus, we see that we are able the derive equations of motion from the Principle of Least Action. Nowhere have we made use of Newton's second law; we have only used the Principle of Least Action and the variational calculus. Thus, the Principle of Least Action can be taken to be an alternate basis for mechanics. We have so far considered only the case of physical situations where a Lagrangian can be defined (*i.e.*, we require the forces to be conservative) and wherein there are no constraints among the degrees of freedom (*i.e.*, there are no constraints at all, or the constraints are holonomic and have already been used to competely eliminate uninteresting degrees of freedom).

Since the Euler-Lagrange equations were already derived in the previous section, the examples we presented then still hold. For physical situations obeying the above conditions – that all non-constraint forces are conservative and that any constrained degrees of freedom have been eliminated – we get the same Euler-Lagrange equations that we would have found by applying the constraint discussion of the previous section. So, no new examples are necessary.

2.2.3 Imposing Constraints in Variational Dynamics

As mentioned above, the derivation of the Euler-Lagrange equations from the Principle of Least Action assumed that there were no constraints among the different degrees of freedom. In this section, we show how to use **Lagrange multipliers** to incorporate constraints and how the multipliers are related to the constraint forces in the system.

Lagrange Multipliers for Standard Calculus Minimization Problems

NOTE: READ THIS SECTION! This material is not in any of the standard texts. It's not important for understanding how to use Lagrange multipliers, but it provides a more satisfactory derivation than I have been able to find.

Let $\{y_k\}$ denote the full set of degrees of freedom of a system; we write it as $\vec{y}$ to indicate a generic M-dimensional vector where M is the number of degrees of freedom of the system before constraints are applied. Let $\{dy_k\}$, or $d\vec{y}$, be a set of displacements of the various degrees of freedom. Let there be j constraint equations of the form $G_p(\vec{y}) - C_p = 0$, $p = 1, \ldots, j$. Each constraint equation specifies a (M-1)-dimensional surface in $\vec{y}$ space. Alternately, we may write

$$0 = dG_p = \sum_k \frac{\partial G_p}{\partial y_k}\, dy_k = \vec{\nabla} G_p \cdot d\vec{y}$$

The above relation says that any differential displacement $d\vec{y}$ that satisfies the constraints must be perpendicular to the gradient of G_p. The gradient of G_p thus defines the normal

vector to the constraint surface. The subspace containing the allowed displacements is called a **tangent subspace**; for $M = 3$, the reason of the naming is clear: the condition $\vec{\nabla} G_p \cdot d\vec{y} = 0$ specifies a plane tangent to the constraint surface at the point $\vec{y}$. With j constraints, we have j constraint surfaces; a solution $\vec{y}^c$ that satisfies the constraints must lie on the intersection of all these surface. Allowed displacements $d\vec{y}$ from the point $\vec{y}^c$ must be perpendicular to all the $\{\vec{\nabla} G_p\}$ at the point $\vec{y}^c$.

Now, let's consider the minimization criterion on a function $H(\vec{y})$, $dH = 0$. In the vector notation, a point $\vec{y}^m$ provides an extremum value of H subject to the constraints if

$$0 = dH = \sum_k \frac{\partial H}{\partial y_k}(\{y_k^m\})\, dy_k = \vec{\nabla} H(\vec{y}^m) \cdot d\vec{y}$$

for all $d\vec{y}$ that satisfy

$$0 = \vec{\nabla} G_p(\vec{y}^m) \cdot d\vec{y}$$

for all p. Note the requirement on the $d\vec{y}$: we only need to consider $d\vec{y}$ that satisfy all the constraints. The condition is satisfied if

$$\vec{\nabla} H(\vec{y}^m) = \sum_p \lambda_p(\vec{y}^m) \vec{\nabla} G_p(\vec{y}^m) \tag{2.27}$$

We can see that this expression is sufficient to minimize H while respecting the constraint condition by simply calculating $\vec{\nabla} H(\vec{y}^m) \cdot d\vec{y}$:

$$\vec{\nabla} H(\vec{y}^m) \cdot d\vec{y} = \left[\sum_p \lambda_p(\vec{y}^m) \vec{\nabla} G_p(\vec{y}^m)\right] \cdot d\vec{y} = \sum_p \lambda_p(\vec{y}^m) \left[\vec{\nabla} G_p(\vec{y}^m) \cdot d\vec{y}\right] = 0$$

where the last equality holds because we are only consideringq $d\vec{y}$ that satisfy the constraints, which mathematically is the statement $\vec{\nabla} G_p(\vec{y}^m) \cdot d\vec{y} = 0$ for all p as explained earlier.

How do we see that the expression is necessary, that it is the minimal possible expression for $\vec{\nabla} H(\vec{y}^m)$? That is straightforward to see geometrically. Geometrically, the constraints require that the allowed $d\vec{y}$ lie in the intersection of the j tangent subspaces created by the constraints. In order to have $dH = 0$ subject to the constraints, the gradient of H must point "out of" this intersection of subspaces – it can have no component "along" the intersection of tangent subspaces. To stay out of this intersection, $\vec{\nabla} H(\vec{y}^m)$ must have a nonzero projection along at least one $\vec{\nabla} G_p(\vec{y}^m)$. We can use proof by contradiction to see this. Suppose $\vec{\nabla} H(\vec{y}^m)$ has zero projection along every $\vec{\nabla} G_p(\vec{y}^m)$. Then $\vec{\nabla} H(\vec{y}^m)$ would be perpendicular to all $\vec{\nabla} G_p(\vec{y}^m)$, which would imply that $\vec{\nabla} H(\vec{y}^m)$ lies in all the tangent subspaces, which implies that it lies in the intersection of the tangent subspaces. That is exactly what we do *not* want. So it must be false to assume that $\vec{\nabla} H(\vec{y}^m)$ has zero projection along every $\vec{\nabla} G_p(\vec{y}^m)$. If that is false, then the expression we have written down is the minimal one that allows $\vec{\nabla} H(\vec{y}^m)$ to have nonzero projection along at least one $\vec{\nabla} G_p(\vec{y}^m)$. Not all the λ_p need be nonzero, only one has to be nonzero.[10]

[10] We shall see later that a given λ_p vanishes when the corresponding constraint force vanishes. That happens if no force is needed to enforce the constraint. For example, if a particle is restricted to live on the plane $x = 0$, is subject to gravity in the z direction, and is given initial condition $x = 0$, then it will continue to satisfy $x = 0$ for all time with no constraint force applied.

The undetermined $\{\lambda_p\}$ can be found because we now have M minimization equations (the M components of Equation 2.27) and j constraint equations (the j equations $\{G_p(\vec{y}) - C_p = 0\}$), enough information to find the M components of $\vec{y}^m$ and the j parameters $\{\lambda_p\}$.

Let us summarize. In the absence of constraints, a location $\{y_k^m\}$ that is a local extremum of a function $H(\{y_k\})$ of a set of M variables $\{y_k\}$ is the location such that each partial derivative at that location vanishes; *i.e.*, $\frac{\partial H}{\partial y_k}(\{y_k^m\}) = 0$ for all k. With constraints, a location that is a local extremum of H is instead a point where $\frac{\partial H}{\partial y_k}(\{y_k^m\}) = \sum_p \lambda_p \frac{\partial G_p}{\partial y_k}(\{y_k^m\})$ and $\{G_p(\{y_k^m\}) - C_p = 0\}$. We have enough equations to find $\{y_k^m\}$ and $\{\lambda_p\}$.

It is not yet clear why the $\{\lambda_p\}$ are called Lagrange multipliers. Here we explain why. Let us define a new function

$$H'(\{y_k\}) = H(\{y_k\}) - \sum_p \lambda_p \left[G_p(\{y_k\}) - C_p\right]$$

Because of the constraints, $H' = H$. But let us seek to minimize H' without applying any constraints. The reader may wonder why we want to do this; but, certainly, we are free to define any function we like and minimize it with or without constraints as we like. The question is whether such a procedure is relevant to our initial problem. An extremum of H' is found when $dH' = 0$:

$$0 = dH' = \sum_k \frac{\partial H}{\partial y_k}\, dy_k - \sum_p \lambda_p \frac{\partial G_p}{\partial y_k}\, dy_k - \sum_p \left[G_p(\{y_k\}) - C_p\right]\, d\lambda_p$$

We have so far not applied the constraints. Let us continue to ignore the constraints and simply assume that all the $\{dy_k\}$ and the $\{d\lambda_p\}$ are independent. Then we end up with $M + j$ equations:

$$\begin{aligned} \frac{\partial H}{\partial y_k} - \sum_p \lambda_p \frac{\partial G_p}{\partial y_k} &= 0 \\ G_p(\{y_k\}) - C_p &= 0 \end{aligned}$$

That is, we recover the minimization condition we found above by direct means, and we recover the constraint equations that we did not apply. To summarize: *When we desire to minimize the function $H(\{y_k\})$ in the presence of constraints $\{G_p(\{y_k\}) - C_p = 0\}$, we can obtain the solution by instead minimizing $H' = H - \sum_p \lambda_p \left[G_p(\{y_k\}) - C_p\right]$ without any constraints among the $\{y_k\}$ and treating the $\{\lambda_p\}$ as additional independent variables.* The whole process is sort of magical, but it is entirely rigorous because we have proved that minimization of H' with respect to the $\{y_k\}$ and $\{\lambda_p\}$ without constraints yields the *same equations* as minimizing H with respect to the $\{y_k\}$ with the constraints applied. If the same equations hold, then the solution must also be the same.

Example 2.7

Let $H(x, y) = x^2 + y^2$. Find the minimum of H subject to the constraint $G(x, y) = y - 2x = 1$.

With such a simple problem, the way you are used to doing it is to solve $G(x, y)$ for y as a function of x, substitute into $H(x, y)$ to make it a function of x only, minimize with respect to x, and finally use the constraint equation to find y at the minimum in x.

To do the problem with Lagrange multipliers, we define

$$\begin{aligned} H'(x,y) &= H(x,y) + \lambda\,[G(x,y) - C] \\ &= x^2 + y^2 + \lambda\,[y - 2x - 1] \end{aligned}$$

Our three equations (2 degrees of freedom plus 1 constraint) are

$$\begin{aligned} 0 &= \frac{\partial H'}{\partial x} = 2\,x - 2\,\lambda \\ 0 &= \frac{\partial H'}{\partial y} = 2\,y + \lambda \\ 0 &= \frac{\partial H'}{\partial \lambda} = y - 2x - 1 \end{aligned}$$

The solution is $x = -\frac{2}{5}$, $y = \frac{1}{5}$, $\lambda = -\frac{2}{5}$.

Lagrange Multipliers in Variational Problems

With the basics of Lagrange multipliers in hand, let's consider the purely mathematical problem of deriving Euler's equation in the presence of constraints between the degrees of freedom. We consider only holonomic constraints at this point.

Our constraint equations are, as before, of the form

$$G_p(\{y_k\}, x) - C_p(x) = 0$$

(Holonomic constraints in mechanics are allowed to be time-dependent, which is why the $\{C_p\}$ are allowed to have x arguments here.) Now, suppose we consider a variation in $\{y_k\}$, $\{\delta y_k\}$. The $\{\delta y_k\}$ are functions of the independent variable x; the quantities $\{\delta y_k(x)\}$ for fixed x are just like the differentials $\{dy_k\}$ we discussed earlier for the calculus minimization problem. Applying the variation $\{\delta y_k(x)\}$ to the constraint yields the j equations

$$\sum_k \frac{\partial G_p(x)}{\partial y_k}\,\delta y_k(x) = 0 \qquad \text{at each } x \text{ independently}$$

So, just as before, the constraints restrict the variations to follow the tangent subspaces of surfaces in $\{y_k\}$ space. The complication here is that, if there is x-dependence in the constraints, these surfaces change with x. But that is only a technical complication; at each value of x, the picture we have from before of constraint surfaces continues to hold.

Now consider the variational extremization problem. We want to find the solution $\{y_k^m\}$ such that for small variations $\{\delta y_k\}$ away from this solution, the variation δI in the functional

$$I[\{y_k\}] = \int_{x_0}^{x_1} dx\, F\left(\{y_k\}, \left\{\frac{dy_k}{dx}\right\}, x\right)$$

vanishes. As we saw before, the requirement and the variation are

$$0 = \delta I = \int_{x_0}^{x_1} dx \sum_{k=1}^{M} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy_k}{dx}}\right)\right] \delta y_k$$

where the sum is over all the degrees of freedom. In the absence of constraints, we used the fact that the $\{\delta y_k\}$ were all independent of each other and arbitrary to infer that the

bracketed quantity for each term in the sum must vanish separately at any value of the independent variable x; this gave independent Euler equations for each y_k that held at all values of x. With constraints, the $\{\delta y_k\}$ no longer are independent and so we cannot infer that each term in the sum vanishes. But, because the constraints are still holonomic, we can infer that the $\{\delta y_k\}$ at different values of x are independent (modulo smoothness limitations). Another way to look at this is that, because the constraints are holonomic, the derivatives $\{\frac{dy_k}{dx}\}$ do not enter the constraints, and so the constraints enforce no relationship between the $\{y_k\}$ at different values of x. The constraints are "instantaneous" relations, not differential equations. Thus, it holds that the entire sum must vanish at any given value of x:

$$0 = \sum_{k=1}^{M} \left[\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial \frac{dy_k}{dx}} \right) \right] \delta y_k \tag{2.28}$$

Note that the $\{\delta y_k\}$ are retained because the constraints imply the $\{\delta y_k\}$ are not all independent. Now that we have an "instantaneous" condition, we can apply the same logic as for the calculus problem. For any given x and $\{y_k\}$, the differential versions of the constraints restrict the $\{\delta y_k\}$ to a subspace at $\{y_k\}$ that satisfies $\sum_k \frac{\partial G_p}{\partial y_k} \delta y_k = 0$ for all p. The above condition Equation 2.28 says that the "vector"

$$\frac{\delta F}{\delta y_k} \equiv \frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial \frac{dy_k}{dx}} \right)$$

must be "perpendicular" to this subspace. (The quantity $\frac{\delta F}{\delta y_k}$ is called the **variational derivative**.) Thus, just as before, the variational derivative must be a linear combination of the gradients of the constraints:

$$\frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial \frac{dy_k}{dx}} \right) = \frac{\delta F}{\delta y_k} = \sum_p \lambda_p(x) \frac{\partial G_p}{\partial y_k}$$

The $\{\lambda_p(x)\}$ are of course the Lagrange multipliers for this problem. They are functions of x because, while the above relation has been derived to hold at each x separately, it is possible for the $\{G_p\}$ to vary with x and thus for the $\{\lambda_p\}$ to vary. Think of it as a set of M equations, similar to Equation 2.27, for each value of x. Thus, the equations that determine the system are

$$\begin{aligned} \frac{\partial F}{\partial y_k} - \frac{d}{dx} \left(\frac{\partial F}{\partial \frac{dy_k}{dx}} \right) - \sum_p \lambda_p(x) \frac{\partial G_p}{\partial y_k} &= 0 \\ G_p(\{y_k\}, x) - C_p(x) &= 0 \end{aligned} \tag{2.29}$$

Now, just as before we realized we could derive the analogous set of relations for the calculus minimization problem by redefining the function to be minimized by adding the constraints with Lagrange multipliers as coefficients and treating the $\{y_k\}$ and $\{\lambda_p\}$ as a larger set of unconstrained, independent variables, we can do a similar thing here. Define

$$\begin{aligned} I'[\{y_k\}] &= I[\{y_k\}] - \int_{x_0}^{x_1} dx \sum_p \lambda_p(x) \left[G_p(\{y_k\}, x) - C_p(x) \right] \\ &= \int_{x_0}^{x_1} dx \left\{ F\left(\{y_k\}, \left\{ \frac{dy_k}{dx} \right\}, x \right) - \sum_p \lambda_p(x) \left[G_p(\{y_k\}, x) - C_p(x) \right] \right\} \end{aligned}$$

As before, $I'[\{y_k\}] = I[\{y_k\}]$ by construction. Now, let us allow *independent* variations $\{\delta y_k\}$ and $\{\delta\lambda_p\}$ and require the resulting variation $\delta I'$ to vanish without application of any constraints (which we are free to do, though as before the relevance is not yet clear). Doing so only requires us to apply the Euler equations, which yields

$$\begin{aligned} \frac{\partial F}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy_k}{dx}}\right) - \sum_p \lambda_p(x)\,\frac{\partial G_p}{\partial y_k} &= 0 \\ G_p(\{y_k\},x) - C_p &= 0 \end{aligned}$$

The first set of equations comes from the $\{\delta y_k\}$ terms, the second from the $\{\delta\lambda_p\}$ terms. We thus recover the equations we obtained directly. Together, the $M+j$ equations are enough to determine the j multipliers $\{\lambda_p(x)\}$ and the motion $\{y_k(x)\}$ in the original M coordinates. To state it in a manner similar to how we did in the simple calculus case, *When we desire to minimize the functional $I[\{y_k\}]$ in the presence of constraints $\{G_p(\{y_k\},x) - C_p(x) = 0\}$, we can obtain the solution by instead minimizing $I' = I - \int_{x_0}^{x_1} dx\, \sum_p \lambda_p\,[G_p(\{y_k\}) - C_p(x)]$ without any constraints among the $\{y_k\}$ and treating the $\{\lambda_p(x)\}$ as additional independent degrees of freedom.*

Lagrange Multipliers in Physical Situations

The application of the preceding material to variational dynamics is straightforward; about all we need to do is replace F by L and x by t. We rewrite the Euler-Lagrange equations with Lagrange multipliers simply for completeness. We have the M Euler-Lagrange equations

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \frac{dq_k}{dt}}\right) + \sum_l \lambda_l \frac{\partial G_l}{\partial q_k} = 0 \tag{2.30}$$

and j holonomic constraint equations

$$G_l(\{q_k\},t) = C_l$$

(The sign on the $\{\lambda_l\}$ is arbitrary, we have changed it so that the $\{\lambda_l\}$ will have the right sign to correspond to constraint forces, as we will later see.)

Example 2.7

Treat the simple pendulum using Lagrange multipliers. That is, rather than selecting the single degree of freedom θ, retain the two nonindependent Cartesian coordinates x and y and apply a constraint between them using Lagrange multipliers.

In terms of the x and y coordinates of the pendulum bob, the Lagrangian is

$$L = \frac{1}{2}\,m\,\left(\dot{x}^2 + \dot{y}^2\right) - m\,g\,y$$

The x and y origin is the pendulum pivot point, x points to the right and y points upward. The constraint equation is

$$G(x,y) = \sqrt{x^2+y^2} = l$$

where l is the length of the pendulum. The resulting Euler-Lagrange equations are

$$\begin{aligned} -m\,\ddot{x} + \lambda\frac{x}{l} &= 0 \\ -m\,g - m\,\ddot{y} + \lambda\frac{y}{l} &= 0 \end{aligned}$$

where we have not only used the constraint to calculate $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ but we have also subsituted $l = \sqrt{x^2 + y^2}$ where possible. We have three equations for the three unknown functions $x(t)$, $y(t)$, and $\lambda(t)$. To understand the meaning of λ, let's solve the problem using vector mechanics and Newton's second law in cartesian coordinates. The forces acting on the mass are gravity and the tension in the rope, $\mathcal{T}$. Newton's second law indicates

$$\begin{aligned} m\,\ddot{x} &= -\mathcal{T}\sin\theta = -\mathcal{T}\,\frac{x}{l} \\ m\,\ddot{y} &= \mathcal{T}\cos\theta - m\,g = -\mathcal{T}\,\frac{y}{l} - m\,g \end{aligned}$$

(Note that $y < 0$, and I think Hand and Finch have a sign error on the cos term in Equation 2.38). Clearly, the multiplier takes on the role of the tension, $\lambda = -\mathcal{T}$, which acts as a constraint force in this problem. The sign difference between the two occurs because $\vec{\mathcal{T}}$ points in the opposite direction as the position coordinate.

Lagrange Multipliers and Constraint Forces

The above example points to a more general principle that Lagrange multipliers and constraint forces are intimately related.

Let us return to the concept of virtual work and d'Alembert's principle, Equation 2.6:

$$\sum_i \left[\sum_j \vec{F}_{ij}^{(nc)} - \dot{\vec{p}}_i\right] \cdot \delta\vec{r}_i = 0$$

On the left side, we earlier had the sum running over all the forces, but we eliminated constraint forces from the sum by assuming that any given constraint force results in zero virtual work from a virtual displacement. We can of course add any subset back in, as long as it is a subset of terms that yield zero virtual work. Let us do so, and also move the $\dot{\vec{p}}_i$ term to the other side:

$$\sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i = \sum_i \left[\sum_j \vec{F}_{ij}^{(nc)}\right] \cdot \delta\vec{r}_i + \sum_m \left[\sum_n \vec{F}_{mn}^{(c)}\right] \cdot \delta\vec{r}_m$$

The use of m and n indices for the constraint force term is only intended to highlight the fact that we may only look at constraint forces on some subset of particles (the m index) and we may only look at a subset of the constraint forces acting on a given particle (the n index). Though, one must be careful to always include all necessary terms pertaining to a given constraint (*e.g.*, in the Atwood's machine problem, there are terms for both masses expressing the rope length constraint). Now, let's rewrite the $\delta\vec{r}_i$ using the chain rule in terms of displacements in generalized coordinates $\{\delta q_k\}$, and also apply the relation between $\dot{\vec{p}}_i$ and derivatives of the kinetic energy, Equation 2.9:

$$\sum_k \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right] \delta q_k = \sum_k \left[\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}\right] \delta q_k = \sum_k \mathcal{F}_k \delta q_k + \sum_k \mathcal{N}_k \delta q_k$$

where we have defined

$$\mathcal{F}_k \equiv \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \qquad \mathcal{N}_k \equiv \sum_{ij} \vec{F}_{ij}^{(c)} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

The definition of $\mathcal{F}_k$ is the same as our original definition of generalized force, Equation 2.5. The definition of $\mathcal{N}_k$ is simply the analogous definition for constraint forces. In order to use a Lagrangian, F_k must be derivable from a potential, $F_k = -\frac{\partial U}{\partial q_k}$. Using this fact, and moving the F_k terms to the left side, lets us rewrite in terms of the Lagrangian:

$$\sum_k \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right]\delta q_k = \sum_k \mathcal{N}_k \delta q_k$$

In the case that the generalized coordinates have been defined so that all constraints have been taken into account, the right side vanishes under our standard assumption that constraint forces do no net work for virtual displacements (*i.e.*, displacements that respect the constraints). Also, when the generalized coordinates account for all the constraints, the δq_k are all independent, so the vanishing of the right side implies that the term for each k on the left side vanishes, which gives us the usual Euler-Lagrange equations.

However, if the generalized coordinates have been defined in a way that they do not incorporate all constraints, then two steps in the above derivation fail: we are not assured the right side vanishes, and we are not assured that the δq_k are all independent. The right side need not vanish because some displacements in this "too large" set of generalized coordinates will violate the constraints and thus allow the constraint forces to do work. The δq_k are not assured to be independent because there are leftover constraints between generalized coordinates.

Fortunately, the Lagrange multiplier procedure allows us to proceed. We know that the terms on the left side are related to the Lagrange multipliers by the Euler-Lagrange equations in the presence of multipliers:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_l}\right) - \frac{\partial L}{\partial q_k} = \sum_p \lambda_p \frac{\partial G_p}{\partial q_k}$$

Substituting this into the left side, we obtain

$$\sum_{k,p} \lambda_p \frac{\partial G_p}{\partial q_k}\,\delta q_k = \sum_k \mathcal{N}_k\,\delta q_k$$

When Lagrange multipliers are used, the δq_k are taken to be independent because the constraints are not enforced at the start but instead fall out of the Euler-Lagrange equations for the $\{\lambda_p\}$ (which do not appear here). So, the two sides must be equal term-by-term, giving

$$\mathcal{N}_k = \sum_p \lambda_p \frac{\partial G_p}{\partial q_k} \tag{2.31}$$

So we see that it is in general true that the Lagrange multipliers give the constraint forces. Using this fact, we can rewrite the Euler-Lagrange equations with the Lagrange multipliers in a more physical form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_l}\right) - \frac{\partial L}{\partial q_k} = \mathcal{N}_k \tag{2.32}$$

That is, when some constraints are not applied at the start and the resulting constraint forces left in the problem, they appear on the right side of the Euler-Lagrange equation

(which normally vanishes). This is very reminiscent of Equation 2.15, where we wrote down how to include *non-conservative* forces in the Euler-Lagrange equation based on going back to the generalized equation of motion, Equation 2.10, as we have done here.

We make a final note that the equation for $\mathcal{N}_k$ implicitly implies that it does not matter how the constraints are written down. For example, with the pendulum we had:

$$\begin{aligned} G(x,y) - C &= \sqrt{x^2 - y^2} - l = 0 \\ \lambda &= -\mathcal{T} \\ \mathcal{N}_x &= \lambda \frac{\partial G}{\partial x} = -\mathcal{T}\,\frac{x}{l} \\ \mathcal{N}_y &= \lambda \frac{\partial G}{\partial y} = -\mathcal{T}\,\frac{y}{l} \end{aligned}$$

But we could have written the constraint differently, resulting in

$$\begin{aligned} G(x,y) - C &= x^2 - y^2 - l^2 = 0 \\ \lambda &= -\frac{1}{2}\,\frac{\mathcal{T}}{l} \\ \mathcal{N}_x &= \lambda \frac{\partial G}{\partial x} = \left(-\frac{1}{2}\,\frac{\mathcal{T}}{l}\right)(2\,x) = -\mathcal{T}\,\frac{x}{l} \\ \mathcal{N}_y &= \lambda \frac{\partial G}{\partial y} = \left(-\frac{1}{2}\,\frac{\mathcal{T}}{l}\right)(2\,y) = -\mathcal{T}\,\frac{y}{l} \end{aligned}$$

We find that λ depends on how the constraint is written, but the constraint force in the generalized coordinate does not (as one would expect!).

Example 2.8

Let's return to the simple pendulum example and calculate the constraint forces.

Sticking with the original x and y coordinates, with the constraint $G(x,y) - C = \sqrt{x^2 + y^2} - l = 0$, we found that the Lagrange multiplier was $\lambda = -\mathcal{T}$. The generalized constraint forces are therefore

$$\begin{aligned} \mathcal{N}_x &= -\mathcal{T}\frac{x}{l} = -\mathcal{T}\sin\theta \\ \mathcal{N}_y &= -\mathcal{T}\frac{y}{l} = \mathcal{T}\cos\theta \end{aligned}$$

as one would expect.

Example 2.9

Consider a hoop of mass M and radius R rolling down an inclined plane (inclination angle α), with the requirement that the axis of rotation always be perpendicular to the slope of the plane – *i.e.*, the hoop is not free to pivot about its point of contact with the plane. Find the equations of motion, the constraint forces, and the angular acceleration.

Define the coordinate system such that x points down the inclined plane and θ describes the rotation of the hoop. The kinetic energy consists of two pieces, one translational and one rotational:

$$T = \frac{1}{2}\,M\dot{x}^2 + \frac{1}{2}\,MR^2\,\dot{\theta}^2$$

The potential energy is

$$U = -M\,g\,x\,\sin\alpha$$

The constraint equation is

$$x - R\,\theta = 0$$

Let's first solve the problem by eliminating the unnecessary degrees of freedom using the constraint. Let's eliminate θ, so then

$$L = M\,\dot{x}^2 + M\,g\,x\,\sin\alpha$$

The Euler-Lagrange equation is

$$\begin{aligned} M\,g\,\sin\alpha - \frac{d}{dt}\,(2\,M\,\dot{x}) &= 0 \\ \ddot{x} = \frac{1}{2}\,g\,\sin\alpha & \end{aligned}$$

The angular acceleration is found by using the constraint equation

$$\ddot{\theta} = \frac{\ddot{x}}{R} = \frac{1}{2}\,\frac{g\,\sin\alpha}{R}$$

Now, let's repeat using Lagrange multipliers. The only reason we do this is so we can find the constraint forces. Our three equations are

$$\begin{aligned} \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda\frac{\partial G}{\partial x} &= 0 \\ \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial G}{\partial \theta} &= 0 \\ G(x,\theta) &= 0 \end{aligned}$$

Written out explicitly, they are

$$\begin{aligned} M\,g\,\sin\alpha - \frac{d}{dt}\,M\,\dot{x} + \lambda &= 0 \\ -\frac{d}{dt}\,M\,R^2\,\dot{\theta} - R\,\lambda &= 0 \\ x - R\,\theta &= 0 \end{aligned}$$

Rearranging gives

$$\ddot{x} = \frac{\lambda}{M} + g\,\sin\alpha \qquad \ddot{\theta} = -\frac{\lambda}{M\,R} \qquad \theta = \frac{x}{R}$$

Solving, we find

$$\ddot{x} = \frac{1}{2}\,g\,\sin\alpha \qquad \ddot{\theta} = \frac{1}{2}\,\frac{g\,\sin\alpha}{R} \qquad \lambda = -\frac{1}{2}\,M\,g\sin\alpha$$

The constraint forces are

$$\begin{aligned} \mathcal{N}_x &= \lambda\,\frac{\partial G}{\partial x} = -\frac{1}{2}\,M\,g\sin\alpha \\ \mathcal{N}_\theta &= \lambda\,\frac{\partial G}{\partial \theta} = \frac{1}{2}\,M\,R\,g\sin\alpha \end{aligned}$$

The x constraint force acts to counter the gravitational acceleration along x, while the θ constraint force exerts the torque needed to make the hoop roll rather than slide.

2.2.4 Incorporating Nonholonomic Constraints in Variational Dynamics

So far we have required constraints be holonomic to make use of them either via substitution and removal of variables or to apply the Lagrange multiplier formalism. In this section, we show how some nonholonomic constraints can be incorporated.

Inequality Constraints

We give an example of how to deal with one type of inequality constraint. We make no claim this is a generic method, but it is instructive.

Consider a pointlike particle sitting on top of a hemisphere of radius R. Let the coordinate system origin be at the center of the hemisphere. The particle thus satisfies $r \geq R$. Suppose the particle is placed at rest at the top of the hemisphere and given an infinitesimal nudge to get it sliding down the hemisphere. We want to determine the dynamics of the particle, specifically the polar angle at which it leaves the hemisphere. The polar angle is taken to be zero at the top of the hemisphere.

We will solve the problem by treating the constraint as an exact equality constraint, $r = R$, and then finding at what angle the constraint force enforcing the constraint – as given by the corresponding Lagrange multiplier – goes to zero. It will become clear that this is the point at which the particle leaves the hemisphere.

The Lagrangian and constraint equation are

$$L = \frac{1}{2}\, m\, \dot{r}^2 + \frac{1}{2}\, m\, r^2\, \dot{\theta}^2 - m\, g\, r\, \cos\theta \qquad\qquad r - R = 0$$

The resulting Euler-Lagrange equations with Lagrange multipliers are:

$$\begin{aligned} r: &\quad m\, r\, \dot{\theta}^2 - m\, g\, \cos\theta - m\, \ddot{r} + \lambda = 0 \\ \theta: &\quad m\, g\, r\, \sin\theta - \frac{d}{dt}\left(m\, r^2\, \dot{\theta}\right) = 0 \\ \lambda: &\quad r - R = 0 \end{aligned}$$

We use the λ equation to substitute for r (also using $\dot{r} = 0$) to obtain

$$\begin{aligned} r: &\quad \lambda = m\, g\, \cos\theta - m\, R\, \dot{\theta}^2 \\ \theta: &\quad m\, R^2 \ddot{\theta} = m\, g\, r\, \sin\theta \end{aligned}$$

We cannot solve the system analytically. But it is clear that λ tells us the force that the hemisphere must exert to counter gravity acting on the particle, after subtracting off the apparent centrifugal force due to the particle's circular motion. λ is positive when the particle is on the hemisphere, but it will go to zero and become negative when the particle leaves the hemisphere. (The equations obviously become invalid as soon as λ goes negative because the constraint is no longer valid.) So we simply need to find the θ at which $\lambda = 0$. So we want $\dot{\theta}$ as a function of θ. We can obtain that by conservation of energy:

$$\begin{aligned} m\, g\, R &= m\, g\, R\, \cos\theta + \frac{1}{2}\, m\, R^2 \dot{\theta}^2 \\ \dot{\theta}^2 &= 2\, \frac{g}{R}\, (1 - \cos\theta) \end{aligned}$$

We could also obtain the same result by a clever integration of the θ equation of motion:

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$
$$\ddot{\theta}\, d\theta = \dot{\theta}\, d\dot{\theta}$$

We can replace $\ddot{\theta}$ on the LHS using the θ equation of motion and then integrate both sides:

$$\frac{g}{R}\int_0^\theta \sin\theta' d\theta' = \frac{1}{2}\left(\dot{\theta}'\right)^2\Big|_0^{\dot{\theta}}$$
$$\frac{g}{R}\left(1-\cos\theta\right) = \frac{1}{2}\dot{\theta}^2$$

With $\dot{\theta}$ written in terms of θ, we can reduce the r equation and obtain λ as a function of θ:

$$\lambda = m\,g\,\cos\theta - m\,R\left[2\,\frac{g}{R}\left(1-\cos\theta\right)\right]$$
$$= m\,g\,\left(3\,\cos\theta - 2\right)$$

So $\lambda = 0$ when $\cos\theta = 2/3$. One can obtain the complete dynamics by calculating $\dot{\theta}$ at this angle and then using this position and velocity as the initial condition for continued motion with no constraint and subject to gravity. That would, for example, let one figure out exactly where the particle hits the ground.

Nonintegrable Constraints

Here we have to step back further into variational dynamics. If it is possible to write the constraint as a differential relation between coordinates, then it is possible to incorporate the constraint in the variation of the action because the variation of the action contains factors δy_k for the variation of the coordinates. The constraint(s) can be incorporated via direct substitution or via Lagrange multipliers.

The problem we will consider is a disk rolling down an inclined plane without slipping, with the disk's motion allowed to have a component transverse to the slope. (If the disk may only roll straight down the slope, the constraint is holonomic – we just did this case in the previous example).

Let the x axis point transverse to the slope and the y axis down the slope. Let ϕ denote the rotation angle of the disk of radius R about its rotation axis, θ denote the angle between the disk's rotation axis and the y axis. Let α be the slope of the plane.

The Lagrangian for the problem is

$$L = \frac{1}{2}MR^2\left[\frac{3}{2}\dot{\phi}^2 + \frac{1}{4}\dot{\theta}^2\right] - m\,g\,y\,\sin\alpha$$

This is the Lagrangian one obtains immediately if one calculates the rotational kinetic energy using the point of contact with the slope as the instantaneous axis of rotation.[11]

[11] One could have also calculated in terms of translation of and rotation about the center of mass and then used the velocity constraints below to rewrite the center-of-mass translation in terms of rotation, which would yield the same L. One might wonder why one should eliminate $\dot{x}$ and $\dot{y}$ immediately using the velocity constraint rather than allowing the Lagrange multiplier procedure to deal with them. One could leave them untouched, it will just result in more equations to deal with later that will reduce to the ones we will eventually find. Feel free to check it yourself.

The constraint equation is based on the non-slip condition:

$$\begin{aligned} \dot{x} &= R\,\dot{\phi}\,\sin\theta \\ \dot{y} &= R\,\dot{\phi}\,\cos\theta \end{aligned}$$

$R\dot{\phi}$ gives the speed of the edge of the disk, and the sin and cos break this motion down into components along the x and y axes. Let's first examine why the constraints are nonholonomic and nonintegrable. The differential versions of the constraints are

$$\begin{aligned} dx &= R\,d\phi\,\sin\theta \\ dy &= R\,d\phi\,\cos\theta \end{aligned}$$

Suppose there were a constraint of the form $f(x, y, \phi, \theta, t) = 0$. The differential would satisfy $df = 0$, *i.e.*,

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial \phi}\,d\phi + \frac{\partial f}{\partial \theta}\,d\theta + \frac{\partial f}{\partial t}\,dt = 0$$

The differential versions of the constraints imply

$$\frac{\partial f}{\partial x} = 1 \quad \frac{\partial f}{\partial \phi} = R\,\sin\theta \quad \frac{\partial f}{\partial y} = 0 \quad \frac{\partial f}{\partial \theta} = 0$$

But then we have

$$\frac{\partial^2 f}{\partial\theta\,\partial\phi} = R\,\cos\theta \quad \frac{\partial^2 f}{\partial\phi\,\partial\theta} = 0$$

That is, the partial derivatives of f do not commute. This is not possible for a smooth function, so we must conclude that f does not exist.

Though the constraints are nonintegrable, we may still make use of the differential versions given above by incorporating them when calculating the variation of the action. When we incorporated a constraint $G(x, y) - C = 0$ by adding to the Lagrangian a term $\lambda\,[G(x, y) - C]$, it resulted in the addition of a term to the variation of the action involving $\lambda \sum_k \frac{\partial G}{\partial y_k}\,\delta y_k$. So let's just directly incorporate a similar term directly in the variation of the action:

$$\delta S = \int_{t_0}^{t_1} dt\,\left\{\left[\left(\frac{\partial L}{\partial y}\right)\delta y + \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}\right)\delta\phi + \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}\right)\delta\theta\right] + \lambda\,(\delta y - R\,\delta\phi\,\cos\theta)\right\}$$

where we have dropped the terms that we know will vanish:

$$\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial \dot{x}} = 0 \quad \frac{\partial L}{\partial \dot{y}} = 0 \quad \frac{\partial L}{\partial \phi} = 0 \quad \frac{\partial L}{\partial \theta} = 0$$

and we have therefore included only a Lagrange multiplier for the second constraint because x and $\dot{x}$ do not appear in the Lagrangian. Clearly, the equations of motion will be

$$\begin{aligned} \frac{\partial L}{\partial y} + \lambda &= 0 \\ -\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \lambda\,R\cos\theta &= 0 \\ -\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0 \end{aligned}$$

Alternatively, we could have instead done a direct substitution of the differential constraint to eliminate y from the variation of the action:

$$\delta S = \int_{t_0}^{t_1} dt \left\{ \left(\frac{\partial L}{\partial y} \right) R \, \delta\phi \, \cos\theta + \left(-\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} \right) \delta\phi + \left(-\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \right) \delta\theta \right\}$$

which would have yielded the equations of motion

$$\begin{aligned} \frac{\partial L}{\partial y} R \, \cos\theta - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} &= 0 \\ -\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} &= 0 \end{aligned}$$

Either set of equations appears to be somewhat problematic in that there are three coordinates left – y, ϕ, and θ – and possibly the multiplier λ, and one too few equations. However, the only dependence on y in L will be in the potential energy $U = -m\,g\,y\,\sin\alpha$, so $\frac{\partial L}{\partial y} = m\,g\,\sin\alpha$ and thus y is in practice eliminated from the equations. Had it not been possible to eliminate y from the equations of motion, we clearly would have had an underdetermined system and another constraint would have been necessary.

2.3 Hamiltonian Dynamics

Noether's Theorem brought to the fore the importance of the canonical momentum (introduced in Section 2.1). Treating coordinates and canonical momenta as variables of equal importance, rather than giving preference to coordinates, leads us to Legendre transformations and the Hamiltonian formulation of dynamics. This material is covered by Hand and Finch, Chapter 5.

I will generally follow a formal tack, presenting examples once the general theoretical result is available. Hand and Finch tend to go with examples first, so some may find their approach more intuitive.

2.3.1 Legendre Transformations and Hamilton's Equations of Motion

We begin with a mathematical interlude on Legendre transformations, which will provide us with a means to generate the Hamiltonian function from the Lagrangian. We derive Hamilton's equations of motion and discuss the importance of moving from the $(q, \dot{q})$ phase space to the (q, p) phase space. **Note that the following assumes no constraints – if there existed constraints, they must be removed via defining a new set of unconstrained generalized coordinates using the constraint equations.**

Legendre Transformations

The **Legendre transformation** is a generic technique for generating a new pair of independent variables (x, z) from an initial pair (x, y). The transformation is completely invertible – in fact, applying it twice returns the initial variable pair (x, y). It will become clear that learning about the general Legendre transform is not necessary for deriving the Hamiltonian function. However, it is also no doubt clear that transformations with special properties are of physical importance. Legendre transformations will become of more general use in the next section when we discuss advanced Hamiltonian mechanics.

Consider a function $A(x, y)$ of two variables x and y. Define a new independent variable z and the function $B(x, y, z)$ by

$$B(x, y, z) \equiv y\, z - A(x, y) \tag{2.33}$$

If we have small changes dx, dy, and dz in x, y, and z, then B changes by

$$\begin{aligned} dB &= y\, dz + z\, dy - \left.\frac{\partial A}{\partial x}\right|_y dx - \left.\frac{\partial A}{\partial y}\right|_x dy \\ &= -\left.\frac{\partial A}{\partial x}\right|_y dx + \left(z - \left.\frac{\partial A}{\partial y}\right|_x\right) dy + y\, dz \end{aligned}$$

where we explicitly state which variables are held fixed when each partial derivative is taken. We can make B a function of x and z only (*i.e.*, eliminate any explicit dependence on y) by making the coeffiicient of dy vanish, which is accomplished by defining

$$z \equiv \left.\frac{\partial A}{\partial y}\right|_x \tag{2.34}$$

The other partial derivatives of B are

$$\left.\frac{\partial B}{\partial x}\right|_z = -\left.\frac{\partial A}{\partial x}\right|_y \qquad \left.\frac{\partial B}{\partial z}\right|_x = y \tag{2.35}$$

B is computed explicitly by inverting Equation 2.34 to find $y(x,z)$ and then plugging in $B(x,z) = B(x,y(x,z),z)$. This puts a condition on A that $\frac{\partial A}{\partial y}$ must be an invertible function of y.

The variable x is called the **passive** variable and y the **active** variable because of their different roles in the transform.

So, what's so great about this particular method for finding a new variable z and a new function $B(x,z)$?

- The process is completely invertible and is in fact its own inverse. If you repeat the Legendre transform on $B(x,z)$, the new variable you find is just y as indicated above by the fact that $y = \frac{\partial B}{\partial z}\big|_x$. The invertibility ensures that no information is lost in the transformation; the self-invertibility is just nice.
- There is a very nice geometric interpretation. Fix the x variable for a second. The new variable z is the slope of the tangent to $A(x,y)$ when considered as a function of y only. Since the tangent line must have value $A(x,y)$ at ordinate value y, the tangent line equation is $yz + b = A(x,y)$ where b is the intersection of the tangent line with the vertical axis. But, by our Legendre transformation, $b = -B(x,z)$. That is, the Legendre transform is a mapping, for each x, from ordinate values y to tangent slope values z, and from the function $A(x,y)$ to the vertical-axis intercept of the line tangent to $A(x,y)$ in the y direction, which is $B(x,z)$.

The Hamiltonian and Hamilton's Equations of Motion

Now, let us apply the Legendre transformation to the Lagrangian $L(q,\dot{q},t)$ with $\dot{q}$ as the active variable and q and t as passive variables. As a preliminary, let us state the conditions for the following derivation:

- We must be able to write a Lagrangian; *i.e.*, there must be a potential energy function. To date we have considered only conservative (position-dependent) potentials, possibly with explicit time dependence (*i.e.*, conservative at any instant in time). Though it is possible to write a Lagrangian with a velocity-dependent potential, such cases will not be considered here. Systems with dissipation (friction) also cannot be described by a Lagrangian.
- All coordinates must be unconstrained; the Legendre transformation does not carry through if there are constrained coordinates. If constraints do exist, it is necessary to use them to rewrite the Lagrangian in terms of unconstrained, independent generalized coordinates.

Following our Legendre transformation formalism, the new active variable will be $p = \frac{\partial L}{\partial \dot{q}}$ and the new function will be

$$H(q,p,t) \equiv \dot{q}\,p - L(q,\dot{q},t)$$

The new function is called the **Hamiltonian** and is now a function of q and p (and t) instead of q and $\dot{q}$ (and t).

We may derive equations of motion for q and p in terms of the Hamiltonian rather than for q in terms of the Lagrangian by making use of the Euler-Lagrange equations and the

properties of the Legendre transformation. Our starting point is the four equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$
$$p = \frac{\partial L}{\partial \dot{q}} \qquad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \qquad \dot{q} = \frac{\partial H}{\partial p}$$

The first equation is just the Euler-Lagrange equation in q and $\dot{q}$. The second equation is the definition of the canonical momentum, which is also the definition of the z variable in the Legendre transformation. The third equation is the relation between partial derivatives of the initial function $A(x, y)$ and the new function $B(x, z)$ with respect to the passive x variable in the Legendre transformation. The final equation arises from figuring out what $\frac{\partial B}{\partial z}$ is in the Legendre transformation or, equivalently, realizing that repeating the Legendre transformation returns the original active variable y. Inserting the second equation in the first and using the third equation, and simply copying the fourth equation, we obtain

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{q} = -\frac{\partial H}{\partial p}$$

which are **Hamilton's equations of motion**.

We can also arrive at Hamilton's equations via a variational principle. If we simply rewrite the variation of the Lagrangian δL in terms of H, we find

$$\begin{aligned}
\delta L &= \dot{q}\,\delta p + p\,\delta\dot{q} - \delta H \\
&= \dot{q}\,\delta p - \dot{p}\,\delta q + \frac{d}{dt}(p\,\delta q) - \left.\frac{\partial H}{\partial q}\right|_p \delta q - \left.\frac{\partial H}{\partial p}\right|_q \delta p \\
&= \left(\dot{q} - \left.\frac{\partial H}{\partial p}\right|_q\right)\delta p - \left(\dot{p} + \left.\frac{\partial H}{\partial q}\right|_p\right)\delta q + \frac{d}{dt}(p\,\delta q)
\end{aligned}$$

As usual, the last term contributes nothing because it is a total differential and $\delta q(t_0) = \delta q(t_1) = 0$. We can invoke the Legendre transformation to explain why the first term vanishes, and this gives us the $\dot{q}$ Hamilton's equation. That leaves us with the δq term; since δq is arbitrary, its coefficient must vanish identically in order for the action to be minimized, which gives us the $\dot{p}$ Hamilton's equation. In this version of the derivation, we invoked the properties of the Legendre transformation (which is pure mathematics) but we did *not* use the Euler-Lagrange equations. Basically, instead of keeping δL in terms of q and $\dot{q}$ and deriving the Euler-Lagrange equation, we wrote it in terms of q and p and obtained Hamilton's equations.

A final approach would be to postulate a new variational principle, which is that q and p should be considered independent, with arbitrary independent variations δq and δp, subject only to the requirement that δq vanish at the endpoints. To minimize the action, then, the coefficients of δq and δp would have to vanish separately, yielding the $\dot{p}$ and $\dot{q}$ Hamilton's equations, respectively. It is self-consistent to forsake the information from the Legendre transformation relation for $\dot{q}$: if this information were kept, it would create a relation between $\frac{d}{dt}\delta q$ and δp, which would violate the assumption of independent variations of q and p.

Independence of q and p and the Hamiltonian Point of View

The last approach above is the true Hamiltonion point of view, in which p is on equal footing to q and the two are treated as independent variables. Some further explanation of why it is justified to assume they are indepedent is warranted, and we can use our discussion of Lagrange multipliers as an analogy.

Recall how, with Lagrange multipliers, we found we could obtain the same equations of motion in two ways:

- Assume the j constraints from the start, using them to define new unconstrained generalized coordinates with $M - j$ degrees of freedom.
- Forget the j constraints at the start, and instead write a new Lagrangian with additional constraint terms $\lambda_j \left[G_j(\{q_k\}, t) - C_p\right]$ and treat the original M coordinates and the j new $\{\lambda_p\}$ coordinates as unconstrained. The equations of motion obtained are a set of M Euler-Lagrange equations and j constraint equations that we proved (recall the geometric argument about the gradient of the constraint functions) are mathematically equivalent to the $M - j$ Euler-Lagrange equations found by applying the constraints at the start. The constraint equations, rather than being explicitly applied from the start, "fall out" of the formalism via the assumption that the λ_p should be treated as dynamical coordinates.

The second formulation was not *a priori* valid – it was valid because we proved that it resulted in a mathematically equivalent set of equations.

Following that example, we make a similar argument for Lagrangian vs. Hamiltonian mechanics. The two mathematically equivalent points of view are

- Lagrangian: assume δq and $\delta \dot{q}$ are related by a time derivative $\frac{d}{dt}\delta q = \delta \dot{q}$ and derive an equation of motion (the Euler-Lagrange equation). Since the equation is a second-order differential equation, it requires two initial conditions ($q(0)$ and $\dot{q}(0)$). The Euler-Lagrange equation can be rewritten via a Legendre transformation as the two sets of first-order Hamiltonian equations. The extra equation $\dot{q} = \left.\frac{\partial H}{\partial p}\right|_q$ is not dynamical, but is rather part of the Legendre transformation – it is analogous to the constraint equations in the Lagrange multiplier formalism. The equation for $\dot{p}$ is the one containing the physics of the Euler-Lagrange equations. The two first-order equations require two initial conditions, so there has been no change of the amount of information.
- Hamiltonian: define a new coordinate $p = \left.\frac{\partial L}{\partial \dot{q}}\right|_q$ and forget the relation between δp and δq at the start, just as we forgot the constraints between coordinates at the start of the Lagrange multiplier formalism. We can then treat q and p as independent variables. As noted above, this assumption would lead us to the same Hamilton's equations, which are identical to the ones we arrive at even if we do not assume this independence. Not only do we arrive at Hamilton's equations, but the first equation, $\dot{q} = \left.\frac{\partial H}{\partial p}\right|_q$, which was initially entirely mathematical based on the definition of p via the Legendre transformation, "falls out" of the formalism by treating q and p as independent variables; the first equation is now in some sense "dynamical" just as the constraint equations became dynamical in the Lagrange multiplier formalism.

The change in point of view may seem strange, but, just as with Lagrange multipliers, we will see later that the new point of view has advantages.

Hamilton's Equations for Multiple Dimensions

It is straightforward to generalize the Legendre transformation to multiple new variables, which yields the Hamiltonian

$$H = \sum_k p_k \, \dot{q}_k - L \tag{2.36}$$

$$H = H(\{q_k\}, \{p_k\}, t) \tag{2.37}$$

The formalism of the previous section follows through for each k separately (recall, we assumed the $\{q_k\}$ were unconstrained generalized coordinates). Let us also calculate the relation between the time derivatives. Just as with the $\{q_k\}$, time is a passive variable in the Legendre transformation, so we are guaranteed

$$\left.\frac{\partial H}{\partial t}\right|_{\{q_k\},\{p_k\}} = - \left.\frac{\partial L}{\partial t}\right|_{\{q_k\},\{\dot{q}_k\}}$$

The total time derivative of H is

$$\begin{aligned} \frac{dH}{dt} &= \sum_k \frac{\partial H}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial H}{\partial p_k} \dot{p}_k + \frac{\partial H}{\partial t} \\ &= \sum_k \frac{\partial H}{\partial q_k} \frac{\partial H}{\partial p_k} - \sum_k \frac{\partial H}{\partial p_k} \frac{\partial H}{\partial q_k} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} \end{aligned}$$

where the second line arises via application of Hamilton's equations. Thus, our generic Hamilton's equations become

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{2.38}$$

Recall that we had proven the last equation in Section 2.1 using the Euler-Lagrange equations. A couple of useful facts, which we also saw before in Section 2.1:

- If there is no explicit time dependence in the Lagrangian (and therefore in the Hamiltonian), then H is conserved (as indicated by the above).
- If the kinetic energy is quadratic in the generalized velocity and there is a conservative potential energy U, then the Hamiltonian is the total energy, $H = T + U$, because the $\sum_k p_k \, \dot{q}_k = 2T$. (See Problem 1.9 in Hand and Finch.)

Examples

Hamilton's equations in practice offer little or no advantage over the Euler-Lagrange equations – instead of M second-order equations with $2M$ initial conditions, we have $2M$ coupled first-order equations with $2M$ initial conditions. The power of the Hamiltonian formulation will become apparent when we use it to prove important theorems of classical mechanics. But, regardless, let us do some examples to elucidate the usage of Hamilton's equations. The critical point to remember is that H **is a function of** $\{q_k\}$ **and** $\{p_k\}$**; while you must go through an intermediate step where** H **is written in terms of** $\{q_k\}$**,** $\{\dot{q}_k\}$ **and** $\{p_k\}$**, you must eliminate the** $\{\dot{q}_k\}$ **before applying Hamilton's equations of**

motion. We refer the reader to Hand and Finch for two more examples, a particle sliding on a parabolic wire and a spherical pendulum.

Example 2.10

Simple harmonic oscillator. You know

$$L = \frac{1}{2}\,m\,\dot{x}^2 - \frac{1}{2}\,k\,x^2$$
$$p = \frac{\partial L}{\partial \dot{x}} = m\,\dot{x}$$

The Hamiltonian is therefore

$$\begin{aligned} H &= \dot{x}\,p - L \\ &= \frac{1}{m}\,p^2 - \frac{1}{2\,m}\,p^2 + \frac{1}{2}\,k\,x^2 \\ &= \frac{1}{2\,m}\,p^2 + \frac{1}{2}\,k\,x^2 \end{aligned}$$

The equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = -k\,x$$

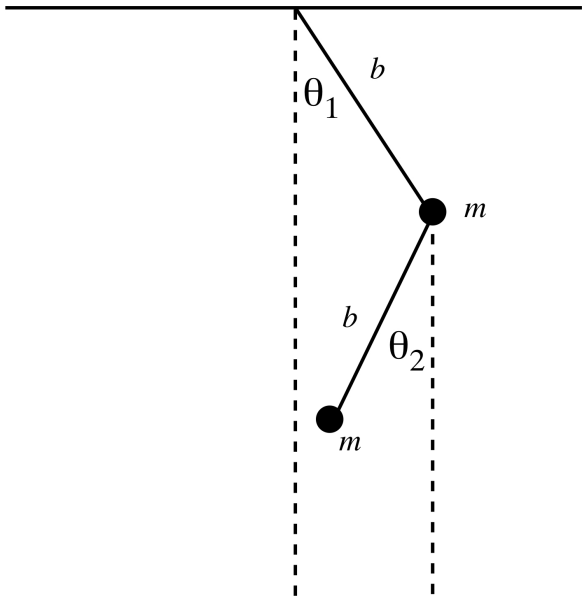
We recover $\dot{x}$ and we obtain Hooke's law, which is just Newton's second law for a spring. We can differentiate the first equation and substitute into the second to obtain

$$m\,\ddot{x} + k\,x = 0$$

which is the Euler-Lagrange equation we would have obtained from the Lagrangian formalism.

Example 2.11

Double pendulum. Consider the double pendulum shown in the following figure, with two masses m and the two pendulum rods both having length b:



The two generalized coordinates are θ_1 and θ_2. The Lagrangian is most easily found by first noting

$$\begin{aligned}\vec{r}_1 &= \hat{x}\, b\, \sin\theta_1 - \hat{y}\, b\, \cos\theta_1 \\ \vec{r}_2 &= \vec{r}_1 + \hat{x}\, b\, \sin\theta_2 - \hat{y}\, b\, \cos\theta_2\end{aligned}$$

The above form makes it easier to evaluate the Lagrangian:

$$\begin{aligned}L &= \frac{1}{2}\, m\, \left(\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_1 + \dot{\vec{r}}_2 \cdot \dot{\vec{r}}_2\right) - m\, g\, (\vec{r}_1 + \vec{r}_2) \cdot \hat{y} \\ &= \frac{1}{2}\, m\, \left(b^2\, \dot{\theta}_1^2 + b^2\, \dot{\theta}_1^2 + b^2\, \dot{\theta}_2^2 + 2\, b^2\, \left[\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right] \dot{\theta_1}\, \dot{\theta_2}\right) \\ &\quad + m\, g\, (b\, \cos\theta_1 + b\, \cos\theta_1 + b\, \cos\theta_2) \\ &= \frac{1}{2}\, m\, b^2\, \left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\, \dot{\theta}_1\, \dot{\theta}_2\, \cos\left(\theta_1 - \theta_2\right)\right) + m\, g\, b\, \left(2\cos\theta_1 + \cos\theta_2\right)\end{aligned}$$

The canonical momenta are

$$\begin{aligned}p_1 &= \frac{\partial L}{\partial \dot{\theta}_1} = m\, b^2\, \left(2\, \dot{\theta}_1 + \dot{\theta}_2\, \cos\left(\theta_1 - \theta_2\right)\right) \\ p_2 &= \frac{\partial L}{\partial \dot{\theta}_2} = m\, b^2\, \left(\dot{\theta}_2 + \dot{\theta}_1\, \cos\left(\theta_1 - \theta_2\right)\right)\end{aligned}$$

Notice the strange form of the momenta due to the coupling of the motions in the two angles. Rewriting the generalized velocities in terms of the momenta gives

$$\begin{aligned}\dot{\theta}_1 &= \frac{1}{2\, m\, b^2}\, \frac{p_1 - p_2\, \cos\left(\theta_1 - \theta_2\right)}{1 - \frac{1}{2}\cos^2\left(\theta_1 - \theta_2\right)} \\ \dot{\theta}_2 &= \frac{1}{m\, b^2}\, \frac{p_2 - \frac{1}{2}\, p_1\, \cos\left(\theta_1 - \theta_2\right)}{1 - \frac{1}{2}\cos^2\left(\theta_1 - \theta_2\right)}\end{aligned}$$

We can rewrite the Lagrangian in a convenient form

$$L = \frac{1}{2}\left(p_1\, \dot{\theta}_1 + p_2 \dot{\theta}_2\right) + m\, g\, b\, \left(2\cos\theta_1 + \cos\theta_2\right)$$

The Hamiltonian is then

$$\begin{aligned}H &= p_1\, \dot{\theta}_1 + p_2\, \dot{\theta}_2 - L \\ &= \frac{1}{2}\left(p_1\, \dot{\theta}_1 + p_2 \dot{\theta}_2\right) - m\, g\, b\, \left(2\cos\theta_1 + \cos\theta_2\right) \\ &= \frac{1}{4\, m\, b^2}\, \frac{p_1^2 + 2\, p_2^2 - 2\, p_1\, p_2\, \cos\left(\theta_1 - \theta_2\right)}{1 - \frac{1}{2}\, \cos^2\left(\theta_1 - \theta_2\right)} - m\, g\, b\, \left(2\cos\theta_1 + \cos\theta_2\right)\end{aligned}$$

We have already derived the first set of Hamilton's equations of motion, so let's derive the

second pair:

$$\begin{aligned}
\dot{p}_1 &= -\frac{\partial H}{\partial \theta_1} \\
&= \frac{1}{2\,m\,b^2}\frac{p_1\,p_2\,\sin(\theta_1-\theta_2)}{1-\frac{1}{2}\,\cos^2(\theta_1-\theta_2)} \\
&\quad -\frac{1}{4\,m\,b^2}\frac{p_1^2+2\,p_2^2-2\,p_1\,p_2\,\cos(\theta_1-\theta_2)}{\left(1-\frac{1}{2}\,\cos^2(\theta_1-\theta_2)\right)^2}\,\cos(\theta_1-\theta_2)\,\sin(\theta_1-\theta_2) \\
&\quad +2\,m\,g\,b\sin\theta_1 \\
\dot{p}_2 &= -\frac{\partial H}{\partial \theta_1} \\
&= -\frac{1}{2\,m\,b^2}\frac{p_1\,p_2\,\sin(\theta_1-\theta_2)}{1-\frac{1}{2}\,\cos^2(\theta_1-\theta_2)} \\
&\quad +\frac{1}{4\,m\,b^2}\frac{p_1^2+2\,p_2^2-2\,p_1\,p_2\,\cos(\theta_1-\theta_2)}{\left(1-\frac{1}{2}\,\cos^2(\theta_1-\theta_2)\right)^2}\,\cos(\theta_1-\theta_2)\,\sin(\theta_1-\theta_2) \\
&\quad +m\,g\,b\sin\theta_1
\end{aligned}$$

This is clearly a case where not only do we gain nothing from the Hamiltonian point of view, but in fact, it is more complicated! Had we just used the Euler-Lagrange equations, we would have found

$$\begin{aligned}
-2\,m\,g\,b\,\sin\theta_1 - \frac{d}{dt}\left(2\,m\,b^2\dot{\theta}_1 + m\,b^2\,\dot{\theta}_2\ \cos(\theta_1-\theta_2)\right) &= 0 \\
-m\,g\,b\,\sin\theta_2 - \frac{d}{dt}\left(m\,b^2\dot{\theta}_1 + m\,b^2\,\dot{\theta}_1\,\cos(\theta_1-\theta_2)\right) &= 0
\end{aligned}$$

which reduce to

$$\begin{aligned}
2\,g\,\sin\theta_1 + 2\,b\,\ddot{\theta}_1 + b\,\ddot{\theta}_2\,\cos(\theta_1-\theta_2) - b\,\dot{\theta}_1\left(\dot{\theta}_1-\dot{\theta}_2\right)\sin(\theta_1-\theta_2) &= 0 \\
g\,\sin\theta_2 + b\,\ddot{\theta}_2 + b\,\ddot{\theta}_1\,\cos(\theta_1-\theta_2) - b\,\dot{\theta}_2\left(\dot{\theta}_1-\dot{\theta}_2\right)\sin(\theta_1-\theta_2) &= 0
\end{aligned}$$

which, while no more analytically solvable than the Hamiltonian version, are certainly much simpler.

2.3.2 Phase Space and Liouville's Theorem

We present the concept of phase space and derive Liouville's theorem on the conservation of phase space density. Analogies to fluid mechanics and classical wave mechanics and quantum mechanics are drawn.

Phase Space

Phase space is simply the $2M$-dimensional space with axes $\{q_k\}$ and $\{p_k\}$. Phase space becomes particularly useful in the Hamiltonian formulation of dynamics because the $\{q_k\}$ and the $\{p_k\}$ are now the degrees of freedom and Hamilton's equations relate time derivatives of the coordinates to partial derivatives in phase space of the Hamiltonian.

Geometrically, you can think of the Hamiltonian as a set of surfaces in phase space, on each of which the Hamiltonian function is constant. You can also think of a single particle as

moving through this space. If the Hamiltonian is conserved – *i.e.*, if there is no explicit time dependence in the Hamiltonian – then these surfaces remain fixed in phase space. A particle moves along these surfaces. In fact, Hamilton's equation tells us the direction of motion in phase space. If we think of phase space as having unit vectors $\hat{q}$ and $\hat{p}$ in the q and p directions, then we have

$$\vec{\nabla}_{qp} H = \hat{q}\,\frac{\partial H}{\partial q} + \hat{p}\,\frac{\partial H}{\partial p} \qquad \hat{q}\,\dot{q} + \hat{p}\,\dot{p} = \hat{q}\,\frac{\partial H}{\partial p} - \hat{p}\,\frac{\partial H}{\partial q}$$
$$\implies \quad \vec{\nabla}_{qp} H \cdot (\hat{q}\,\dot{q} + \hat{p}\,\dot{p}) = 0$$

The vector $\hat{q}\,\dot{q} + \hat{p}\,\dot{p}$ is the direction in which the particle's position in phase space evolves. So we see that the direction of evolution is normal to the phase-space gradient out of the surface of constant H. In this case, we are thinking about a 2-dimensional phase space, so the evolution direction is uniquely determined (up to a sign) by requiring that it be perpendicular to $\vec{\nabla}_{qp} H$, as it has to be for H to be conserved.

The classic example is the 1-D simple harmonic oscillator. The value of H, denoted by E, specifies an ellipse in the 2-D phase space with semimajor axes $x_m = \sqrt{2\,E/k}$ and $p_m = \sqrt{2\,E\,m}$. The vector $\vec{\nabla}_{qp} H$ is the outward normal to the ellipse at any given point. Thus, the velocity vector is the tangent to the ellipse at the point, indicating that the position in phase space simply evolves along the ellipse.

In more than one spatial dimension, one obtains the generalization of the above equations:

$$\vec{\nabla}_{qp} H = \sum_k \left[\hat{q}_k\,\frac{\partial H}{\partial q_k} + \hat{p}_k\,\frac{\partial H}{\partial p_k}\right] \qquad \sum_k \left[\hat{q}_k\,\dot{q}_k + \hat{p}_k\,\dot{p}_k\right] = \sum_k \left[\hat{q}_k\,\frac{\partial H}{\partial p_k} - \hat{p}_k\,\frac{\partial H}{\partial q_k}\right]$$
$$\implies \vec{\nabla}_{qp} H \cdot \sum_k \left[\hat{q}_k\,\dot{q}_k + \hat{p}_k\,\dot{p}_k\right] = 0$$

This statement is simpler than it looks. Notice how the different dimensions (the index k) decouple: the gradient has some projection into a given $q_k - p_k$ plane, as does the velocity vector, and those projections are individually orthogonal. So the geometric interpretation is analogous to that we had for a single spatial dimension, where the velocity vector is proportional to the tangent to surfaces of constant H in each 2-D phase-space projection.

If the Hamiltonian is like a SHO in all dimensions (coupled or uncoupled), then the motion is simply ellipses in each 2-D phase space projection. For two spatial dimensions, this can be visualized as the two axes of a 2-torus. Some path is followed on the torus that is a combination of circulatory motion in the two dimensions separately.

Systems of particles can be considered to be a "gas" or "fluid" moving in phase space. The way the fluid evolves through phase space over time is described by Hamilton's equations.

Liouville's Theorem

One of the most interesting things about Hamiltonian dynamics and phase space is Liouville's theorem, which basically says that not only can a system of particles be treated like a fluid in phase space, but it also holds that the fluid has incompressible flow.[12] This has interesting consequences, and can even be considered a precursor of the uncertainty principle in quantum mechanics. We derive Liouville's theorem in this section.

[12] This is not the same as saying the fluid is incompressible; we shall discuss this point in detail in the next section.

Let's consider a system of particles with so many particles that they can be viewed as defining a density field in phase space. The defining relation is that, the number of particles with phase space coordinates in some differential volume $d^M q\, d^M p$ at a point $(\{q_k\}, \{p_k\})$ is given by

$$dN = \rho(\{q_k\}, \{p_k\}, t)\, d^M q\, d^M p$$

where there is in general time dependence as the particles move through phase space over time under the influence of Hamilton's equations. Correspondingly, the total number of particles is just the integral over phase space

$$N = \int d^M q\, d^M p\, \rho(\{q_k\}, \{p_k\}, t)$$

Conservation of particles requires the time derivative of the integral to vanish.

Now, we want to determine the evolution of the phase space density $\rho(\{q_k\}, \{p_k\}, t)$, which we will write as $\rho(\vec{q}, \vec{p}, t)$, *as we flow along with the particles*. That is, we don't want to sit at some fixed point in phase space and see how the phase-space density at the point varies with time; instead, we want to follow the particles along their trajectories in phase space and see how the density around the trajectory changes with time.

Since our definition of ρ is actually via a volume,

$$\rho(\vec{q}, \vec{p}, t) = \frac{dN}{d^M q\, d^M p}$$

and we want to follow a fixed set of particles as they move through phase space, the numerator is held constant. We calculate the change in density by calculating the change in the volume $d^M q\, d^M p$ that they occupy. To do this, consider $2\,M + 1$ particles that follow trajectories in phase space given by

- $(\vec{q}_0(t), \vec{p}_0(t))$
- $(\vec{q}_{2k-1}(t), \vec{p}_{2k-1}(t)) = (\vec{q}_0(t), \vec{p}_0(t)) + \hat{q}_k\, dq_k(t)$
- $(\vec{q}_{2k}(t), \vec{p}_{2k}(t)) = (\vec{q}_0(t), \vec{p}_0(t)) + \hat{p}_k\, dp_k(t)$

where $k = 1 \ldots M$. The $2M$ particles are separated from the reference particle at $(\vec{q}_0(t), \vec{p}_0(t))$ by time-varying vectors $\hat{q}_k\, dq_k(t)$ and $\hat{p}_k\, dp_k(t)$. They are at the various corners of a small volume element $d^M q\, d^M p$. If we calculate how their separation evolves in time, we will have calculated how the volume element evolves in time (under the assumption that the time dt is short enough that the particles follow simple straight paths). That evolution is

$$\begin{aligned}
dq_k(t+dt) - dq_k(t) &= [q_{2k-1,k}(t+dt) - q_{0,k}(t+dt)] - [q_{2k-1,k}(t) - q_{0,k}(t)] \\
&= [q_{2k-1,k}(t+dt) - q_{2k-1,k}(t)] - [q_{0,k}(t+dt) - q_{0,k}(t)] \\
&= \left.\frac{dq_k}{dt}\right|_{\vec{q}_{2k-1}, \vec{p}_{2k-1}} dt - \left.\frac{dq_k}{dt}\right|_{\vec{q}_0, \vec{p}_0} dt \\
&= \left.\frac{dq_k}{dt}\right|_{\vec{q}_0 + \hat{q}_k\, dq_k, \vec{p}_0} dt - \left.\frac{dq_k}{dt}\right|_{\vec{q}_0, \vec{p}_0} dt \\
&= \frac{\partial \dot{q}_k}{\partial q_k}\, dq_k\, dt
\end{aligned}$$

and similarly

$$dp_k(t+dt) - dp_k(t) = \frac{\partial \dot{p_k}}{\partial p_k}\, dp_k\, dt$$

where we drop the explicit indication of where the derivatives are evaluated because at this point the difference between evaluating at any of the corners of the volume element $d^M q d^M p$ is a second-order differential. Rewriting these in multiplicative fashion so we can apply them to the volume element and using Hamilton's equations gives

$$\begin{aligned} dq_k(t+dt) &= \left(1 + \frac{\partial \dot{q_k}}{\partial q_k}\, dt\right) dq_k(t) = \left(1 + \frac{\partial^2 H}{\partial q_k\, \partial p_k}\, dt\right) dq_k(t) \\ dp_k(t+dt) &= \left(1 + \frac{\partial \dot{p_k}}{\partial p_k}\, dt\right) dp_k(t) = \left(1 - \frac{\partial^2 H}{\partial p_k\, \partial q_k}\, dt\right) dp_k(t) \end{aligned}$$

So, now we can calculate the evolution of ρ:

$$\begin{aligned} &\rho(\vec{q}(t+dt), \vec{p}(t+dt), t+dt) - \rho(\vec{q}(t), \vec{p}(t), t) \\ &= \frac{dN}{d^M q(t+dt)\, d^M p(t+dt)} - \frac{dN}{d^M q(t)\, d^M p(t)} \\ &= \frac{dN}{d^M q(t)\, d^M p(t)} \left(\frac{1}{\prod_k \left(1 + \frac{\partial^2 H}{\partial q_k\, \partial p_k}\, dt\right)\left(1 - \frac{\partial^2 H}{\partial p_k\, \partial q_k}\, dt\right)} - 1 \right) \\ &= \frac{dN}{d^M q(t)\, d^M p(t)} \sum_k \left(-\frac{\partial^2 H}{\partial q_k\, \partial p_k} + \frac{\partial^2 H}{\partial p_k\, \partial q_k} \right) dt \\ &= 0 \end{aligned}$$

where in the next-to-last step we have Taylor expanded the denominator, discarding any products of the second-order partial derivatives, and in the last step we have made use of the fact that partial derivatives of the Hamiltonian commute. If we divide both sides by dt, we have on the left side the total time derivative of ρ.

Thus, we have **Liouville's Theorem** for the evolution of the phase space density,

$$\frac{d\rho}{dt}(\{q_k\}, \{p_k\}, t) = 0 \tag{2.39}$$

The phase space density is constant along the particle trajectories.

The Fluid Interpretation of Liouville's Theorem

To see clearly the fluid interpretation of Liouville's theorem, we must expand out the differential above or the total derivative using the chain rule: If we write out the total derivative using the chain rule, we have

$$\rho(\vec{q}(t+dt), \vec{p}(t+dt), t+dt) - \rho(\vec{q}(t), \vec{p}(t), t) = \frac{\partial \rho}{\partial t}\, dt + \sum_k \left[\frac{\partial \rho}{\partial q_k}\, \dot{q_k}\, dt + \frac{\partial \rho}{\partial p_k}\, \dot{p_k}\, dt \right]$$

or

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_k \left[\frac{\partial \rho}{\partial q_k}\, \dot{q_k} + \frac{\partial \rho}{\partial p_k}\, \dot{p_k} \right]$$

The second term can be rewritten to give

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \left[\dot{\vec{q}} \cdot \vec{\nabla}_q + \dot{\vec{p}} \cdot \vec{\nabla}_p\right] \rho = \frac{\partial \rho}{\partial t} + \dot{\vec{\xi}} \cdot \vec{\nabla}_\xi \rho$$

where in the first step we have simply rewritten the sums over k as dot products (note the q and p subscripts on $\vec{\nabla}$, which specify which variables the gradient is taken with respect to), and in the second step we have defined a new composite coordinate $\vec{\xi}$, with

$$\xi_k = \begin{cases} q_{(k+1)/2} & k = \text{odd} \\ p_{k/2} & k = \text{even} \end{cases}$$

This is known as *symplectic notation*; we will return to this term later. Four points can be made about this form:

- The derivative expression given is completely analogous to the one we would find in fluid mechanics, where $\dot{\vec{\xi}}$ would be replaced by the fluid velocity $\vec{v}$ and the gradient $\vec{\nabla}_\xi$ would be replaced by the simple spatial gradient $\vec{\nabla}_r$. The derivative operator $\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}_r$ is generically called the "convective", "advective", "Lagrangian", "substantial", "substantive", "material", or "Stokes" derivative due to its use in fluid mechanics to calculate differentials "moving along" with the fluid.
- Liouville's theorem can be rewritten in a slightly different way using the convective form. Let's separate the two pieces of the convective derivative:

  $$0 = \frac{d\rho}{dt} \quad \Longrightarrow \quad 0 = \frac{\partial \rho}{\partial t} + \dot{\vec{\xi}} \cdot \vec{\nabla}_\xi \rho \quad \Longrightarrow \quad \frac{\partial \rho}{\partial t} = -\dot{\vec{\xi}} \cdot \vec{\nabla}_\xi \rho$$

  The last form says that if, instead of moving along with the phase space flow, you sit at one point in phase space and watch the phase space density at that point change with time, then the rate at which it changes, $\frac{\partial \rho}{\partial t}$, is given by the negative of the gradient of the phase space density along the flow direction multiplied by the flow speed.
- Liouville's theorem is not a trivial result of conservation of particle number. Conservation of particle number simply states

  $$\frac{\partial}{\partial t} \int_V \rho \, dV_{qp} + \int_S dA_{qp} \; \hat{n} \cdot \rho \dot{\vec{\xi}} = 0$$

  i.e., the rate of change of the number of particles in a volume V in phase space is just related to the net flow into the volume through its surface S. Gauss's theorem lets us rewrite the above as

  $$\frac{\partial}{\partial t} \int_V \rho \, dV_{qp} + \int_V dV_{qp} \; \vec{\nabla}_\xi \cdot \left(\rho \dot{\vec{\xi}}\right) = 0$$

  Since the volume is not changing in time (we consider a volume V fixed in phase space, not one moving with the flow), we may move the time derivative inside the integral:

  $$\int_V dV_{qp} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla}_\xi \cdot \left(\rho \dot{\vec{\xi}}\right)\right] = 0$$

  Finally, because the volume V is arbitrary, the integrand must vanish at any point in phase space:

  $$\frac{\partial \rho}{\partial t} + \vec{\nabla}_\xi \cdot \left(\rho \dot{\vec{\xi}}\right) = 0$$

The above equation is the **continuity equation** and simply states conservation of particle number. We need to know nothing about Hamiltonian dynamics to derive it To get from the continuity equation to Liouville's theorem, let's expand out the divergence term:

$$0 = \frac{\partial \rho}{\partial t} + \vec{\nabla}_\xi \cdot \left(\rho \, \dot{\vec{\xi}} \right) = \frac{\partial \rho}{\partial t} + \dot{\vec{\xi}} \cdot \vec{\nabla}_\xi \rho + \rho \left(\vec{\nabla}_\xi \cdot \dot{\vec{\xi}} \right)$$

We need for the third term to vanish to obtain Liouville's theorem. It turns out that it vanishes because of Hamilton's equations:

$$\vec{\nabla}_\xi \cdot \dot{\vec{\xi}} = \sum_k \left[\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \dot{p}_k}{\partial p_k} \right] = \sum_k \left[\frac{\partial^2 H}{\partial q_k \, \partial p_k} - \frac{\partial^2 H}{\partial p_k \, \partial q_k} \right] = 0$$

Thus, we are left with

$$0 = \frac{\partial \rho}{\partial t} + \dot{\vec{\xi}} \cdot \vec{\nabla}_\xi \rho$$

i.e., Liouville's theorem. Liouville's theorem is thus a consequence of both conservation of particle number and Hamilton's equations. We implicitly assumed conservation of particle number in our derivation: we followed particular particles along, assuming that they could not vanish. And we had to use Hamilton's equations in our original derivation of Liouville's theorem. So we have obtained an alternate derivation of Liouville's theorem, though one that rests on the same physics. The derivation is actually really the same in that one can make a term-by-term correspondence between the two; the language we use here is just more sophisticated and results in a more compact derivation.

- Combining the above two points, we can see that there is a very nice analogy between phase space flow and incompressibility in fluid mechanics. There are two kinds of incompressibility in fluid mechanics, and it is the latter one that corresponds to phase space flow:
  - *incompressible fluid*: An incompressible fluid has an unchangeable density: $\rho =$ constant. Therefore, $\vec{\nabla}_r \rho = 0$, $\frac{\partial \rho}{\partial t} = 0$, and $\frac{d\rho}{dt} = 0$. We can use the continuity equation to also infer that $\vec{\nabla}_r \cdot \vec{v} = 0$:

    $$0 = \frac{\partial \rho}{\partial t} + \vec{\nabla}_r \cdot (\rho \, \vec{v}) = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}_r \rho + \rho \, \vec{\nabla}_r \cdot \vec{v}$$

    Incompressibility of the fluid implies that the first two terms vanish, so (assuming we don't have a trivial fluid with $\rho = 0$)

    $$\vec{\nabla}_r \cdot \vec{v} = 0$$

    We thus see than an incompressible fluid is very uninteresting. The phase space density is *not* an incompressible fluid.
  - *incompressible flow*: The definition of a fluid with incompressible flow is

    $$\vec{\nabla}_r \cdot \vec{v} = 0$$

    We derived above that the phase space flow obeys the analogous equation thanks to Hamilton's equations. Thus, there is a perfect correspondence between the

phase space fluid and a fluid with incompressible flow. Just as we derived Liouville's theorem from the continuity equation with the additional condition based on Hamilton's equations that $\vec{\nabla}_\xi \cdot \dot{\vec{\xi}} = 0$, we could derive the analogy of Liouville's theorem for incompressible flows:

$$0 = \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla}_r \rho$$

We note than an incompressible fluid always has incompressible flow (because we showed above $\vec{\nabla}_r \cdot \vec{v} = 0$ for an incompressible fluid), but of course a fluid with incompressible flow need not be an incompressible fluid.

Liouville's Pseudo-Theorem

What happens when a system does not obey Hamilton's equations? How does the phase space density evolve? Returning to the derivation, had we not used Hamilton's equations, we would have found

$$\begin{aligned}
&\rho(\vec{q}(t+dt), \vec{p}(t+dt), t+dt) - \rho(\vec{q}(t), \vec{p}(t), t) \\
&= \frac{dN}{d^M q(t)\, d^M p(t)} \left(\frac{1}{\prod_k \left(1 + \frac{\partial \dot{q_k}}{\partial q_k}\, dt\right) \left(1 + \frac{\partial \dot{p_k}}{\partial p_k}\, dt\right)} - 1 \right) \\
&= -\frac{dN}{d^M q(t)\, d^M p(t)} \sum_k \left(\frac{\partial \dot{q_k}}{\partial q_k} + \frac{\partial \dot{p_k}}{\partial p_k} \right) dt \\
&= -\rho(\vec{q}(t), \vec{p}(t), t) \sum_k \left(\frac{\partial \dot{q_k}}{\partial q_k} + \frac{\partial \dot{p_k}}{\partial p_k} \right) dt
\end{aligned}$$

which can be rewritten

$$\frac{d\rho}{dt} + \rho \sum_k \left(\frac{\partial \dot{q_k}}{\partial q_k} + \frac{\partial \dot{p_k}}{\partial p_k} \right) = 0$$

which is solved by

$$\rho(\vec{q}(t), \vec{p}(t)) = \rho(\vec{q}(0), \vec{p}(0)) \exp\left(-\int_0^t dt' \sum_k \left[\frac{\partial \dot{q_k}}{\partial q_k} + \frac{\partial \dot{p_k}}{\partial p_k} \right] \right)$$

That is, the phase space density either shrinks or expands depending on the sign of the "deviation from Hamiltonianness".

Liouville's Theorem and Future Topics

We rewrite the total derivative of the phase space density in a form that we will come back to later:

$$\begin{aligned}
\frac{d\rho}{dt} &= \frac{\partial \rho}{\partial t} + \sum_k \left[\frac{\partial \rho}{\partial q_k} \dot{q_k} + \frac{\partial \rho}{\partial p_k} \dot{p_k} \right] \\
&= \frac{\partial \rho}{\partial t} + \sum_k \left[\frac{\partial \rho}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial \rho}{\partial p_k} \frac{\partial H}{\partial q_k} \right]
\end{aligned}$$

Again, in the second line we have made use of Hamilton's equations to calculate the time derivatives of the coordinates and momenta. The quantity in brackets in the second term is

the **Poisson bracket** of ρ and H. For those of you who have taken quantum mechanics, the above expression is similar to the expression for the total time derivative of an observable; observables that commute with the Hamiltonian and have no explicit time dependence are conserved. We will explore the parallels between Poisson brackets and quantum mechanical commutators more in the next section.

Finally, we point out the interesting fact that Liouville's theorem foreshadows the uncertainty principle of quantum mechanics. Liouville's theorem allows you to trade volume between the q dimension and the p dimension. This is analogous to the quantum mechanical uncertainty relation $\Delta p \Delta x = \hbar$. If you squeeze the volume of a system in position space (say, by focusing a beam of charged particles so it has a small transverse spatial extent), then you pay a price in increase dispersion in momentum space (the transverse momenta become spread over a large range of values). This is not the uncertainty principle *per se* because each particle's position and momentum are perfectly determined. But the ensemble acts something like a quantum mechanical wave function. The reason this occurs is because Liouville's theorem makes the ensemble act in some ways like a classical wave, and classical waves suffer diffraction when their phase space in some dimension is restricted. The uncertainty principle in quantum mechanics is the same phenomenon, which arises because we treat particles as wavefunctions rather than ideal points in quantum mechanics.

2.4 Topics in Theoretical Mechanics

In this last section on formal analytical mechanics, we discuss canonical transformations, Poisson Brackets, action-angle variables, and the Hamilton-Jacobi equation. These are primarily theoretical developments that are interesting primarily as a prelude to the development of quantum mechanics. Those continuing to work in dynamical systems will find practical use for this material.

This material derives from Hand and Finch Chapter 6 and Goldstein Chapter 9, but many of the derivations differ have significant additional detail and the order is different. Thornton covers none of this material.

2.4.1 Canonical Transformations and Generating Functions

Canonical transformations are transformations of the coordinates and momenta (q, p) that preserve Hamilton's equations (though with a different Hamiltonian). They can be generated by **generating functions**, which are classifiable into four types.

Types of Transformations

The transformations we have worked with to date, transformations that affect only the coordinates, are of the form

$$Q = Q(q(t), t)$$

These are called **point transformations**. It can be shown that under a point transformation, a system that obeys the Euler-Lagrange equations in the original coordinates continues to obey them in the new coordinates (though with a new Lagrangian).

More generally, we can consider **contact transformations**, which are of the form

$$Q = Q(q(t), p(t), t) \qquad P = P(q(t), p(t), t)$$

An arbitrary contact transformation may not preserve Hamilton's equations. We denote those contact transformations that do preserve Hamilton's equations as **canonical transformations**.

Generating Functions

We can derive an algorithm for generating canonical transformations. The algorithm consists, essentially, of constructing the transformation so that Hamilton's equations automatically hold true for the new Hamiltonian as a function of the transformed coordinates and momenta.

In terms of the Hamiltonian, the action integral in the original coordinates is

$$S_{qp} = \int_{t_0}^{t_1} dt \, [p\,\dot{q} - H(q, p, t)]$$

where $_{qp}$ indicates that the integrand is a function of q and p. We consider independent variations δq and δp and require the action be minimized with respect to these variations:

$$\begin{aligned} 0 = \delta S_{qp} &= \int_{t_0}^{t_1} dt \left[p\,\delta\dot{q} + \dot{q}\,\delta p - \frac{\partial H}{\partial q}\,\delta q - \frac{\partial H}{\partial p}\,\delta p \right] \\ &= \int_{t_0}^{t_1} dt \left[\left(-\dot{p} - \frac{\partial H}{\partial q} \right) \delta q + \left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p \right] + [p\,\delta q]|_{t_0}^{t_1} \end{aligned}$$

where we have used $\delta\dot{q} = \frac{d}{dt}\,\delta q$ and done the usual integration-by-parts trick. Normally, we assume the variations δq and δp are independent and arbitrary except for the requirement that $\delta q(t_0) = 0$ and $\delta q(t_1) = 0$; these requirements give us Hamilton's equations.

To guarantee that the Hamiltonian $\widetilde{H}(Q,P)$ in our new coordinates (Q,P) also satisfies Hamilton's equations, we require that the action for this new Hamiltonian is minimized when Hamilton's equations in the new coordinates hold. We first must calculate the variation of the action in the new coordinates:

$$\delta S_{QP} = \int_{t_0}^{t_1} dt \left[\left(-\dot{P} - \frac{\partial \widetilde{H}}{\partial Q} \right) \delta q + \left(\dot{Q} - \frac{\partial \widetilde{H}}{\partial P} \right) \delta P \right] + \left[P\,\delta Q \right]\Big|_{t_0}^{t_1}$$

We want to show that the integral term vanishes for arbitrary variations δQ and δP so that we recover Hamilton's equations in the new Hamiltonian. We shall make the enlightened guess that a *sufficient* condition for that term to vanish is that the *Lagrangians* (not the Hamiltonians) in the two coordinate systems differ by a total derivative of a function $F(q,Q,t)$ and that $P = \frac{\partial F}{\partial Q}$. We shall prove later that this condition is *necessary* and that F fully determines (and is fully determined by) the transformation. Supposing this condition to be true, it implies

$$P\,\dot{Q} - \widetilde{H}(Q,P,t) = p\,\dot{q} - H(q,p,t) - \frac{d}{dt}\,F(q,Q,t)$$

(we have chosen the sign for the total derivative with foreknowledge of what will be convenient later.) Integrating over time thus gives

$$S_{QP} = S_{qp} - F\big|_{t_0}^{t_1}$$

Now, calculate the variation of the action for variations δq and δp such that $\delta q = 0$ at the endpoints. There will be corresponding variations δQ and δP, but we don't know how the condition $\delta q = 0$ at the endpoints translates to δQ and δP at the endpoints. So we have

$$\begin{aligned}
&\int_{t_0}^{t_1} dt \left[\left(-\dot{P} - \frac{\partial \widetilde{H}}{\partial Q} \right) \delta q + \left(\dot{Q} - \frac{\partial \widetilde{H}}{\partial P} \right) \delta P \right] + \left[P\,\delta Q \right]\Big|_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} dt \left[\left(-\dot{p} - \frac{\partial H}{\partial q} \right) \delta q + \left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p \right] + \left[p\,\delta q \right]\Big|_{t_0}^{t_1} - \frac{\partial F}{\partial q}\,\delta q\Big|_{t_0}^{t_1} - \frac{\partial F}{\partial Q}\,\delta Q\Big|_{t_0}^{t_1} \\
&= -\frac{\partial F}{\partial Q}\,\delta Q\Big|_{t_0}^{t_1}
\end{aligned}$$

where in the last line we have made use of $\delta q = 0$ at the endpoints and $\delta S_{qp} = 0$. The partial derivatives of F arise from propagating the variations δq and δQ through F using the chain rule. We want the two surface terms to be equal. Since δQ is arbitrary and will in general not vanish at the endpoints, these terms are equal if and only if $P = -\frac{\partial F}{\partial Q}$. If that is true – which we assumed it would be – then we have the desired condition that the integral in the first line vanishes alone, which implies Hamilton's equations for $\widetilde{H}(Q,P,t)$ hold.

Furthermore, now knowing that Hamilton's equations hold for $\widetilde{H}(Q,P,t)$, we may apply the inverse contact transformation – $q(Q,P,t)$ and $p(Q,P,t)$. Following through the above logic results in the condition $p = \frac{\partial F}{\partial q}$ (a sign flip appears because now $\frac{d}{dt}\,F$ appears with the opposite sign).

To summarize, we have demonstrated that a *sufficient* condition that a contact transformation $(q,p) \leftrightarrow (Q,P)$ be a canonical transformation is that there is a function $F(q,Q,t)$ such that

$$p = \frac{\partial F}{\partial q} \qquad P = -\frac{\partial F}{\partial Q} \tag{2.40}$$
$$\widetilde{L}(Q,\dot{Q},t) = L(q,\dot{q},t) - \frac{d}{dt}F(q,Q,t)$$

We have glossed over one point, though. What does the last line actually mean? If we have a transformation $(q,p) \leftrightarrow (Q,P)$, then should we not obtain $\widetilde{L}(Q,\dot{Q},t)$ by simply writing $L(q,\dot{q},t)$ in terms of Q and $\dot{Q}$ using the transformation; *i.e.*, $\widetilde{L}(Q,\dot{Q},t) = L(q(Q,\dot{Q},t),\dot{q}(Q,\dot{Q},t),t)$ (where we have rewritten the transformation as $(q,\dot{q}) \leftrightarrow (Q,\dot{Q})$ because we want $\dot{q}$ and $\dot{Q}$ for the Lagrangian)? Yes and no. While it is true that this would be the natural way to obtain $\widetilde{L}$, we are free to add a total time derivative of a function of the coordinates to any Lagrangian without affecting the action. So we choose to use the above equation with the additional $\frac{d}{dt}F$ term as the *definition* of the transformed Lagrangian. In addition to providing clarification, this definition has the advantage of reducing the set of assumptions for a transformation to be canonical. Now all we need is that the transformation $(q,p) \leftrightarrow (Q,P)$ be derivable from a function F via the aforementioned partial differential equations, and then we *define* the transformed Lagrangian to include $-\frac{d}{dt}F$.

F is called a **generating function**. The two partial differential equations for p and P give two equations in four variables, q, p, Q, and P, which can be solved to find the transformation $Q(q,p,t)$ and $P(q,p,t)$ (or vice versa) if the mixed second partial derivatives $\frac{\partial^2 F}{\partial q \partial Q}$ and $\frac{\partial^2 F}{\partial Q \partial q}$ are nonzero everywhere.[13]

Moreover, we can now demonstrate that the existence of a generating function as decribed above is a *necessary* condition for a transformation to be canonical; *i.e.*, every canonical transformation is derivable from a generating function. Suppose one has a canonical transformation. One can rewrite the transformation equations in the form $p = p(q,Q,t)$ and $P = P(q,Q,t)$ simply by solving the transformation equations for p and P.[14] These functions we then take to be the partial derivatives of the prospective generating function F; *i.e.*, assume $p(q,Q,t) = \frac{\partial F}{\partial q}$ and $P(q,Q,t) = \frac{\partial F}{\partial Q}$. This is a pair of coupled partial differential equations for F. Does a solution F always exist? Yes, and we can see this as follows.

Let us first consider only time-independent canonical transformations. Consider a closed path in phase space $(q(t),p(t))$ (not necessarily a physically allowed path). For some small displacement along that path, we may calculate $p\,dq = \int dp\,dq$. Viewed geometrically, the path is a curve $p(q)$ in the qp plane and $p\,dq$ is the area under that curve. If we now integrate $p\,dq$ over the entire closed path, $\oint p\,dq$, we obtain the area in qp space enclosed by the path. We will show later that the Jacobian determinant of any canonical transformation is unity.

[13] This condition ensures the transformation $(q,p) \leftrightarrow (Q,P)$ is invertible. One can understand this condition by realizing that it requires $\frac{\partial P}{\partial q} \neq 0$ and $\frac{\partial p}{\partial Q} \neq 0$. (If one vanishes, then the other must due to commutativity of the partial derivatives; lack of commutativity would imply F is a truly pathological function.) If both these derivatives vanish, then P is a function of Q only and p is a function of q only, $P = P(Q)$ and $p = p(q)$. There is then no connection between (q,p) and (Q,P) – there is no canonical transformation. (For example, consider $F(q,Q) = q^2 + Q^2$.)

[14] This statement is not always true! Consider, for example, any kind of point transformation. Such transformations only provide a relation between q and Q. To obtain the relation between p and P, one must explicitly rewrite L in terms of Q and then obtain the new canonical momenta P from partial derivatives of L. In such cases, one must pursue an alternate version of this proof using a generating function that relies on different arguments. These are the other types of generating functions discussed in the following section.

So, if one maps a closed path in (q,p) to a closed path in (Q,P) and calculates the areas $\oint p\,dq$ and $\oint P\,dQ$, they are equal. Thus, $\oint (p\,dq - P\,dQ) = 0$ for any closed path. If we consider two points 0 and 1 on that path, at (q_0,p_0) and (q_1,p_1) in the qp phase space and (Q_0,P_0) and (Q_1,P_1) in the QP phase space, the vanishing of the closed path integral is equivalent to the statement that $\int_0^1 (p\,dq - P\,dQ)$ is independent of the path from 0 to 1. The integral must then only depend on the endpoints, so $\int_0^1 (p\,dq - P\,dQ) = F(q_1,Q_1) - F(q_0,Q_0)$, which implies that the integrand is a perfect differential, $p\,dq - P\,dQ = dF$. Thus, we see that there is indeed always a function $F(q,Q)$ that satisfies the partial differential equations.

For time-dependent transformations, one must break down the transformation into two transformations. This first is the transformation $(Q(q,p,t=0), P(q,p,t=0))$. That is, we just use the form of the transformation at $t=0$; this is now a time-independent transformation, so we know a generating function can be found. Second, we will show later that time evolution over an infinitesimal time dt can be viewed as a canonical transformation with generating function $F_2(q,P) = q\,P + dt\,H(q,P)$ (we shall discuss later generating functions with arguments other than q and Q) where H is the Hamiltonian in the original coordinates. Time evolution over a finite time is built up from infinitesimal transformations, so a finite time evolution is also a canonical transformation for which there exists a generating function. Note that finite time evolution to a specific time t is not a time-dependent transformation in the usual sense because we fix the time we want to evolve to.

Finally, because we have defined a Lagrangian that differs from what one would get by simple substitution using the contact transformation, we must find the corresponding Hamiltonian. We do this by returning to the relations we obtained between the integrands of the action integrals (the Lagrangians):

$$\begin{aligned} P\,\dot{Q} - \widetilde{H}(Q,P,t) &= p\,\dot{q} - H(q,p,t) + \frac{dF}{dt} \\ \widetilde{H}(Q,P,t) &= H(q,p,t) - \frac{\partial F}{\partial q}\,\dot{q} - \frac{\partial\, F}{\partial Q}\dot{Q} + \frac{dF}{dt} \\ \widetilde{H}(Q,P,t) &= H(q(Q,P,t),p(Q,P,t),t) + \frac{\partial}{\partial t}\,F(q(Q,P,t),Q,t) \end{aligned} \tag{2.41}$$

where we have used $p = \frac{\partial F}{\partial q}$ and $P = -\frac{\partial F}{\partial Q}$ and the chain rule for $\frac{dF}{dt}$ and, in the last line, we explicitly write q and p as functions of Q, P, and t to indicate that all q's and p's should be removed to obtain the new $\widetilde{H}(Q,P,t)$. Note that the final term is a *partial* derivative.

Example 2.12

$F = q\,Q$. Then $P = -\frac{\partial F}{\partial Q} = -q$ and $p = \frac{\partial F}{\partial q} = Q$; that is, we simply exchange coordinate and momentum (with signs). Obviously, this will result in $\tilde{H}(Q,P) = H(-P,Q)$.

Example 2.13

Simple harmonic oscillator. The Hamiltonian for a simple harmonic oscillator is

$$H = \frac{1}{2}\left(p^2 + \omega^2\,q^2\right)$$

(in this, q has been defined to be $q = \sqrt{m}\,x$, which implies $p = \sqrt{m}\,\dot{x}$ and gives the above Hamiltonian with $\omega^2 = \frac{k}{m}$.) Choose the generating function to be (with foresight!)

$$F(q,Q) = \frac{1}{2}\,\omega\,q^2 \cot 2\,\pi\,Q$$

This gives

$$p = \frac{\partial F}{\partial q} = \omega\, q\, \cot 2\,\pi\, Q$$
$$P = -\frac{\partial F}{\partial Q} = \pi\, \omega\, q^2\, \csc^2 2\,\pi\, Q$$

We need to write (q, p) in terms of (Q, P) to explicitly do the transformation. If we invert the P equation, we get

$$q = \sqrt{\frac{P}{\pi\, \omega}}\, \sin\, 2\,\pi\, Q$$
$$p = \sqrt{\frac{\omega\, P}{\pi}}\, \cos\, 2\,\pi\, Q$$

Notice the extremely symmetric form of the equations. If we make the above substitutions into the Hamiltonian, we find

$$\tilde{H}(P, Q) = \frac{\omega}{2\pi}\, P$$

Let's now determine the equations of motions for P and Q. We have

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = \frac{\omega}{2\,\pi} \qquad \dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = 0$$

Q is a cyclic coordinate. We can trivially integrate to find

$$Q = \frac{\omega\, t}{2\,\pi} \qquad P = P_0 = const$$

We have explicitly found Q and P as functions of time, so we may rewrite q and p in terms of these solutions:

$$q = \sqrt{\frac{P_0}{\pi\, \omega}}\, \sin\, \omega\, t$$
$$p = \sqrt{\frac{\omega\, P_0}{\pi}}\, \cos\, \omega\, t$$

The reader will recognize that we have transformed from simple position and momentum to phase (Q) and energy (P) of the oscillatory motion. The energy depends only on the oscillator amplitude. This kind of transformation is going to have obvious use when dealing with mechanical or electromagnetic waves.

One comment on the "obviousness" of the transformation. One clearly would not have pulled this transformation out of thin air. However, some physical insight might have led us there. We know by other means that the total energy of an oscillator is constant and independent of phase. Clearly, this energy is related to the maximum amplitude of the oscillator and/or its maximum momentum. Realizing this, one would be encouraged to look for a transformation that made the phase a cyclic coordinate. Finding the above transformation, though, would obviously take some playing around.

Other Forms of the Generating Function

We made a particular choice to make F a function of q and Q. Given the symmetry between coordinates and canonical momenta, it is likely that we could equally well write F as a function of (q, P), (p, P) or (p, Q). Sometimes it will be convenient to do so. The obvious technique by which to change to these different pairs of variables is via Legendre transformation. However, do not assume that, because we are transforming the generating function to depend on a different pair of independent variables, the new pair are conjugate to each other (*i.e.*, satisfy Hamilton's equations)! **These different generating functions are simply different ways to generate the same canonical transformation** $(q, p) \to (Q, P)$**; the conjugate pairs remain the same regardless of how the transformation is generated!** This point is not made sufficiently strongly in Hand and Finch, so watch out!

For reference, we summarize here the four kinds of generating functions and the transformation equations that are derived from them.

$$\begin{aligned}
F_1: &\quad F_1(q,Q) \qquad P = -\frac{\partial F_1}{\partial Q} \qquad p = \frac{\partial F_1}{\partial q} \\
F_2: &\quad F_2(q,P) \qquad p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P} \\
F_3: &\quad F_3(p,Q) \qquad P = -\frac{\partial F_3}{\partial Q} \qquad q = -\frac{\partial F_3}{\partial p} \\
F_4: &\quad F_4(p,P) \qquad q = -\frac{\partial F_4}{\partial p} \qquad Q = \frac{\partial F_4}{\partial P}
\end{aligned} \tag{2.42}$$

We make the very important point that not every canonical transformation can be derived from generating functions of all four types. In some cases, the Legendre transformation from one type to another will yield zero. This relates to the footnote in the previous section regarding the validity of the assumption that one can write a canonical transformation in the form $p = p(q, Q, t)$ and $P = P(q, Q, t)$. Point transformations violate this assumption because they do not mix q and p. There would be no information about p and P in a F_1-type generating function, making it impossible to generate the transformation from such a generating function. Such a transformation can be derived from only F_2 and F_3 type generating functions.

For canonical transformations that do not suffer from such pathologies, let us derive the relations between the different types of generating functions:

- F_3: we want to change variables from (q, Q) to (p, Q). The Legendre transformation is obviously (notice that we introduce the $q\,p$ term with sign opposite to the standard Legendre transformation, which will introduce additional signs elsewhere):

  $$F_3(p, Q, t) = F_1(q, Q, t) - q\,p$$

  The partial derivative relations arising from the transformation are

  $$\frac{\partial F_3}{\partial Q} = \frac{\partial F_1}{\partial Q} = -P \qquad \frac{\partial F_3}{\partial p} = -q \qquad p = \frac{\partial F_1}{\partial q}$$

  where the third relation comes from the assumption that $\frac{\partial F_3}{\partial q} = 0$, the standard condition we apply to derive the Legendre transformation (*i.e.*, to get rid of q). Note

that the Legendre transformation (albeit with the nonstandard sign) returns to us the relation $p = \frac{\partial F_1}{\partial q}$ that we had earlier derived. Rewriting the above relations in the more standard form (like Equation 2.40),

$$P = -\frac{\partial F_3}{\partial Q} \qquad q = -\frac{\partial F_3}{\partial p} \qquad p = \frac{\partial F_1}{\partial q} \tag{2.43}$$

We now see that the sign choice in the Legendre transformation was necessary to avoid changing the form of the relation for P. We have picked up an additional relation that was the defining relation in tranforming from F_1 to F_3.

Example 2.13 continued

Let's rewrite our canonical transformation for the harmonic oscillator in the F_3 form. We don't necessarily need to find F_3, but let's calculate it explicitly so we can see how it differs from F_1. We follow through the usual Legendre transformation technique. First, we need to find the new variable in terms of the old variable using the condition $p = \frac{\partial F_1}{\partial q}$ that partially defined the transformation (we have already done this calculation for other reasons in the earlier part of this example, but we follow it through so the logical structure is clear):

$$p = \frac{\partial F_1}{\partial q} = \omega\, q \cot 2\,\pi\, Q$$

Next, we invert that relation to find $q(p, Q)$:

$$q = \frac{p}{\omega} \tan 2\,\pi\, Q$$

Now, we can calculate F_3 directly using the Legendre transformation and substituting in our formula $q(p, Q)$.

$$\begin{aligned} F_3(p,Q) &= \frac{1}{2}\,\omega\, q^2 \cot 2\,\pi\, Q - q\, p \\ &= \frac{1}{2}\,\omega \left(\frac{p}{\omega} \tan 2\,\pi\, Q\right)^2 \cot 2\,\pi\, Q - p \left(\frac{p}{\omega} \tan 2\,\pi\, Q\right) \\ &= -\frac{p^2}{2\,\omega} \tan 2\,\pi\, Q \end{aligned}$$

Note that, an alternative technique for finding F_3 once we have $q(p, Q)$ is to make use of $q = -\frac{\partial F_3}{\partial p}$. This relation was derived directly above from the forward transformation. One could alternatively have derived $q = -\frac{\partial F_3}{\partial p}$ from the reverse Legendre transformation because the defining relation to eliminate q is exactly this derivative relation. So, we have

$$\begin{aligned} q &= \frac{p}{\omega} \tan 2\,\pi\, Q = -\frac{\partial F_3}{\partial q} \\ F_3(p,Q) &= -\frac{p^2}{2\,\omega} \tan 2\,\pi\, Q \end{aligned}$$

Finally, using our third partial derivative relation, we have

$$\begin{aligned} P &= -\frac{\partial F_3}{\partial Q} \\ &= \frac{p^2}{2\,\omega} \left(1 + \tan^2 2\,\pi\, Q\right)\, 2\,\pi \\ &= \frac{\pi\, p^2}{\omega} \sec^2 2\,\pi\, Q \end{aligned}$$

Summarizing, the transformation generated by F_3 is

$$\begin{aligned} q(p,Q) &= \frac{p}{\omega}\tan 2\pi Q \\ P(p,Q) &= \frac{\pi p^2}{\omega}\sec^2 2\pi Q \end{aligned}$$

We can check that this transformation is algebraically the same as our original F_1 transformation by: 1) inverting the first line, which gives $p(q,Q) = \omega q \cot 2\pi Q$, which was what we had from F_1; and 2) substituting $p(q,Q)$ in to $P(p,Q)$, we get $P = \pi\omega q^2 \csc^2 2\pi Q$, which is also what we had from F_1.

- F_4: Here, we transform the generating function from (p,Q) to (p,P). The Legendre transformation is (choice of sign to be explained below)

$$F_4(p,P,t) = F_3(p,Q,t) + QP \tag{2.44}$$

The relations between partial derivatives are

$$\frac{\partial F_4}{\partial p} = \frac{\partial F_3}{\partial p} = -q \qquad \frac{\partial F_4}{\partial P} = Q \qquad \frac{\partial F_3}{\partial Q} = -P$$

where again the last term arises because of the condition that F_4 not depend on Q. The last term also makes it clear that we had to choose the sign on the QP term as we did in order to have self-consistency with the F_3 equations. Rewriting in standard form (like Equation 2.40) gives

$$q = -\frac{\partial F_4}{\partial p} \qquad Q = \frac{\partial F_4}{\partial P} \qquad P = -\frac{\partial F_3}{\partial Q} \tag{2.45}$$

Example 2.13 continued

Let's rewrite the harmonic oscillator canonical transformation using the F_4 form. Using the partial derivative equation relating the old and new independent variable,

$$P = -\frac{\partial F_3}{\partial Q} = \frac{\pi p^2}{\omega}\sec^2 2\pi Q$$

we invert to obtain

$$Q = \frac{1}{2\pi}\sec^{-1}\sqrt{\frac{\omega P}{\pi p^2}}$$

We can directly substitute to find F_4 (using $1+\tan^2\theta = \sec^2\theta$):

$$\begin{aligned} F_4(p,P) &= F_3(p,Q) + QP \\ &= -\frac{p^2}{2\omega}\tan 2\pi Q + QP \\ &= -\frac{p^2}{2\omega}\tan\sec^{-1}\sqrt{\frac{\omega P}{\pi p^2}} + \frac{P}{2\pi}\sec^{-1}\sqrt{\frac{\omega P}{\pi p^2}} \\ &= -\frac{p^2}{2\omega}\sqrt{\frac{\omega P}{\pi p^2} - 1} + \frac{P}{2\pi}\sec^{-1}\sqrt{\frac{\omega P}{\pi p^2}} \end{aligned}$$

which is just a mess! Integration of $Q = \frac{\partial F_4}{\partial P}$ would have yielded the same mess. Clearly, this is not a particularly convenient choice for a generating function. We will not bother to grind through the algebra needed to prove that the transformation generated by the above function is equivalent to the one we have seen before.

- F_2: The final transformation of the generating function gives $F_2(q, P)$, though we start from F_1:

$$F_2(q, P, t) = F_1(q, Q, t) + Q\,P \tag{2.46}$$

The relations between partial derivatives are

$$\frac{\partial F_2}{\partial q} = \frac{\partial F_1}{\partial q} = p \qquad \frac{\partial F_2}{\partial P} = Q \qquad \frac{\partial F_1}{\partial Q} = -P$$

where again the last term arises because of the condition that F_2 not depend on Q. Again, consistency of the last term with the original version of the transformation (F_1) forced the choice of sign for the $Q\,P$ term. Rewriting in standard form (like Equation 2.40) gives

$$p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P} \qquad P = -\frac{\partial F_1}{\partial Q} \tag{2.47}$$

Example 2.13 continued

Let's rewrite the harmonic oscillator canonical transformation using the F_1 form. Using the partial derivative equation relating the old and new independent variable,

$$P = -\frac{\partial F_1}{\partial Q} = \pi\,\omega\,q^2\ \csc^2 2\,\pi\,Q$$

we invert to obtain

$$Q = \frac{1}{2\,\pi}\csc^{-1}\sqrt{\frac{P}{\omega\,\pi\,q^2}}$$

We can see this is going to devolve into a mess similar to what we had for F_4, but let's plow ahead and do the direct substitution to find F_2 (using $1 + \cot^2\theta = \csc^2\theta$):

$$\begin{aligned}
F_2(q, P) &= F_1(q, Q) + Q\,P \\
&= -\frac{\omega\,q^2}{2}\cot 2\,\pi\,Q + Q\,P \\
&= -\frac{\omega\,q^2}{2}\cot\csc^{-1}\sqrt{\frac{P}{\omega\,\pi\,q^2}} + \frac{P}{2\,\pi}\csc^{-1}\sqrt{\frac{P}{\omega\,\pi\,q^2}} \\
&= -\frac{\omega\,q^2}{2}\sqrt{\frac{P}{\omega\,\pi\,q^2} - 1} + \frac{P}{2\,\pi}\csc^{-1}\sqrt{\frac{P}{\omega\,\pi\,q^2}}
\end{aligned}$$

which is as expected, a mess! Integration of $Q = \frac{\partial F_2}{\partial P}$ would have yielded the same mess. Again, we have an inconvenient generating function, but nevertheless it will be consistent with the canonical transformation.

Theoretically Interesting Generating Functions

Here we consider a couple of generic generating functions that are of particular interest from the theoretical point of view.

- **Point transformation generating function**: If we have a point transformation, which can generically be written $\{Q_l\} = \{f_l(\{q_k\})\}$, ($M$ functions of M variables) then we can use a generating function of the F_2 form (recall our earlier note that it is impossible to derive a point transformation from a F_1-type generating function):

$$F_2(\{q_k\}, \{P_l\}) = \sum_n f_n(\{q_k\})\, P_n$$

Using Equations 2.47, we find

$$p_l = \frac{\partial F_2}{\partial q_l} = \sum_n P_n \frac{\partial f_n}{\partial q_l} \qquad Q_l = \frac{\partial F_2}{\partial P_l} = f_l(\{q_k\})$$

The relation $P = -\frac{\partial F_1}{\partial Q}$ is not used because it was only needed to define a F_2-type transform in terms of a F_1-type transform; here, we start with the F_2-type transform. We see that we reproduce the desire point transformation, and thus we see that any point transformation is a canonical transformation. We also see how to find the new canonical momenta without having to go through the Lagrangian.

- **Infinitesimal Canonical Transformation**: Let $F_2 = q\,P + \epsilon\, G(q, P)$. Then we have from Equations 2.47 again

$$p = \frac{\partial F_2}{\partial q} = P + \epsilon \frac{\partial G}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P} = q + \epsilon \frac{\partial G}{\partial P}$$

Rewriting to find P in terms of (q, p), we have

$$Q = q + \epsilon \frac{\partial G}{\partial P} \qquad P = p - \epsilon \frac{\partial G}{\partial q}$$

If we take $\epsilon = dt \to 0$ and $G(q, P) = H(q, P) \approx H(q, p)$, we see that the second terms in the two equations are the increments in q and p expected for evolution over a time dt based on Hamilton's equations. This generating function generates time evolution, which is a canonical transformation. Note that this same infinitesimal technique can be used to generate a variety of other transformations, such as translation, rotation, etc.

2.4.2 Symplectic Notation

We define symplectic notation and restate some of our earlier results in that form. We make this transition in order to make use of the more geometric point of view for the remainder of our discussion of theoretical mechanics. This material is covered in the appendix to Chapter 6 of Hand and Finch.

Definition of Symplectic Notation

As we did with regard to Liouville's theorem, we are free to define the vector $\vec{\xi}$ with $(k = 1, \ldots, 2M)$

$$\xi_k = \begin{cases} q_{(k+1)/2} & k = \text{even} \\ p_{k/2} & k = \text{odd} \end{cases} \tag{2.48}$$

The partial derivatives involved in Hamilton's equation $\frac{\partial}{\partial q_k}$ and $\frac{\partial}{\partial p_k}$ simply become the gradient with respect to ξ, $\vec{\nabla}_\xi$. We define the matrix $\mathbf{\Gamma}$ by

$$\mathbf{\Gamma}_{ij} = (j-i)\,\delta_{|i-j|,1} \qquad \text{for } i \text{ odd and } j \text{ even} \tag{2.49}$$

For example, a 4-dimensional phase space has

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

With $\mathbf{\Gamma}$, Hamilton's equations are written

$$\frac{d\vec{\xi}}{dt} = \mathbf{\Gamma}\vec{\nabla}_\xi H \tag{2.50}$$

Derivatives and the Jacobian in Symplectic Notation

A contact transformation generates a new set of phase space coordinates $\vec{\Xi}$. The Jacobian of transformation is $\mathbf{J}$

$$J_{ij} = \frac{\partial \Xi_i}{\partial \xi_j} \tag{2.51}$$

An example for 2 phase-space dimensions is

$$\mathbf{J} = \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}$$

$\mathbf{J}$ gives the transformation of differential line and volume elements:

$$d\vec{\Xi} = \mathbf{J}\, d\vec{\xi} \tag{2.52}$$

$$\prod_k d\Xi_k = \prod_k \left(\sum_l J_{kl}\, d\xi_l \right) = [\det \mathbf{J}] \prod_l d\xi_l \tag{2.53}$$

To understand how the Jacobian transforms gradients, consider differential displacements $d\vec{\xi}$ and $d\vec{\Xi}$ that yield the same change in H:

$$\begin{aligned} (\vec{\nabla}_\xi H)^T d\vec{\xi} = dH &= (\vec{\nabla}_\Xi H)^T d\vec{\Xi} \\ (\vec{\nabla}_\xi H)^T d\vec{\xi} &= (\vec{\nabla}_\Xi H)^T \mathbf{J}\, d\vec{\xi} \\ (\vec{\nabla}_\xi H)^T &= (\vec{\nabla}_\Xi H)^T \mathbf{J} \\ \vec{\nabla}_\xi H &= \mathbf{J}^T \vec{\nabla}_\Xi H \end{aligned}$$

That is, gradients are equivalent if appropriately transformed with $\mathbf{J}$.

Canonical Transformation and Symplectic Jacobians

We can obtain a condition on the Jacobian for canonical transformations. If $\vec{\Xi}$ is a canonical transformation of $\vec{\xi}$, then Hamilton's equations hold for $\vec{\xi}$ and $\vec{\Xi}$:

$$\frac{d\vec{\xi}}{dt} = \mathbf{\Gamma}\, \vec{\nabla}_\xi H \qquad \frac{d\vec{\Xi}}{dt} = \mathbf{\Gamma}\, \vec{\nabla}_\Xi H$$

Restricting to transformations with no explicit time dependence, $\vec{\Xi} = \vec{\Xi}(\vec{\xi})$, we may rewrite the left equation:

$$\begin{aligned} \frac{d\vec{\xi}}{dt} &= \mathbf{\Gamma}\,\vec{\nabla}_\xi H \\ \frac{\mathbf{J}^{-1}d\vec{\Xi}}{dt} &= \mathbf{\Gamma}\,\mathbf{J}^T\vec{\nabla}_\Xi H \\ \frac{d\vec{\Xi}}{dt} &= \mathbf{J}\,\mathbf{\Gamma}\,\mathbf{J}^T\vec{\nabla}_\Xi H \end{aligned}$$

Thus, in order for Hamilton's equations to be satisfied in the Ξ coordinates, we require

$$\mathbf{J}\,\mathbf{\Gamma}\,\mathbf{J}^T = \mathbf{\Gamma} \tag{2.54}$$

A matrix that satisfies the above condition is called **symplectic**. *Symplectic* derives from the Greek for "intertwined;" clearly, the intertwining of the q and p coordinates by Hamilton's equations motivates the use of the term. Note that the above condition is equivalent to the condition $\det \mathbf{J} = 1$, the Jacobian determinant is unity. Hence, the phase space volume element is preserved under canonical transformations. Note also that $\mathbf{J}^T \neq \mathbf{J}^{-1}$ in general; in fact, if you assume that such an equality did hold, you would restrict yourself to a very specific canonical transformation.

2.4.3 Poisson Brackets

We define Poisson brackets and explore their useful characteristics.

Definition

The **Poisson Bracket** of two functions of F and G of the coordinates and canonical momenta q and p is defined to be

$$[F,G]_{\vec{q},\vec{p}} = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right] \tag{2.55}$$

In symplectic notation, the Poisson bracket is written

$$[F,G]_{\vec{\xi}} = \left[\vec{\nabla}_\xi F\right]^T \mathbf{\Gamma}\vec{\nabla}_\xi G \tag{2.56}$$

The $\mathbf{\Gamma}$ matrix provides the necessary negative signs.

Important Properties and Applications

- **Invariance under canonical transformations**
  The most useful property of Poisson brackets is that their value is invariant under canonical transformations. This is trivial to see in symplectic notation:

$$\begin{aligned} [F,G]_{\vec{\xi}} &= \left[\vec{\nabla}_\xi F(\vec{\xi})\right]^T \mathbf{\Gamma}\ \left[\vec{\nabla}_\xi G(\vec{\xi})\right] \\ &= \left[\mathbf{J}^T\vec{\nabla}_\Xi \tilde{F}(\vec{\Xi})\right]^T \mathbf{\Gamma}\ \left[\mathbf{J}^T\vec{\nabla}_\Xi \tilde{G}(\vec{\Xi})\right] \\ &= \left[\vec{\nabla}_\Xi \tilde{F}(\vec{\Xi})\right]^T \mathbf{J}\,\mathbf{\Gamma}\,\mathbf{J}^T\left[\vec{\nabla}_\Xi \tilde{G}(\vec{\Xi})\right] \\ &= \left[\vec{\nabla}_\Xi \tilde{F}(\vec{\Xi})\right]^T \mathbf{\Gamma}\left[\vec{\nabla}_\Xi \tilde{G}(\vec{\Xi})\right] \\ &= [\tilde{F},\tilde{G}]_{\vec{\Xi}} \end{aligned}$$

where $\tilde{F}$ and $\tilde{G}$ refer to F and G after the transformation, and where the penultimate line required that $\mathbf{J}$ be symplectic; *i.e.*, that the transformation characterized by $\mathbf{J}$ be canonical.

- **Provide test for whether a transformation is canonical**

  Poisson brackets can be used to test whether a transformation is canonical. Clearly, $[Q_k, P_l]_{\vec{Q},\vec{P}} = \delta_{kl}$. Therefore, by the above theorem, if the transformation is canonical, it must hold that

$$[Q_k(\vec{q},\vec{p}), P_l(\vec{q},\vec{p})]_{\vec{q},\vec{p}} = \delta_{kl} \tag{2.57}$$

  It turns that the above is not just a necessary but it is also a sufficient condition for a transformation to be canonical. This statement is identical to the condition that $\mathbf{J}$ be symplectic.

- **Provide time evolution of functions of coordinates**

  Proof not available in Hand and Finch

  Another useful property of the Poisson bracket is that the Poisson bracket of any function with the Hamiltonian gives the time derivative of that function. We saw this for the phase space density in connection with Liouville's theorem, but can see easily that it holds generally. Let $F(\vec{\xi})$ be an arbitrary function of the symplectic coordinate $\vec{\xi}$. Then

$$\begin{aligned}
\frac{dF}{dt} &= \frac{\partial F}{\partial t} + \left[\vec{\nabla}_\xi F\right]^T \frac{d\vec{\xi}}{dt} \\
&= \frac{\partial F}{\partial t} + \left[\vec{\nabla}_\xi F\right]^T \mathbf{\Gamma}\, \vec{\nabla}_\xi H \\
&= \frac{\partial F}{\partial t} + [F, H]_{\vec{\xi}}
\end{aligned}$$

  where we employed the chain rule in the first line, Hamilton's equations in symplectic notation in the second line, and the definition of the Poisson bracket in the last line. A corollary of the above result is that any function F whose Poisson bracket with the Hamiltonian vanishes – *i.e.*, $[F, H]_{\vec{\xi}} = 0$ – is conserved unless it has explicit time dependence. Moreover, one can demonstrate the **Hamiltonian version of Noether's Theorem**: if a quantity I is conserved due to invariance of the Lagrangian under a particular point transformation, then $[I, H] = 0$ and the quantity is also conserved in Hamiltonian dynamics. This will be a homework problem.

- **Alternate form of Liouville's theorem**

  Liouville's theorem can be written down in Poisson bracket form. We take for the function $F(\vec{\xi})$ the phase space density written as a function of symplectic coordinates, $\rho(\vec{\xi})$. It holds that

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + [\rho, H]_{\vec{\xi}}$$

  Liouville's theorem told us that $d\rho/dt = 0$. Thus, we may write Liouville's theorem as

$$\frac{\partial \rho}{\partial t} + [\rho, H]_{\vec{\xi}} = 0$$

There is no additional content here, it is simply an alternate way of writing Liouville's theorem. If one works back from the Poisson bracket to symplectic notation, one recovers the form of Liouville's theorem obtained when it was proved,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \dot{\vec{\xi}} \cdot \vec{\nabla}_\xi \rho$$

- **Geometrical Interpretation:** The reader will no doubt see that the Poisson bracket looks a bit like a vector cross product. In fact, for a single physical dimension, for which the phase space is two-dimensional, the Poisson bracket $[F, G]$ looks exactly like the cross product of $\vec{\nabla}_\xi F$ and $\vec{\nabla}_\xi G$:

$$[F, G]_{\vec{q},\vec{p}} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} = \left(\vec{\nabla}_\xi F\right)_q \left(\vec{\nabla}_\xi G\right)_p - \left(\vec{\nabla}_\xi F\right)_p \left(\vec{\nabla}_\xi G\right)_q$$

  Of course, in a 2D phase space, there is no third dimension that the cross product can point into, so the analogy is imperfect. But, clearly, the value of the Poisson bracket is the same as the projection of the cross-product of two vectors along the direction of their cross product. In more than one spatial dimension, the Poisson bracket is just the sum over these "cross-product" components over all k.

  It is instructive to talk about our various applications of Poisson brackets in terms of the geometrical picture.

  - The vectors $\vec{\nabla}_\xi F$ and $\vec{\nabla}_\xi G$ point normal to surfaces of constant F and G. Recall that cross products take on their maximum values when the two vectors are perpendicular. We see the same behavior here. For example, when $F = q_k$ and $G = p_k$, the surfaces of constant F and G are just lines of fixed q_k and p_k. The Poisson bracket of q_k and p_k is 1, reflecting the fact that the two surface normals are perfectly perpendicular. In a 2D phase space, the Poisson bracket $[F, G]$ is $|\vec{\nabla}_\xi F||\vec{\nabla}_\xi G| \sin\theta_{FG}$ where θ_{FG} is the angle in the qp plane between $\vec{\nabla}_\xi F$ and $\vec{\nabla}_\xi G$. For more than one spatial dimension, the Poisson bracket is simply the sum over k of such terms for each k.
  - Invariance of Poisson brackets under canonical transformation therefore implies that such transformations preserve the angle between the gradient vectors and thus between the surfaces they define. This is something like conformal mapping, which also preserves angles.
  - The way in which Poisson brackets can be used to test whether a transformation is canonical is a special case of the above property. But it is a very important special case, because it implies that the surfaces in qp phase space corresponding to surfaces of constant value of the transformed coordinates Q and P must always intersect at right angles. This gives a geometric interpretation of the condition for a transformation to be canonical.
  - Time evolution: Given a function F, we thus see that time evolution occurs only to the extent that surfaces of constant F are normal to surfaces of constant H. This holds in particular for $F = q_k$ or $F = p_k$. If surfaces of constant F and constant H coincide, we of course expect no time evolution of F.

Clearly, the above properties foreshadow many properties of commutators in quantum mechanics. $[Q, P] = 1$ will become $[q, p] = i\hbar$. The time evolution of observables in quantum mechanics is determined by an equation identical to the one above describing the time derivative of classical quantities.

2.4.4 Action-Angle Variables and Adiabatic Invariance

Action-angle variables are used to simplify the description of systems whose motion is periodic, writing it in terms of coordinates in which the motion is periodic (angle variables) and conjugate momenta that are constant (action variables). Our derivation will deviate from Hand and Finch by not basing it on the Hamilton-Jacobi formalism. We will provide the Hamilton-Jacobi derivation of action-angle variables later. We have supplemented the derivation with some motivation from Goldstein. There are some nice figures in Hand and Finch Section 6.5 illustrating the phase space for the simple pendulum that will be useful to look at when considering the geometrical interpretation of the action variable (as well as to understand periodicity).

The derivation of action-angle variables is admittedly opaque. For this reason, we provide a pre-summary of the results. Reading this first will motivate the reader to understand the derivation that follows it and give some clue as to where it is headed, and also serves as a short summary of the useful results. The pre-summary also serves as a cookbook so that those who have not studied the previous material on canonical transformations and generating functions can still appreciate and know how to make use of action-angle variables and adiabatic invariance. Though such readers will not find it useful to wade through the derivation that follows, parts of the examples may be instructive and should be studied.

Note: Goldstein's angle variable is defined with period 1 instead of $2\,\pi$, and his action variable differs by a canceling factor of $2\,\pi$. No physics is affected, but detailed results may differ by factors of $2\,\pi$.

Types of Periodic Motion for 1-Dimensional Systems

If we have a 1-dimensional system whose energy is conserved, $H(q,p) = \alpha_1$, then we can eliminate time as an independent variable and instead write $p = p(q,\alpha_1)$. This describes an orbit in the 2-dimensional phase space of the system. Two types of orbits serve to classify the two kinds of periodic motion:

- **libration**: The orbit is a closed path in phase space, which implies that the motion is periodic in both p and q. This occurs if the initial position lies between zeros of the kinetic energy (or equivalently, if the motion is in a bounded potential). The simple harmonic oscillator is an example of such a system.
- **rotation**: Here, the requirement is only that p be a periodic function of q, but the orbit need not be closed. This implies that the system's configuration is invariant under translations in the q variable by the period q_0. The classic example of this motion is rigid-body rotation. q in general can increase without bound, and the periodicity refers to the periodicity of p in q only.

A useful example of a system that can execute both kinds of motion is a simple pendulum. If $E < m\,g\,l$, then the pendulum's motion is oscillatory, *i.e.*, of the libration type. If $E > m\,g\,l$, then the pendulum can rotate 360 degrees around its pivot point and it executes rotation motion.

A Motivating Example

Given a system of one of the above types, we will follow along what we did in the simple harmonic oscillator case and use a F_1-type generating function $\widetilde{W}(q,\psi)$ to try to transform to a set of canonical variables (ψ, I) such that I is constant and ψ cycles through $2\,\pi$ during every period of the motion. Recall that the F_1-type generating function for the simple

harmonic oscillator was

$$F(q,Q) = \frac{1}{2}\,\omega\,q^2\cot 2\,\pi\,Q$$

with the resulting relationships between canonical variables

$$\begin{aligned}
q &= \sqrt{\frac{P}{\pi\,\omega}}\,\sin 2\,\pi\,Q \\
p &= \frac{\partial F}{\partial q} = \omega\,q\,\cot 2\,\pi\,Q = \sqrt{\frac{\omega\,P}{\pi}}\,\cos 2\,\pi\,Q \\
P &= -\frac{\partial F}{\partial Q} = \pi\,\omega\,q^2\,\csc^2 2\,\pi\,Q = \frac{2\,\pi}{\omega}\,E \\
Q &= \frac{\omega\,t}{2\,\pi}
\end{aligned}$$

where E is the constant conserved energy. A relationship that was not seen earlier is ($\oint$ refers to an integral over one period on (q,p))

$$P = \oint dq\,p$$

which can be seen by simply substituting in the above relations:

$$\begin{aligned}
\oint dq\,p &= \oint d\left(\sqrt{\frac{P}{\pi\,\omega}}\,\sin 2\,\pi\,Q\right)\sqrt{\frac{\omega\,P}{\pi}}\,\cos 2\,\pi\,Q = 2\,P\oint dQ\,\cos^2 2\,\pi\,Q \\
&= \frac{P}{\pi}\int_0^T \omega\,dt\,\cos^2\omega\,t = \frac{P}{\pi}\int_0^{2\,\pi} d\theta\,cos^2\theta \\
&= P
\end{aligned}$$

Note that we were able to treat P as constant because we knew from the previous analysis it would be. In this example, our desired variables (ψ, I) would be

$$\psi = 2\,\pi\,Q = \omega\,t \qquad I = \frac{P}{2\,\pi} = \frac{E}{\omega}$$

(We insert the $2\,\pi$ for ψ to get the desired period, but, to ensure (ψ, I) are canonical variables, we must factor out a corresponding $2\,\pi$ from P to get I.)

Pre-Summary and Cookbook

In the following section, we shall generalize the above example. The basic result we will obtain is that, for any 1-dimensional periodic system, we may always define conjugate action and angle variables analogous to P and Q above and that they always obey the following:

- The action variable I is constant and has the value

$$I = \oint p(E,q)\,dq \tag{2.58}$$

  where one obtains $p(E,q)$ by inverting $H(p,q)$ to obtain $p(H,q)$ and then using the fact that energy is conserved for a periodic system, so that H is a constant with value E set by the initial conditions, $E = H(p(t=0), q(t=0))$. The integral is thus explicit for any initial condition (though whether it is analytically integrable depends on the specific problem).

- The angle variable evolves linearly in time at rate ω:

$$\psi = \omega\, t + \psi_0 \quad \text{with} \quad \omega \equiv \frac{\partial H(I)}{\partial I} \tag{2.59}$$

  where $H(I)$ is just the Hamiltonian rewritten in terms of I instead of in terms of p and q. ψ_0 is set by initial conditions.

The full proof will demonstrate that it is always possible to define I and ψ for any periodic system and that they always satisfy the above.

We will also demonstrate that I is an *adiabatic invariant*, which means that, if some normally constant parameter in H is changed sufficiently slowly, then I is constant to first order in the change in the parameter. An example of a "normally constant" parameter is the natural frequency ω in a SHO, or the mass m and spring constant k that combine to give ω. By "sufficiently slowly," we mean that, if the changing parameter is α, the rate of change of α must satisfy $\dot{\alpha}\, T/\alpha \ll 1$ where T is the period of the system. That is, the fractional change in α in one period is small compared to 1. By "constant to first order," we mean that $\dot{I} \propto \dot{\alpha}^2$.

Adiabatic invariance of I is how these concepts prove useful because they provide one quantity, I, that is constant even when parameters in the problem and the total energy are changing. The classic example is a pendulum with a slowly lengthening bob length $l(t)$. If the parameter changes sufficiently slowly, we do not need to solve the full problem with the rheonomic constraint that gives us $l(t)$ to find out how the energy, oscillation amplitude, or maximum speed of the pendulum change with time; we simply use the fact that I is constant to relate the rate of change of E to the rate of change of the bob length l, and then we can obtain the rate of change of the amplitude A or the maximum speed v from the rate of change of E.

The geometric interpretation of all of the above in phase space is also interesting. I, by definition, is the area enclosed in phase space by the phase space orbit and ψ is the angular position of the system on that orbit in the pq plane. Adiabatic invariance has a clever geometric interpretation – even as a parameter in the problem changes, causing the orbit shape to change (*e.g.*, the maximum q and p will change), the area of the orbit is preserved as it changes shape.

The Full Derivation

For a general periodic system, we would like a generating function $\widetilde{W}(q, \psi)$ that generates a canonical transformation leading to variables (ψ, I) that behave as above – I constant and ψ cycling through $2\,\pi$ for each period of the system. Any such valid F_1-type generating function will yield the relations

$$p = \frac{\partial \widetilde{W}}{\partial q} \qquad I = -\frac{\partial \widetilde{W}}{\partial \psi}$$

We therefore take these relationships as a requirement on the as-yet undetermined $\widetilde{W}$. It implies that the differential of $\widetilde{W}$ is

$$d\widetilde{W} = p\, dq - I\, d\psi$$

Now, we want I to be a constant and the system to be periodic in ψ with period $2\,\pi$. We will assume both conditions hold and see if we run into any contradictions. We can find a formula for I by integrating the above. The first step is

$$\oint d\widetilde{W} = \oint p\,dq - \oint I\,d\psi$$

where $\oint$ denotes integration over one period. The assumed constancy of I lets us pull it out of the integral. We know that the system is periodic in q, and we assume it is periodic in ψ. Therefore, $\widetilde{W}(q,\psi)$ is also periodic. So

$$\begin{aligned}\widetilde{W}(T) - \widetilde{W}(0) &= \oint p\,dq - I\oint d\psi \\ 0 &= \oint p\,dq - 2\,\pi\,I \\ I &= \frac{1}{2\,\pi}\oint p\,dq\end{aligned}$$

where T is the period in time. The reduction of $\oint d\psi$ is a result of the assumed periodicity in ψ with period $2\,\pi$. So far, no contradictions – we have obtained a formula for I which indeed yields a constant; the expression is constant because, for any periodic system, the integral of any function of the periodic variables over a period is independent of time. With I now seen to be constant, we can write an expression for $\widetilde{W}$ by integrating its differential:

$$\widetilde{W}(q,\psi) = \int p(q,I)\,dq - I\,\psi$$

We have written explicit arguments for p now. Because the motion is periodic, p is determined entirely by q and one constant of integration. We have already shown that I is a constant derived from the orbit, so it can be used as the constant of integration.

Note that we have not checked whether $\widetilde{W}$ as defined is a valid generating function. It must satisfy the mixed second-derivative condition

$$\frac{\partial^2\widetilde{W}(q,\psi)}{\partial\psi\,\partial q} = \frac{\partial^2\widetilde{W}(q,\psi)}{\partial q\,\partial\psi} \qquad \Longleftrightarrow \qquad \frac{\partial p}{\partial\psi} = -\frac{\partial I}{\partial q}$$

We will in fact not check this for $\widetilde{W}$; we will check it for the generating function $\widetilde{W}_2$ that we obtain below from $\widetilde{W}$ by Legendre transformation. If $\widetilde{W}_2$ is a valid generating function, then so must $\widetilde{W}$ be because the two are related by a Legendre transformation.

We wish to find an explicit formula for ψ next. The natural thing is to go from our F_1-type generating function $\widetilde{W}(q,\psi)$ to a F_2-type generating function $\widetilde{W}_2(q,I)$ via Legendre transformation since I has been seen to be constant and we have an expression for it; since ψ will be dependent in the F_2 scheme, an expression for it will become available.

$$\begin{aligned}\widetilde{W}_2(q,I) &= \widetilde{W}(q,\psi) + \psi\,I \\ &= \int p(q,I)\,dq\end{aligned}$$

Let's first check that $\widetilde{W}_2$ satisfies the mixed second derivative rule:

$$\frac{\partial^2\widetilde{W}_2(q,I)}{\partial I\,\partial q} = \frac{\partial}{\partial I}\frac{\partial}{\partial q}\int p(q,I)\,dq = \frac{\partial}{\partial I}\,p(q,I)$$

$$\frac{\partial^2\widetilde{W}_2(q,I)}{\partial q\,\partial I} = \frac{\partial}{\partial q}\frac{\partial}{\partial I}\int p(q,I)\,dq = \frac{\partial}{\partial q}\int dq\,\frac{\partial}{\partial I}\,p(q,I) = \frac{\partial}{\partial I}\,p(q,I)$$

Indeed it does. We were allowed to move $\partial/\partial I$ inside the integral because q and I are independent when dealing with the $\widetilde{W}_2$ generating function: q and I are the independent variables, p and ψ are the dependent variables. Then, as noted above, since $\widetilde{W}_2$ is a valid generating function, so is $\widetilde{W}$.

Now, we could use the standard Legendre transformation rules to obtain an explicit formula for ψ:

$$\psi = \frac{\partial \widetilde{W}_2(q,I)}{\partial I} = \frac{\partial}{\partial I}\int p(q,I)\,dq = \int dq\,\frac{\partial p(q,I)}{\partial I}$$

We shall show below that, after the canonical transformation, H has no ψ dependence and thus is a function of I alone, $H = H(I)$. This would allow us to obtain $p(q,I)$ via H: $H = H(p,q)$ also, so we should be able find $p = p(q,H)$ and therefore $p = p(q,H(I)) = p(q,I)$. We could then do the integral and take $\frac{\partial}{\partial I}$, or alternatively take $\frac{\partial}{\partial I}$ and do the integral, to obtain ψ. But it may not always be possible to do the integral analytically. It turns out there is an easier way to obtain $\psi(t)$ using the fact that (ψ, I) are canonical variables by construction. Hamilton's equations in the new variables are

$$\dot{\psi} = \frac{\partial H(\psi,I)}{\partial I} \qquad \dot{I} = -\frac{\partial H(\psi,I)}{\partial \psi}$$

I is constant by construction. Therefore, $\dot{I} = 0$ and thus $\frac{\partial H}{\partial \psi} = 0$. Therefore, the canonically transformed H has no ψ dependence, $H = H(I)$ only. Consequently, $\dot{\psi} = \frac{\partial H}{\partial I}$ can depend only on I. Moreover, because I is constant in time, we know $\frac{\partial H}{\partial I}$ is constant in time. Putting this all together gives

$$\dot{\psi} = \frac{\partial H(I)}{\partial I} \equiv \omega = \text{constant}$$

Therefore,

$$\psi = \omega\, t + \psi_0$$

We thus obtain an explicit formula for ψ that is valid independent of the details of the problem. The Legendre transformation formula for ψ would yield this, but one would have to perform the integral explicitly using the particular $p(q,I)$ function.

We belabor one point, the functional dependences of ω. We explicitly noted above that $\omega = \omega(I)$: ω is not constant as a function of I. But ω *is* constant in *time* because I is constant in time. The constancy of ω in time provides the simple evolution of the angle variable ψ. Now, in some cases, ω may indeed be independent of I – for example, in the SHO, $\omega = \sqrt{k/m}$ depends only on the parameters of the problem, not on I, which comes from the initial conditions. But, more generally, ω may be a function of I; one example is the non-small-angle pendulum – the period of the motion (which we shall see below is given by ω) is dependent on the initial conditions.

So, we end up with $I = \frac{1}{2\,\pi}\oint p\,dq$ as the constant canonical momentum and $\psi = \omega\, t + \psi_0$ as the linearly evolving conjugate coordinate. I is called the **action** variable (because it always has units of action, momentum × position) and ψ the **angle** variable (because action has the same units as angular momentum, so the conjugate variable has angular units).

We can confirm that ψ does indeed change by $2\,\pi$ over one period, as we assumed to start with:

$$\Delta\psi = \oint \frac{\partial\psi}{\partial q}\,dq = \oint \frac{\partial^2 \widetilde{W}_2}{\partial q\,\partial I}\,dq = \frac{\partial}{\partial I}\oint \frac{\partial \widetilde{W}_2}{\partial q}\,dq = \frac{\partial}{\partial I}\oint p\,dq = 2\,\pi$$

because $I = \frac{1}{2\,\pi}\oint p\,dq$. If T is the temporal period of the motion, we are thus guaranteed

$$\begin{aligned} \omega\,T &= \Delta\psi = 2\,\pi \\ T &= \frac{2\,\pi}{\omega} = 2\,\pi\left(\frac{\partial H(I)}{\partial I}\right)^{-1} \end{aligned}$$

That is, to determine the period of the motion, we only need to know $\frac{\partial H(I)}{\partial I}$; we don't need to fully solve the equations of motion.

A final interesting feature is that the line integral $\oint p\,dq$ can be rewritten as an area integral. Recall that $p = p(q, I)$, so that $\oint p\,dq$ is really just the area between the line $p = 0$ and the curve $p = p(q, I)$. If the orbit is closed, then one gets contributions from the $p > 0$ region during one half of the period and the $p < 0$ region during the other half, while q goes from one extreme of its motion to the other and back. So one gets the area enclosed by the contour $p = p(q, I)$. If the orbit is not closed, one only gets the area on one side of the line $p = 0$. So, for *closed orbits*,

$$I = \frac{1}{2\,\pi}\int\int dp\,dq \tag{2.60}$$

Example 2.14: Action-angle variables for the simple harmonic oscillator.

We have already done this above using the generating function that was used earlier, but we can find the action-angle variables without that function also. The Hamiltonian is (again, absorbing the mass into the coordinate)

$$H = \frac{1}{2}\left(p^2 + \omega^2\,q^2\right) \equiv E$$

and has constant value E. Therefore, we may find $p = p(q, E)$:

$$p = \sqrt{2\,E - \omega^2\,q^2}$$

The action variable is thus

$$\begin{aligned} I &= \frac{1}{2\,\pi}\oint p\,dq = \oint dq\,\sqrt{2\,E - \omega^2\,q^2} \\ &= \frac{2\,E}{2\,\pi\,\omega}\int_0^{2\pi} d\sin\theta\ \sqrt{1 - \sin^2\theta} \\ &= \frac{2\,E}{2\,\pi\,\omega}\int_0^{2\pi} \cos^2\theta\,d\theta \\ &= \frac{2\,E}{2\,\pi\,\omega}\int_0^{2\pi} \frac{1}{2}\left(1 + \cos 2\,\theta\right)\,d\theta \\ &= \frac{E}{\omega} \end{aligned}$$

where the integral has been done using the substitution $q = \frac{\sqrt{2\,E}}{\omega}\sin\theta$. The oscillation period is

$$T = 2\,\pi\left(\frac{\partial H(I)}{\partial I}\right)^{-1} = \frac{2\,\pi}{\omega}$$

as expected.

It is amusing to note that the action-angle variable formalism can be used to derive the generating function for the canonical transformation of the simple harmonic oscillator from position and momentum to energy and phase, which was the generating function that would have been very difficult to guess *a priori*. Details are provided in Hand and Finch Section 6.5. In general, this is how one can derive generating functions systematically rather than guessing them.

Additional Examples: The pendulum example is worked out in Hand and Finch Section 6.5.

Multiple-Dimensional Action-Angle Variables

Suffice it to say that one can generalize action-angle variables to multiple coordinate dimensions if there is some coordinate system in which the periodic motion is separable into periodic motion in each dimension. Looking forward to our discussion of the Hamilton-Jacobi equation, this is equivalent to the statement that the Hamilton-Jacobi equation is separable in some coordinate system. We will then be able to write Hamilton's Characteristic Function as a sum of terms, $W(\vec{q},\vec{\alpha}) = \sum_k W_k(q_k,\vec{\alpha})$, each of which generates its own canonical transformation. Separability implies that the period of the motion in the different coordinates may be different.

Adiabatic Invariance

Another useful property of action variables is that they are **adiabatically invariant** – they remain fixed for sufficiently small time variations in the Hamiltonian, even if the total energy changes more quickly.

To prove this, consider a Hamiltonian that is dependent on time only through some parameter α (*e.g.*, the spring constant of the spring in the simple harmonic oscillator), with α varying slowly. We require the variation be slow enough so that, in a single period of the motion, $\Delta E/E \ll 1$. The generating function necessarily also becomes a function of α and thus of time. The canonical transformation from (q,p) to (ψ,I) therefore changes the Hamiltonian as follows (remember, this refers back to our introduction to the use of generating functions):

$$\begin{aligned}\tilde{H}(\psi,I,\alpha) &= H(p(I,\psi,\alpha),q(I,\psi,\alpha),\alpha) + \frac{\partial\widetilde{W}(q,I,\alpha)}{\partial t}\\ &= H(p(I,\psi,\alpha),q(I,\psi,\alpha),\alpha) + \frac{\partial\widetilde{W}(q,I,\alpha)}{\partial\alpha}\,\dot{\alpha}\end{aligned}$$

where the second term has been rewritten using $\frac{\partial}{\partial\alpha}$ because the explicit time dependence in the entire problem is only through $\alpha(t)$. Let us calculate $\dot{I}$ from Hamilton's equations in the new Hamiltonian:

$$\dot{I} = -\frac{\partial\tilde{H}}{\partial\psi} = -\frac{\partial H}{\partial\psi} - \frac{\partial^2\widetilde{W}}{\partial\,\psi\,\partial\,\alpha}\,\dot{\alpha}$$

The first term vanishes because the Hamiltonian is cyclic in ψ. This does not mean that the Hamiltonian before the canonical transformation is time-independent; rather, it simply means that the Hamiltonian, when expressed in terms of (ψ, I) still has no direct dependence on ψ. As an example, consider the harmonic oscillator, which is rewritten as $H = \omega\, I$ in terms of action-angle variables. The lack of ψ dependence does not change if we allow ω to vary slowly with time, though of course H is now time-dependent. We average $\dot{I}$ over one period:

$$-\left\langle \dot{I} \right\rangle = \frac{1}{2\,\pi} \int_0^{2\,\pi} d\psi\, \dot{\alpha}\, \frac{\partial^2 \widetilde{W}}{\partial\, \psi\, \partial\, \alpha}$$

We neglect higher order variations in α beyond $\dot{\alpha}$, so $\dot{\alpha}$ can be pulled out of the integral and the integral done trivially:

$$-\left\langle \dot{I} \right\rangle = \frac{\dot{\alpha}}{2\,\pi} \left[\frac{\partial \widetilde{W}(q, \psi + 2\,\pi, \alpha(T))}{\partial \alpha} - \frac{\partial \widetilde{W}(q, \psi, \alpha(0))}{\partial \alpha} \right]$$

Recall that $\widetilde{W}$ is periodic in ψ because the motion is periodic in ψ. So we have

$$\begin{aligned} -\left\langle \dot{I} \right\rangle &= \frac{\dot{\alpha}}{2\,\pi} \left[\frac{\partial \widetilde{W}(q, \psi, \pi, \alpha(T))}{\partial \alpha} - \frac{\partial \widetilde{W}(q, \psi, \alpha(0))}{\partial \alpha} \right] \\ &= T\, \frac{\dot{\alpha}^2}{2\,\pi}\, \frac{\partial^2 \widetilde{W}}{\partial \alpha^2} \approx 0 \end{aligned}$$

That is, because $\left\langle \dot{I} \right\rangle \propto \dot{\alpha}^2$, it varies much more slowly in time than the Hamiltonian (which depends linearly on $\dot{\alpha}$ via the generating function). To first order $\dot{\alpha}$, we can take I to be constant. This is of great benefit, in that it gives us one conserved quantity even when the Hamiltonian is (slowly) time-varying.

Example 2.15: Harmonic oscillator with varying ω (*e.g.*, varying spring constant or varying mass). If ω is fixed, we have

$$E = \omega\, I$$

Adiabatic invariance implies I is constant to first order even as ω and E vary. Thus, though the energy of the system is changing, it is changing in a very well-defined way, proportional to ω with the constant of proportionality being I. Recall that the solution to the problem is (Example 2.13 and 2.14, also rewritten in terms of action-angle variables at the start of this section)

$$\begin{aligned} q &= \sqrt{\frac{2\,I}{\omega}}\, \sin\, \psi \\ p &= \sqrt{2\,I\,\omega}\, \cos\, \psi \end{aligned}$$

We thus see that the effect on the motion of varying ω is simply to vary the shape of the phase space ellipse but to otherwise leave the motion unchanged. In particular, the phase of the oscillator is not disturbed because the phase is contained in the sin and cos terms that are independent of ω. The lack of first-order time-dependence can to some extent be seen by considering the area enclosed by the above orbit. The orbit is an ellipse with

semimajor axes $\sqrt{\frac{2I}{\omega}}$ and $\sqrt{2I\omega}$, so the area is $2\pi I$ as one would expect. I is to first order independent of the variation of ω, so the area of the orbit stays fixed. The ellipse changes shape – if ω is increased, the maximum amplitude of the motion decreases and the maximum momentum increases – but its area is preserved. This is analogous to Liouville's theorem.

Other Examples: see Hand and Finch Section 6.5 for the pendulum example.

2.4.5 The Hamilton-Jacobi Equation

The Hamilton-Jacobi equation makes use of a special canonical transformation to convert the standard Hamiltonian problem of $2M$ first-order ordinary differential equations in $2M$ variables into a single first-order partial differential equation with $M+1$ partial derivatives with respect to the $\{q_k\}$ and time.

The Goal

We propose to do a slightly crazy thing – we want a canonical transformation from some arbitrary set of generalized coordinates $(\vec{q},\vec{p})$ to some new set $(\vec{Q},\vec{P})$ such that all the $\vec{Q}$ and $\vec{P}$ are constant. One way to guarantee this is to require that $\tilde{H}(\vec{Q},\vec{P})=0$. Recalling our equation for the transformation of the Hamiltonian under a canonical transformation, Equation 2.41, we are requiring there be a generating function F such that

$$0=\tilde{H}(\vec{Q},\vec{P})=H(\vec{q},\vec{p})+\frac{\partial F}{\partial t}$$

This is essentially a differential equation for F. Is it possible to find such a function F?

The Formal Solution – the Hamilton-Jacobi Equation

Since the new momenta will be constant, it is sensible to make F a function of the type F_2, $F=S(\vec{q},\vec{P})$. The $\vec{p}$ thus satisfy $p_k=\frac{\partial S}{\partial q_k}$. Our condition on the generating function is thus the partial differential equation, also known as the **Hamilton-Jacobi Equation**,

$$H\left(\vec{q},\frac{\partial S(\vec{q},\vec{P})}{\partial \vec{q}},t\right)+\frac{\partial S(\vec{q},\vec{P})}{\partial t}=0 \tag{2.61}$$

S is known as **Hamilton's Principal Function**. Since the $\vec{P}$ are constants, this is a partial differential equation in $M+1$ independent variables $\vec{q}$ and t for the function S. We are in this case choosing not to consider the partial derivatives $\frac{\partial S}{\partial \vec{q}}=\vec{p}$ to be independent of $\vec{q}$.[15] Since we have $M+1$ independent variables, there are $M+1$ constants of integration. One of these is the constant offset of S, which is physically irrelevant because physical quantities depend only on partial derivatives of S. Since a solution S of this equation will generate a transformation that makes the M components of $\vec{P}$ constant, and since S is a function of the $\vec{P}$, the $\vec{P}$ can be taken to be the M constants.[16]

Independent of the above equation, we know that there must be M additional constants to specify the full motion. These are the $\vec{Q}$. The existence of these extra constants is not

[15] As we have discussed many times before, it is our choice whether to impose this "constraint" at the beginning or the end of solving the problem; if we did not impose this constraint at the beginning, we would have to carry along the M constraint equations $\vec{p}=\frac{\partial S}{\partial \vec{q}}$ and apply them at the end.

[16] One could choose an arbitrary information-preserving combination of the $\vec{P}$ to instead be the constants, but clearly the choice we have made is the simplest one.

implied by the Hamilton-Jacobi equation, since it only needs $M + 1$ constants to find a full solution S. The additional M constants exist because of Hamilton's equations, which require $2\,M$ initial conditions for a full solution.

Since the $\vec{P}$ and $\vec{Q}$ are constants, it is conventional to refer to them with the symbols $\vec{\alpha} = \vec{P}$ and $\vec{\beta} = \vec{Q}$. The full solution $(\vec{q}(t), \vec{p}(t))$ to the problem is found by making use of the generating function S and the initial conditions $\vec{q}(0)$ and $\vec{p}(0)$. Recall the generating function partial derivative relations are

$$\vec{p} = \frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial \vec{q}} \tag{2.62}$$

$$\vec{\beta} = \frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial \vec{\alpha}} \tag{2.63}$$

The constants $\vec{\alpha}$ and $\vec{\beta}$ are found by applying the above equations at $t = 0$:

$$\vec{p}(t=0) = \left.\frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial \vec{q}}\right|_{t=0, \vec{q}(t=0), \vec{\alpha}} \tag{2.64}$$

$$\vec{\beta} = \left.\frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial \vec{\alpha}}\right|_{t=0, \vec{q}(t=0), \vec{\alpha}} \tag{2.65}$$

The latter two equations let us determine the $\vec{\alpha}$ and $\vec{\beta}$ from $\vec{q}(t=0)$ and $\vec{p}(t=0)$, and then the former two equations give us (implicitly, at least) $\vec{q}(t)$ and $\vec{p}(t)$.

We can more directly determine what S is by evaluating its total time derivative:

$$\begin{aligned} \frac{dS}{dt} &= \frac{\partial S}{\partial t} + \sum_k \frac{\partial S}{\partial q_k}\,\dot{q}_k \\ &= -H + \sum_k p_k\,\dot{q}_k \\ &= L \end{aligned}$$

where we arrived at the second line by using one of the generating function partial derivative relations $p_k = \frac{\partial S}{\partial q_k}$ and the Hamilton-Jacobi equation. We thus see that the generating function S is just the indefinite integral of the Lagrangian:

$$S = \int dt\, L$$

This is an interesting result – that the action integral is the generator of the canonical transformation that corresponds to time evolution. It comes back as a defining principle in field theory, quantum mechanics, and quantum field theory, and, to some extent, in the Feynman path integral formulation of quantum mechanics. It is not of particular practical use, though, since to calculate the indefinite integral we must already know the solution $\vec{q}(t), \vec{p}(t)$.

When H is Conserved – Hamilton's Characteristic Function and the Abbreviated Action

Let us consider the common case of H being conserved. This certainly occurs if H has no explicit time dependence, though that is not a necessary condition. Since H is a constant, we know that S can be written in the form

$$S(\vec{q}, \vec{\alpha}, t) = W(\vec{q}, \vec{\alpha}) - E\,t \tag{2.66}$$

where $E = H$ is the time-independent value of H. That the above rewriting is possible is seen by simply calculating $\frac{\partial S}{\partial t}$; the Hamilton-Jacobi equation is satisfied because H is constant. The definition implies that partial derivatives of S and W with respect to $\vec{q}$ are identical, so the Hamilton-Jacobi equation can be rewritten in the form (known as the **restricted Hamilton-Jacobi equation**)

$$H\left(\vec{q}, \frac{\partial W(\vec{q}, \vec{P})}{\partial \vec{q}}\right) = E \tag{2.67}$$

The function W is known as **Hamilton's Characteristic Function**. W can be rewritten in a more physical manner:

$$\begin{aligned} W &= S + E\,t = \int dt\ (L + H) \\ &= \int dt \sum_k p_k\,\dot{q}_k = \int d\vec{q}\cdot\vec{p} \end{aligned} \tag{2.68}$$

which is known as the **abbreviated action**.

W is more valuable than as just another interesting theoretical quantity. The restricted Hamilton-Jacobi equations looks like a canonical transformation of the Hamiltonian by a F_2 generating function because we have an equation where something (E) equals the Hamiltonian. The reason it must be a F_2 function is because the momenta are replaced by $\frac{\partial W}{\partial \vec{q}}$ in the original Hamiltonian.

It should first be realized that the canonical momenta $\vec{P}$ generated by W may not be the same as the $\vec{\alpha}$ generated by S; after all, W is a different function from S. But, clearly, W is very close to S, differing only by the term $E\,t$. One possible choice of the new momenta $\vec{P}$ is to say $P_1 = E$ and leave the remainder unchanged. That is, suppose we had solved for S and found the M constant momenta $\vec{\alpha}$. We have in the relation between S and W another constant E. Since there are only M constants to be specified to define S (neglecting the offset term), E must be some combination of those M constants. That is, of the $M + 1$ constants E and $\vec{\alpha}$, only M are independent. The solution S chooses the $\vec{\alpha}$ as the independent ones and E as the derived one. But we are free to instead make E an independent one and rewrite α_1 in terms of E and the remainder of $\vec{\alpha}$. This is not the *only* choice we could have made, but obviously it is a simple one.

Let us explore whether W does indeed qualify as a generating function and what transformation it generates. Does the above choice of the relation between the momenta $\vec{\alpha}$ from the Hamilton-Jacobi equation and the moment $\vec{P}$ of the restricted Hamilton-Jacobi equation work – does it generate a canonical transformation that makes $\tilde{H}$ simply equal to the canonical momentum E? We can see that the only difference between the transformations generated by S and by W is in P_1 and Q_1. The remaining P_j are left unchanged by definition. The corresponding Q_j are seen to be the same by calculating what they would be if W is indeed a generating function:

$$Q_j = \frac{\partial W}{\partial P_j} = \frac{\partial}{\partial P_j}\,(S + E\,t) = \frac{\partial S}{\partial P_j} + t\,\frac{\partial E}{\partial P_j} = \beta_j \qquad j \neq 1$$

where $\beta_j = \frac{\partial S}{\partial P_j}$ is the Q_j from our original S transformation and the $\frac{\partial E}{\partial P_j}$ term vanishes for $j \neq 1$ because E is P_1. Since the Q_j and P_j are expressly unchanged for $j > 1$, and neither

appears in $\tilde{H} = E$ (the Hamiltonian after the canonical transformation to $(\vec{Q}, \vec{P})$ generated by W), Hamilton's equations are trivially satisfied and the transformation generated by W is canonical in the $j > 1$ coordinates.

What about $j = 1$? The generating function would give us for Q_1:

$$Q_1 = \frac{\partial W}{\partial P_1} = \frac{\partial S}{\partial P_1} + t = \frac{\partial S}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial P_1} + t = \beta_1 \frac{\partial \alpha_1}{\partial P_1} + t \equiv \beta_1' + t$$

The statement $\frac{\partial S}{\partial P_1} = \frac{\partial S}{\partial \alpha_1}\frac{\partial \alpha_1}{\partial P_1}$ is not obvious. The reason it holds is because we know that P_1 can written in terms of α_1 and the other momenta α_j, and vice versa, and that, when all other momenta are held constant, variations in α_1 and P_1 are therefore directly related. Essentially, the derivatives all become one-dimensional and standard chain rule arguments then apply. Does this form for Q_1 show that the transformation is canonical? This form gives $\dot{Q}_1 = 1$. Hamilton's equations in the new coordinates would give $\dot{Q}_1 = \frac{\partial \tilde{H}}{\partial P_1} = 1$, so yes on the first of Hamilton's equations. The other Hamilton's equation, $\dot{P}_1 = -\frac{\partial \tilde{H}}{\partial Q_1}$, is trivially satisfied because both sides vanish: the left side because H is conserved, and the right side because Q_1 does not appear in $\tilde{H}$. So, then, we have shown that the transformation generated by W is indeed canonical.

To summarize: we have found that, when H is conserved and has value E, if $W(q, E, \alpha_2, \ldots, \alpha_M)$ satisfies the equation

$$H\left(\vec{q}, \frac{\partial W(\vec{q}, E, \alpha_2, \ldots, \alpha_M)}{\partial \vec{q}}, t\right) = E$$

then W generates a canonical transformation that transforms H into $\tilde{H} = E$, with the transformation of the coordinates having the generic functional form

$$p_k = \frac{\partial W(\vec{q}, E, \alpha_2, \ldots, \alpha_M)}{\partial q_k} \tag{2.69}$$

$$Q_k = \frac{\partial W(\vec{q}, E, \alpha_2, \ldots, \alpha_M)}{\partial \alpha_k} = \beta_k \qquad k > 1 \tag{2.70}$$

$$Q_1 = \frac{\partial W(\vec{q}, E, \alpha_2, \ldots, \alpha_M)}{\partial E} + t = \beta_1' + t \tag{2.71}$$

that incorporates initial conditions via the equations

$$p_k(t=0) = \left.\frac{\partial W(\vec{q}, E, \alpha_2, \ldots, \alpha_M)}{\partial q_k}\right|_{q(t=0), E, \alpha_2, \ldots, \alpha_M} \tag{2.72}$$

$$\beta_k = \left.\frac{\partial W(\vec{q}, E, \alpha_2, \ldots, \alpha_M)}{\partial \alpha_k}\right|_{q(t=0), E, \alpha_2, \ldots, \alpha_M} \tag{2.73}$$

$$\beta_1' = \left.\frac{\partial W(\vec{q}, E, \alpha_2, \ldots, \alpha_M)}{\partial E}\right|_{q(t=0), E, \alpha_2, \ldots, \alpha_M} \tag{2.74}$$

The distinction between $k = 1$ and $k > 1$ is due to the choice of P_1 as the energy. Note that, in all the equations, the time dependence is nonexistent or very simple. For the more general transformation and generating function S, while the Q_k were all constant, S could have had explicit time dependence, so there might have been an explicit time dependence

in the relation between $\vec{q}$, $\vec{\alpha}$, and $\vec{\beta}$. Now, we are guaranteed that there can be such explicit time dependence in only the Q_1 equation. And, since the Q_1 equation is so simple, we can use it to eliminate t completely as an independent variable. We can pick a coordinate (call it q_1 for specificity) as the independent variable and parameterize the evolution of the remaining q_k and all the p_k in terms of it. The resulting relations are called **orbit equations** because they describe the shapes of the particle paths in phase space.

Note that we could have made a more complicated choice of how to relate E to the canonical momenta. Instead of picking a transformation that makes $\tilde{H} = P_1$ a simple function of only the first canonical momentum, we could have chosen $\tilde{H} = \tilde{H}(\vec{P})$ a more complicated (but still time-independent) function of all the momenta. This is a choice, based on what is convenient for the problem. If we had chosen this route, some or all of the equations for Q_k would have had a linear time dependence in them. The system remains fairly simple. This kind of situation is presented in the example on the anisotropic simple harmonic oscillator below.

There is a very nice table in Section 10.3 of Goldstein comparing and contrasting the S and W transformations.

Separability

The abstract discussion above has made no attempt to demonstrate in what cases the equations are actually explicity solvable. One sufficient, but not necessary, condition for solubility is that the Hamilton-Jacobi equation be **separable**; that is, if Hamilton's Principal Function can be written as a sum of M terms, each depending on only one of the original coordinates and time,

$$S(\vec{q}, \vec{\alpha}, t) = \sum_k S_k(q_k, \vec{\alpha}, t) \tag{2.75}$$

then we can explicitly show how to convert the partial differential equation to M ordinary differential equations using **separation of variables**. If S can be written in the above fashion, we are guaranteed that H can also be written this way because H is, theoretically at least, derived from S through L. The Hamilton-Jacobi equation then becomes

$$\sum_k \left[H_k \left(q_k, \frac{\partial S_k}{\partial q_k}, \vec{\alpha}, t \right) + \frac{\partial S_k(q_k, \vec{\alpha}, t)}{\partial t} \right] = 0$$

Since each term in the sum depends on a different coordinate (and the same constants $\vec{\alpha}$), each term must separately vanish, yielding a set of M uncoupled equations

$$H_k \left(q_k, \frac{\partial S_k}{\partial q_k}, \vec{\alpha}, t \right) + \frac{\partial S_k(q_k, \vec{\alpha}, t)}{\partial t} = 0 \tag{2.76}$$

If the H_k are individually conserved (*e.g.*, none have any explicit time dependence), then we can further deduce

$$S_k(q_k, \vec{\alpha}, t) = W_k(q_k, \vec{\alpha}) - \alpha_k t$$

and therefore

$$H_k \left(q_k, \frac{\partial W}{\partial q_k}, \vec{\alpha} \right) = \alpha_k \tag{2.77}$$

where we have obviously chosen to take the M constants α_k to be the values of the individually conserved H_k rather than taking $\alpha_1 = \sum_k H_k$ and leaving the remaining α_k to be some linear combination of the H_k. By this choice, the total energy is $E = \sum_k \alpha_k$.

Separability of H in the above fashion is guaranteed if the Hamiltonian meets the **Staeckel Conditions**:

1. The Hamiltonian is conserved (as we already discussed above).
2. The Lagrangian is no more than a quadratic function of the generalized velocities, so the Hamiltonian has the form

$$H = \frac{1}{2} \left(\vec{p} - \vec{a}\right)^T \, \underline{T}^{-1} \, \left(\vec{p} - \vec{a}\right) + V(\vec{q})$$

   where the a_k are functions only of the conjugate coordinate, $a_k = a_k(q_k)$ only, $\underline{T}$ is a symmetric, invertible matrix, and $V(\vec{q})$ is a potential energy function that depends only on the coordinates.
3. The potential energy can be written in the form

$$V(\vec{q}) = \sum_k \frac{V(q_k)}{\underline{T}_{kk}}$$

4. And the final, inscrutable condition: If we define a matrix $\underline{\phi}$ by

$$\sum_l \delta_{kl} \, \underline{\phi}^{-1}_{kl} = \frac{1}{\underline{T}_{kk}}$$

   with

$$\frac{\partial W_k}{\partial q_k} - a_k = \sum_{lm} 2 \, \delta_{kl} \, \phi_{lm} \, \gamma_m$$

   where $\vec{\gamma}$ is an unspecified constant vector. The diagonal elements of $\underline{\phi}$ and $\underline{\phi}^{-1}$ may depend only on the associated coordinate.

We will not attempt to prove these conditions; they are proven in Appendix D of the second edition of Goldstein.

Examples

Example 2.16: Simple harmonic oscillator

The simple harmonic oscillator Hamiltonian is (using the same form as used in Example 2.13):

$$H = \frac{1}{2} \left(p^2 + \omega^2 \, q^2\right) \equiv E$$

where we have explicitly written the conserved value of H as E. The Hamilton-Jacobi equation for this Hamiltonian is

$$\frac{1}{2} \left[\left(\frac{\partial S}{\partial q} \right)^2 + \omega^2 \, q^2 \right] + \frac{\partial S}{\partial t} = 0$$

where we have made the substitution $p = \frac{\partial S}{\partial q}$ in keeping with the assumption that S is a F_2 generating function. Since H is conserved, we may proceed to the Hamilton's Characteristic Function, writing $\frac{\partial S}{\partial t} = \alpha$ a constant:

$$\frac{1}{2}\left[\left(\frac{\partial W}{\partial q}\right)^2 + \omega^2\, q^2\right] = \alpha$$

Obviously, $\alpha = E$ since the left side of the equation is the Hamiltonian. The above equation is directly integrable since it is now a differential equation in q only:

$$\begin{aligned}
\frac{\partial W}{\partial q} &= \sqrt{2\,E - \omega^2\, q^2} \\
W &= \sqrt{2\,E}\int dq\,\sqrt{1 - \frac{\omega^2\, q^2}{2\,E}} \\
S &= -E\,t + \sqrt{2\,E}\int dq\,\sqrt{1 - \frac{\omega^2\, q^2}{2\,E}}
\end{aligned}$$

We neglect to perform the integration (which is entirely doable via trigonometric substitution, resulting in a cos function) because we only need the partial derivatives of S. We have already evaluated the one constant canonical momentum, so let us obtain the corresponding constant β for the linearly evolving coordinate using Equation 2.71:

$$\begin{aligned}
t + \beta &= \frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial E} \\
&= \sqrt{\frac{1}{2\,E}}\int \frac{dq}{\sqrt{1 - \frac{\omega^2\, q^2}{2\,E}}} \\
&= \frac{1}{\omega}\,\arcsin\frac{\omega\, q}{\sqrt{2\,E}}
\end{aligned}$$

which is easily inverted to give

$$q = \frac{\sqrt{2\,E}}{\omega}\,\sin\left(\omega\, t + \phi\right)$$

where $\phi = \omega\,\beta$ is just a rewriting of the constant. We can now obtain p from Equation 2.70:

$$\begin{aligned}
p &= \frac{\partial W}{\partial q} = \sqrt{2\,E}\sqrt{1 - \frac{\omega^2\, q^2}{2\,E}} \\
&= \sqrt{2\,E}\,\cos\left(\omega\, t + \phi\right)
\end{aligned}$$

Finally, we need to connect the constants E and ϕ (α and β in the formal derivation) with the initial conditions. We could go back to Equations 2.65 and 2.65, but it is easier to simply make use of the solutions directly. From the Hamiltonian, we see

$$E = \frac{1}{2}\left(p_0^2 + \omega^2\, q_0^2\right)$$

and from the solutions for $q(t)$ and $p(t)$ we see

$$\tan\phi = \omega\,\frac{q_0}{p_0}$$

Hamilton's Principal Function in fact generates the same canonical transformation as we saw in Example 2.13, converting from position and momentum to energy and phase. The energy is conserved and the phase evolves linearly with time. If we want, we can recover Hamilton's Principal Function explicitly by substituting the solution into our integral form for S and integrating:

$$\begin{aligned} S &= -E\,t + \sqrt{2\,E}\int dq\,\sqrt{1-\frac{\omega^2\,q^2}{2\,E}} \\ &= -E\,t + \sqrt{2\,E}\int \left[d\left(\frac{\sqrt{2\,E}}{\omega}\,\sin\left(\omega\,t+\phi\right)\right)\right]\sqrt{1-\sin^2\left(\omega\,t+\phi\right)} \\ &= -E\,t + 2\,E\int dt\,\cos^2\left(\omega\,t+\phi\right) \\ &= 2\,E\int dt\left[\cos^2\left(\omega\,t+\phi\right)-\frac{1}{2}\right] \end{aligned}$$

where notice that we not only had to substitute for q but for dq also. If we calculate the Lagrangian directly, we have

$$\begin{aligned} L &= \frac{1}{2}\left(p^2-\omega^2\,q^2\right) \\ &= E\left(\cos^2\left(\omega\,t+\phi\right)-\sin^2\left(\omega\,t+\phi\right)\right) \\ &= 2\,E\left[\cos^2\left(\omega\,t+\phi\right)-\frac{1}{2}\right] \end{aligned}$$

and we see that $S=\int dt\,L$ explicitly.

Example 2.17: Two-dimensional anisotropic harmonic oscillator

An anisotropic two-dimensional harmonic oscillator has different spring constants (and therefore different characteristic frequencies) in the two dimensions. It thus does not trivially separate into cylindrical coordinates. The Hamiltonian is

$$H=\frac{1}{2}\left(p_x^2+p_y^2+\omega_x^2\,x^2+\omega_y^2\,y^2\right)\equiv E$$

The Hamiltonian is clearly separable in Cartesian coordinates, so we are led to a Hamilton's Principal Function of the form

$$S(x,y,\alpha_x,\alpha_y,t)=W_x(x,\alpha_x)+W_y(y,\alpha_y)-(\alpha_x+\alpha_y)\,t$$

where $E=\alpha_x+\alpha_y$. Here we have chosen to do the problem symmetrically in the two momenta rather than pick one to be the energy. Since the system is separable, we may go directly to Equation 2.77:

$$\begin{aligned} \frac{1}{2}\left[\left(\frac{\partial W_x}{\partial x}\right)^2+\omega_x^2\,x^2\right] &= \alpha_x \\ \frac{1}{2}\left[\left(\frac{\partial W_y}{\partial y}\right)^2+\omega_y^2\,y^2\right] &= \alpha_y \end{aligned}$$

Each equation is just a 1-dimensional oscillator equation, so the solution is

$$\begin{aligned} x &= \frac{\sqrt{2\,\alpha_x}}{\omega_x}\,\sin\left(\omega_x\,t+\phi_x\right) \\ p_x &= \sqrt{2\,\alpha_x}\,\cos\left(\omega_x\,t+\phi_x\right) \\ y &= \frac{\sqrt{2\,\alpha_y}}{\omega_y}\,\sin\left(\omega_y\,t+\phi_y\right) \\ p_y &= \sqrt{2\,\alpha_y}\,\cos\left(\omega_y\,t+\phi_y\right) \end{aligned}$$

Example 2.18: Isotropic two-dimensional harmonic oscillator

Here, $\omega_x = \omega_y$ so we write the Hamiltonian in polar coordinates (r,θ):

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \omega^2\,r^2\right) \equiv E$$

where $p_\theta = r^2\,\dot{\theta}$ is the canonical momentum conjugate to θ. The problem is not trivially separable because the second term depends on both r and p_θ, mixing the two coordinates. However, because the Hamiltonian is cyclic in θ, we are assured that $p_\theta = \alpha_\theta$ is constant. This is the condition we need to be able to separate variables: Hamilton's Principal Function must be writeable as a sum of terms $S_k(q_k,\vec{\alpha},t)$, which we can indeed do:

$$S(r,\theta,E,\alpha_\theta) = W_r(r,E,\alpha_\theta) + W_\theta(\theta,E,\alpha_\theta) - E\,t$$

where here we do not try to have symmetric constants because there is no symmetry between r and θ. The reduced Hamilton-Jacobi equation is

$$\frac{1}{2}\left[\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{\alpha_\theta^2}{r^2} + \omega^2\,r^2\right] = E$$

There is no equation for θ because there are no terms in H that depend on θ. We can trivially obtain W_θ by making use of

$$\begin{aligned} &\frac{\partial W_\theta}{\partial\theta} = p_\theta = \alpha_\theta \\ &W_\theta = \alpha_\theta\,\theta \end{aligned}$$

At this point, we are essentially done with the formal part, and we need only to solve the above differential equation for r mathematically. But that equation is not trivially integrable, so we pursue a different path (with some foreknowledge of the solution!). The solution is no doubt going to involve simple harmonic motion in the two dimensions, so we try

$$\begin{aligned} x &= \frac{\sqrt{2\,\alpha}}{\omega}\,\sin\left(\omega\,t+\phi\right) \\ p_x &= \sqrt{2\,\alpha}\,\cos\left(\omega\,t+\phi\right) \\ y &= \frac{\sqrt{2\,\alpha}}{\omega}\,\sin\omega\,t \\ p_y &= \sqrt{2\,\alpha}\,\cos\omega\,t \end{aligned}$$

where we include a phase factor for x only because there can be only one undetermined constant of integration for this first-order differential equation. The resulting polar and angular coordinate and momenta solutions are

$$\begin{aligned} r &= \frac{\sqrt{2\,E}}{\omega}\sqrt{\sin^2(\omega\,t+\phi)+\sin^2\omega\,t} \\ p_r &= \dot{r} \\ \theta &= \arctan\left[\frac{\sin\omega\,t}{\sin(\omega\,t+\phi)}\right] \\ p_\theta &= r^2\,\dot{\theta} = \alpha_\theta \end{aligned}$$

For completeness, let us calculate p_θ explicitly from θ to see what form it has.

$$\begin{aligned} p_\theta &= r^2\,\dot{\theta} \\ &= \frac{2\,E}{\omega^2}\left[\sin^2(\omega\,t+\phi)+\sin^2\omega\,t\right]\frac{d}{dt}\arctan\left[\frac{\sin\omega\,t}{\sin(\omega\,t+\phi)}\right] \\ &= \frac{2\,E}{\omega^2}\left[\sin^2(\omega\,t+\phi)+\sin^2\omega\,t\right]\left\{1+\frac{\sin^2\omega\,t}{\sin^2(\omega\,t+\phi)}\right\}^{-1}\frac{d}{dt}\left[\frac{\sin\omega\,t}{\sin(\omega\,t+\phi)}\right] \\ &= \frac{2\,E}{\omega^2}\,\frac{\sin^2(\omega\,t+\phi)+\sin^2\omega\,t}{1+\frac{\sin^2\omega\,t}{\sin^2(\omega\,t+\phi)}}\left\{\frac{\omega\,\cos\omega\,t}{\sin(\omega\,t+\phi)}-\frac{\sin\omega\,t}{\sin^2(\omega\,t+\phi)}\,\omega\,\cos(\omega\,t+\phi)\right\} \\ &= \frac{2\,E}{\omega}\left[\cos\omega\,t\,\sin(\omega\,t+\phi)-\sin\omega\,t\cos(\omega\,t+\phi)\right] \\ &= \frac{2\,E}{\omega}\,\sin\phi \end{aligned}$$

which is very nice. If we consider $\phi = 0$, so the two oscillators are perfectly in phase, then the motion is along a straight line through the origin:

$$\begin{aligned} r &= \frac{2\sqrt{E}}{\omega}\,|\sin\omega\,t| \\ p_r &= \sqrt{2\,E}\cos\omega\,t \\ \theta &= \arctan 1 = \frac{\pi}{4} \\ p_\theta &= 0 \end{aligned}$$

If we consider $\phi = \frac{\pi}{2}$, the two oscillators are $\pi/2$ out of phase and the motion is circular:

$$\begin{aligned} r &= \frac{\sqrt{2\,E}}{\omega}\sqrt{\cos^2\omega\,t+\sin^2\omega\,t} = \frac{\sqrt{2\,E}}{\omega} \\ p_r &= 0 \\ \theta &= \omega\,t \\ p_\theta &= \frac{2E}{\omega} = r^2\,\omega \end{aligned}$$

as one would expect for perfect circular motion.

Additional Examples: See Hand and Finch Section 6.5 – there are a couple of gravitational examples.

Deriving Action-Angle Variables via the Hamilton-Jacobi Formalism

Earlier, we introduced the concept of action-angle variables for 1-dimensional systems with a conserved Hamiltonian by using F_1-type and F_2-type generating functions to transform to coordinates (ψ, I) such that I is constant and ψ evolves linearly with time. This is obviously quite analogous to the Hamilton-Jacobi formalism, so it is interesting to repeat the derivation using those results.

We have a 1-dimensional system for which the Hamiltonian is conserved. We can then certainly use the Hamilton-Jacobi formalism, specializing to the conserved Hamiltonian case. That formalism gives us the new generating function, Hamilton's Characteristic Function (see Equation 2.68)

$$W = \int p\, dq$$

and tells us that the transformation generated by W makes the Hamiltonian identically equal to the constant momentum variable α_1 and yields a linearly evolving coordinate $Q_1 = t + \beta_1$ where β_1 is set by initial conditions. Now, since I is not identically equal to E, this exact equation for Q_1 does not hold for ψ. But we can find ψ. First, the explicit form of ψ is given by the generating function, which yields the relation

$$\psi = \frac{\partial W}{\partial I} = \frac{\partial}{\partial I} \int p\, dq$$

Where does I come in? Remember that we wrote earlier $p = p(q, \alpha_1)$. We now have two constants I and α_1 when we know there is only one independent constant, so we can choose I as that constant and rewrite α_1 in terms of I. So

$$\psi = \frac{\partial}{\partial I} \int p(q, I)\, dq$$

The time evolution of ψ is found by Hamilton's equations

$$\dot{\psi} = \frac{\partial H(I)}{\partial I} \equiv \omega$$

which is just some constant determined by the particular functional form of $H(I)$.

Hamilton-Jacobi Theory, Wave Mechanics, and Quantum Mechanics

Think about S as a surface in configuration space (the M-dimensional space of the system coordinates $\vec{q}$). The Hamilton-Jacobi equation finds a function S such that the momenta are $p_k = \frac{\partial S}{\partial q_k}$, which can be written $\vec{p} = \vec{\nabla}_q S$. The gradient of a function gives a vector field normal to surfaces over which the function value is constant. So the momentum vector is normal to surfaces of constant S. The momentum vector is also tangent to the particle trajectory in configuration space. So the particle trajectories are always normal to the surfaces of constant S. For conserved Hamiltonians, where we write $S = W - E\, t$, the shapes of the surfaces are given by Hamilton's Characteristic Function W and they move linearly in time due to the $-E\, t$ term.

In optics, a classical electromagnetic propagates in a similar fashion. There are surfaces of constant phase ϕ. These are like the surfaces of constant S. The wave propagates by following the normal to the surfaces of constant phase just as the mechanical system

follows the trajectories perpendicular to the surfaces of constant S. Geometrical optics rays are analogous to the particle trajectories, as these rays are always perpendicular to surfaces of constant phase. The geometrical optics wavevector $\vec{k}$ is the obvious analogy to the momentum $\vec{p}$.

Thus, in the way that a classical wave's phase advances linearly in time – the surfaces of constant phase propagate linearly outward according to the wave's phase velocity – similarly the surfaces of constant S propagate forward at a speed determined by the total energy E. The action integral S appears to have the characteristics of a wave phase function ϕ.

In the pre-quantum era, this analogy was little more than a mathematical curiosity, perhaps providing some practical benefit by allowing the flow of solutions between mechanics and optics. The power of the analogy becomes clear only when we consider how it relates to quantum mechanics.

In quantum mechanics, a particle is described not by a simple trajectory $\vec{q}(t)$, but rather by a wave function $\psi(\vec{x},t)$ that can have nonzero value at all points in space and time. If we let $\rho(\vec{x},t) = |\psi(\vec{x},t)|^2$, then ρ describes the probability of finding the particle at position $\vec{x}$ at time t. The classical trajectory is given by $\vec{q}(t) = \int d^3x\, \vec{x}\, |\psi(\vec{x},t)|^2$; *i.e.*, it is the first moment of the wave function. In one dimension, the wave function satisfies Schrodinger's equation,

$$-\frac{\hbar^2}{2\,m}\,\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi = i\,\hbar\,\frac{\partial\psi}{\partial t}$$

where $V(x)$ is the classical potential in which the particle moves. The wave function clearly must have the form $\psi(x,t) = \sqrt{\rho(x,t)}\,e^{i\,\phi(x,t)}$. Given the analogy between S and a classical wave's phase ϕ above, let us in fact assume $\phi = S/\hbar$. As $\hbar \to 0$, the phase of the wave oscillates faster and faster, which we might expect would hide the wave characteristics (waves are most "wave-like" in the long-wavelength limit) as is necessary in this limit. If we rewrite $\psi(x,t) = \sqrt{\rho(x,t)}\,e^{\frac{i}{\hbar}\,S(x,t)}$, Schrodinger's equation becomes

$$-\frac{\hbar^2}{2\,m}\left\{\frac{\partial^2\sqrt{\rho}}{\partial x^2} + \frac{2\,i}{\hbar}\,\frac{\partial\sqrt{\rho}}{\partial x}\,\frac{\partial S}{\partial x} - \frac{1}{\hbar^2}\,\sqrt{\rho}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{i}{\hbar}\,\sqrt{\rho}\,\frac{\partial^2 S}{\partial x^2}\right\} e^{\frac{i}{\hbar}\,S} + V(x)\,\sqrt{\rho}\,e^{\frac{i}{\hbar}\,S}$$
$$= i\,\hbar\left\{\frac{\partial\sqrt{\rho}}{\partial t} + \frac{i}{\hbar}\,\sqrt{\rho}\,\frac{\partial S}{\partial t}\right\} e^{\frac{i}{\hbar}\,S}$$

The equation contains terms up to second order in $\hbar$; neglect all terms containing $\hbar$ or $\hbar^2$. This leaves (canceling the common $\sqrt{\rho}\,e^{\frac{i}{\hbar}\,S}$)

$$\frac{1}{2\,m}\left(\frac{\partial S}{\partial x}\right)^2 + V(x) + \frac{\partial S}{\partial t} = 0$$

We thus recover the Hamilton-Jacobi equation for a particle propagating in a potential $V(x)$.

Chapter 3

Oscillations

Oscillations are a physical phenomenon seen in a wide variety of physical systems. They are especially important in that they describe the motion of a system when perturbed slightly from an equilibrium. We work through the basic formalism of simple linear oscillations, both natural and driven, consider coupled oscillating systems and show how to decompose them in terms of normal coordinates, and apply the theory to oscillations of continuous systems to introduce wave phenomena.

3.1 The Simple Harmonic Oscillator

The linear simple harmonic oscillator (SHO) is the foundation of the theory of oscillations. We discuss equilibria in physical systems and how small oscillations about equilibria can in most cases be described by the SHO equation. We investigate the effect of damping and driving forces on the SHO, using the opportunity to introduce Green's functions.

We follow Hand and Finch Chapter 3 for the most part, though we make small changes here and there. Most intermediate mechanics texts will have a similar discussion of the basics, though most will not discuss Green's functions.

3.1.1 Equilibria and Oscillations

Types of Equilibria

Recall in Section 1.1.3 that the equilibrium points of a conservative system (one with a potential energy function) can be found by requiring $\vec{\nabla} U = 0$, and that the stability of the equilibria is determined by the sign of the second derivatives. In one dimension, the types of equilibria are

- **stable equilibrium**: $d^2U/dx^2 > 0$
- **unstable equilibrium**: $d^2U/dx^2 < 0$
- **saddle point**: $d^2U/dx^2 = 0$

With rheonomic systems, the stability of an equilibrium point may change in time.

Equilibria from the Lagrangian Point of View

Consider a Taylor expansion around an equilibrium point $(q_0, \dot{q}_0)$ of an arbitrary 1-dimension Lagrangian to second order in the generalized coordinates and velocities:

$$L \approx A + B\,q + C\,\dot{q} + D\,q^2 + E\,q\,\dot{q} + F\,\dot{q}^2$$

We neglect A since it is a constant offset and does not affect the dynamics. The constants are given by assorted partial derivatives:

$$B = \left.\frac{\partial L}{\partial q}\right|_{(q_0,\dot{q}_0)} \qquad C = \left.\frac{\partial L}{\partial \dot{q}}\right|_{(q_0,\dot{q}_0)}$$

$$D = \left.\frac{1}{2}\frac{\partial^2 L}{\partial q^2}\right|_{(q_0,\dot{q}_0)} \qquad E = \left.\frac{\partial^2 L}{\partial\,\dot{q}\,\partial\,q}\right|_{(q_0,\dot{q}_0)} \qquad F = \left.\frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}^2}\right|_{(q_0,\dot{q}_0)}$$

$B = 0$ in order for $(q_0, \dot{q}_0)$ to be an equilibrium position. The Euler-Lagrange equation for the system is

$$\frac{d}{dt}\left(C + E\,q + 2\,F\,\dot{q}\right) = 2\,D\,q + E\,\dot{q}$$

$$\ddot{q} - \frac{D}{F}\,q = 0$$

We would have found the same equation of motion had we started with

$$L \approx D\,q^2 + F\,\dot{q}^2$$

This is just the simple harmonic oscillator Lagrangian with characteristic frequency

$$\omega^2 \equiv -\frac{D}{F}$$

Now, let's rescale the time units to get rid of the coefficients: define $\beta = \sqrt{\left|\frac{F}{D}\right|}$ and define a new time coordinate τ by $t = \beta\,\tau$. The equation of motion and Lagrangian becomes

$$\ddot{q} \pm q = 0 \qquad L = \frac{1}{2}\left(\dot{q}^2 \mp q^2\right) \tag{3.1}$$

where the sign is the opposite of the sign of $\frac{D}{F}$. For reasonable Lagrangians, F is the coefficient of the kinetic energy term and thus $F > 0$ holds.

Restricting to conservative, scleronomic systems, we are assured that the sign of D is set by the sign of the second derivative of the potential energy, so stability is, as we found before, determined by the shape of the potential energy function:

$$\begin{array}{lllll} \text{stable:} & \ddot{q} + q = 0 & D < 0 & \frac{\partial^2 U}{\partial q^2} > 0 \\ \text{unstable:} & \ddot{q} - q = 0 & D > 0 & \frac{\partial^2 U}{\partial q^2} < 0 \end{array}$$

Thus, we see that working from the Lagrangian perspective yields, as we would expect, the same conditions for stable and unstable equilibria as we found from the Newtonian perspective. More importantly, we see the central reason for studying the SHO in detail: *the Taylor expansion of (almost) any scleronomic, conservative, one-dimensional system about a stable equilibrium looks like a simple harmonic oscillator.* We will not show it here, but this conclusion extends to multi-dimensional systems, too.

The Hamiltonian

Finally, we note that the above Taylor expansion also lets us write an approximate Hamiltonian for the system. Starting from the original L, we have

$$\begin{aligned} p &\approx \frac{\partial L}{\partial \dot{q}} = C + E\,q + 2\,F\,\dot{q} \\ H &= p\,\dot{q} - L \\ &\approx C\,\dot{q} + E\,q\,\dot{q} + 2\,F\,\dot{q}^2 - A - B\,q - C\,\dot{q} - D\,q^2 - E\,q\,\dot{q} - F\,\dot{q}^2 \\ &= F\,\dot{q}^2 - D\,q^2 \end{aligned}$$

or, after rescaling

$$H = \frac{1}{2}\left(\dot{q}^2 \pm q^2\right) \tag{3.2}$$

where the sign again determines the stability, $+$ is stable, $-$ is unstable. Since we earlier demonstrated that the C and E terms do not affect the dynamics, we are free to drop them and redefine the momentum simply as

$$p = 2\,F\,\dot{q}$$

3.1.2 Solving the Simple Harmonic Oscillator

Characteristics of the Equation of Motion

The equation of motion for the stable simple harmonic oscillator has been written

$$\ddot{q} + q = 0$$

This is a linear, second-order differential equation with constant coefficients. It is:

- **homogeneous**: (the right side vanishes), so solutions can be scaled by a constant and still satisfy the equation.
- **linear**: Because the equation is linear in q and its time derivatives, a linear combination $a\,q_1(t) + b\,q_2(t)$ of two solutions is also a solution.

Conventional Solution

You are no doubt well aware that the general solution of this equation of motion is a sinusoid, which can be written in two forms:

$$a(t) = A\,\sin(t+\phi) = A'\cos t + B'\sin t$$

where the two sets of coefficients are related to each other and the initial conditions by

$$A\,\sin\phi = A' = q(0) \qquad A\,\cos\phi = B' = \dot{q}(0)$$

The period of oscillation is $T = 2\,\pi$, the frequency is $\nu = 1/2\,\pi$, and the angular frequency is $\omega = 1$. For arbitrary $\omega \neq 1$, these become $\nu = \omega/2\,\pi$ and $T = 2\,\pi/\omega$.

Complex Solution

Given the sinusoidal nature of the solution, we could equally well have written it as the real part of a complex function:

$$q_c(t) = \mathcal{A}_c\,e^{it} = A_c\,e^{i\phi}\,e^{it} = A_c\,[\cos(t+\phi) + i\,\sin(t+\phi)] \tag{3.3}$$

where $\mathcal{A}_c = A_c\,e^{i\phi}$ (and thus $A_c = |\mathcal{A}_c|$). The initial conditions are

$$\begin{aligned} q(0) &= \mathcal{R}[q_c(0)] = A_c\,\cos\phi = \mathcal{R}[\mathcal{A}_c] \\ \dot{q}(0) &= \mathcal{R}[\dot{q}_c(0)] = -A_c\,\sin\phi = -\mathcal{I}[\mathcal{A}_c] \end{aligned}$$

or, equivalently,

$$\mathcal{A}_c = q(0) - i\,\dot{q}(0)$$

where $\mathcal{R}[\,]$ and $\mathcal{I}[\,]$ take the real and imaginary part of a complex number. When a physical solution is needed, we simply take $\mathcal{R}[q(t)]$.

We note that a shift of the time origin by t_0 is now just a simple phase factor,

$$\mathcal{A}_c = [q(t_0) - i\,\dot{q}(t_0)]\,e^{-it_0}$$

Note the sign in the phase factor.

3.1.3 The Damped Simple Harmonic Oscillator

Most simple harmonic oscillators in the real world are damped – mechanical oscillators, electrical oscillators, etc. Thus, the damped simple harmonic oscillator is the next important system to study.

Introducing Damping

We assume that a damping force linear in velocity is applied to the harmonic oscillator. For a mechanical oscillator, this could be a frictional force. For an electrical oscillator, this could be a resistive element. Regardless, the frictional force is assumed to be of the form

$$F_{damp} = -\frac{\dot{q}}{Q}$$

The dimensionless constant Q is called the **quality factor** and is ubiquitous in the description of oscillatory systems with damping.

It is nontrivial to introduce frictional forces via the Lagrangian formalism. Since we have already written an equation of motion, it is straightforward to include the damping term via Newtonian mechanics. The equation of motion of the **damped simple harmonic oscillator** is

$$\ddot{q} + \frac{\dot{q}}{Q} + q = 0 \tag{3.4}$$

This is obviously not the most generic frictional force that one can think of. It is, however, the kind that is encountered in most damped electrical circuits and thus is of great practical interest.

It is instructive to go back to a dimensional version of the equation to understand the physical meaning of Q. The equation of a motion of a mechanical oscillator subject to a linear damping force is

$$\begin{aligned} m\,\ddot{x} + b\,\dot{x} + k\,x &= 0 \\ \ddot{x} + \frac{b}{m}\,\dot{x} + \frac{k}{m}\,x &= 0 \end{aligned}$$

If time is rescaled by $\beta = \sqrt{\frac{m}{k}} = \omega^{-1}$ (*i.e.*, $t = \beta\,\tau$), then $Q^{-1} = \frac{b/m}{\omega} = \frac{b}{\sqrt{k\,m}}$. Q decreases as the damping constant increases, and Q increases as the spring constant or mass increase relative to the damping constant (both of which will tend to make the damping constant less important, either because there is a larger force for a given displacement or because the moving mass has more inertia).

The equation of motion of a series LRC oscillator circuit (where q represents the charge on the capacitor and $\dot{q}$ is the current flowing in the circuit) is[1]

$$\begin{aligned} L\,\ddot{q} + \dot{q}\,R + \frac{q}{C} &= 0 \\ \ddot{q} + \frac{\dot{q}}{L/R} + \frac{1}{LC}\,q &= 0 \\ \ddot{q} + \frac{\dot{q}}{\tau_{damp}} + \omega^2\,q &= 0 \end{aligned}$$

[1]The choice of $\tau_{damp} = L/R$ instead of $\tau_{damp} = R\,C$ is determined by the correspondence $b \leftrightarrow R$, $m \leftrightarrow L$.

When we rescale time by using $t = \beta\,\tau$ with $\beta = \sqrt{L\,C} = \omega^{-1}$, we obtain

$$\ddot{q} + \frac{\dot{q}}{\omega\,\tau_{damp}} + q = 0$$

giving $Q = \omega\,\tau_{damp} = 2\,\pi\,\tau_{damp}/T$ where $T = 2\,\pi/\omega$ is the oscillation period. That is, the quality factor can be thought of as the ratio of the damping timescale to the oscillation timescale; if the damping timescale is many oscillation timescales, then the resonator has high Q.

A note on units: for reference, we rewrite the various equations with all the ωs in place so the units are clear.

$$F_{restore} = -k\,x$$

$$F_{damp} = -b\,\dot{x} = \frac{m\,\omega}{Q}\,\dot{x} = \frac{\sqrt{k\,m}}{Q}\,\dot{x}$$

$$\ddot{x} + \frac{\omega}{Q}\,\dot{x} + \omega^2\,x = 0$$

Correspondence with Other Textbooks

Unfortunately, different textbooks define Q in different ways. We list here the three possible conventions so the reader may understand the correspondence between the results derived here and those in other textbooks. The three choices essentially consist of whether Q is defined relative to the natural frequency, the damped characteristic frequency, or the amplitude resonant frequency. Explicitly, the three choices are (using the example of a mechanical resonator with frictional damping force with coefficient b):

- Relative to natural frequency:

  $$Q = \frac{\omega}{b/m} \qquad \text{where} \qquad \omega = \sqrt{\frac{k}{m}} = \text{natural frequency}$$

  This can also be written

  $$Q = \omega\,\tau_{damp}$$

  This is the convention used in these notes.

- Relative the damped oscillation frequency:

  $$Q = \frac{\omega'}{b/m} \qquad \text{where} \qquad \omega' = \sqrt{\frac{k}{m} - \frac{1}{4}\left(\frac{b}{m}\right)^2} = \omega\sqrt{1 - \frac{1}{4}\left(\frac{b/m}{\omega}\right)^2}$$

  which can be written

  $$Q = \omega'\,\tau_{damp}$$

- Relative to the amplitude resonant frequency (defined later in Section 3.1.5):

  $$Q = \frac{\omega_r}{b/m} \qquad \text{where} \qquad \omega_r = \sqrt{\frac{k}{m} - \frac{1}{2}\left(\frac{b}{m}\right)^2} = \omega\sqrt{1 - \frac{1}{2}\left(\frac{b/m}{\omega}\right)^2}$$

  which can be written

  $$Q = \omega_r\tau_{damp}$$

  This is the convention used in Thornton.

It seems obvious that the first equation is most sensible because it discusses the ratio of damping and natural timescales; the other definitions refer to the ratio of the damping to the damped characteristic or the resonant frequency timescales, both of which already incorporate a damping correction; they are thus "moving targets." Moreover, the first definition results in Q giving in very simple form the amplitude and energy decay times. And, of course, in the limit of high Q, the three definitions become identical.

Damped Solutions

Returning to the formal problem: we can try to solve via an exponential solution of the kind tried in the undamped case, $q = e^{i\alpha t}$. One ends up with the equation

$$\left(-\alpha^2 + i\,\frac{\alpha}{Q} + 1\right) e^{i\alpha t} = 0$$

which is solved by the algebraic relation

$$\alpha = \frac{i}{2\,Q} \pm \sqrt{1 - \frac{1}{4\,Q^2}}$$

The solution will be

$$q(t) = \exp\left(-\frac{t}{2\,Q}\right) \exp\left(\pm i\,t\,\sqrt{1 - \frac{1}{4\,Q^2}}\right)$$

There is always a decay term. Depending on the sign of the discriminant, the second term may be oscillatory or decaying or it may simply give 1. We examine these cases separately:

- **Underdamped**: $Q > \frac{1}{2}$. In this case, the discriminant is positive and we obtain oscillatory motion from the second term. The complex solution is given by

$$q_c(t) = \mathcal{A}_c\,\exp\left(-\frac{t}{\tau_d}\right) \exp\left(\pm i\,\omega'\,t\right) \qquad \tau_d \equiv 2\,Q \qquad \omega' \equiv \sqrt{1 - \frac{1}{4\,Q^2}} \tag{3.5}$$

  The oscillation frequency is shifted. The decay time is simply related to the quality factor. The physical version of the solution is found by taking the real part,

$$q(t) = A_c\,\exp\left(-\frac{t}{\tau_d}\right) \cos\left(\omega'\,t + \phi\right) \qquad \mathcal{A}_c = |\mathcal{A}_c|\;e^{i\phi}$$

  Note that the shift in the frequency will be negligible for $Q \gg \frac{1}{2}$. The shift is quadratically suppressed; for large Q, Taylor expansion of the radical gives

$$\omega' \approx 1 - \frac{1}{8\,Q^2}$$

- **Overdamped**: $Q < \frac{1}{2}$. In this case, the discriminant becomes negative and the second term also is decaying. There are actually two possible decay times due to the sign freedom for the radical. The general solution is

$$q(t) = A\,\exp\left(-\frac{t}{\tau_{d,+}}\right) + B\,\exp\left(-\frac{t}{\tau_{d,-}}\right) \qquad \tau_{d,\mp}^{-1} = \frac{1}{2\,Q} \pm \sqrt{\frac{1}{4\,Q^2} - 1} \tag{3.6}$$

The inversion of the sign in subscripting the solution is so that the + subscript goes with the larger time constant. We refer to the decay constants as times (unlike Hand and Finch) because it's more intuitive. We have two possible decay times, one "fast" and one "slow". In the extreme limit of $Q \ll \frac{1}{2}$, the two solutions are extremely different:

$$\tau_{d,+} \approx \frac{1}{Q} \qquad \tau_{d,-} \approx Q$$

- **Critically damped**: $Q = \frac{1}{2}$. The discriminant vanishes and the two solutions become degenerate. The degenerate time constant is $\tau_d = 2\,Q = 1$ (*i.e.*, the damping time becomes $T/2\,\pi$ where T is the undamped oscillation period). However, we must have another solution because the differential equation is second-order. We can motivate a second solution by considering the limit $Q \to \frac{1}{2}$. The two decay time constants are

$$\tau_{d,\mp}^{-1} = 1 \pm \sqrt{\epsilon}$$

where $|\epsilon| \ll 1$ is the small but nonzero radical. Consider two superpositions of the two solutions

$$\begin{aligned}
q_1(t) &= \exp\left(-\frac{t}{\tau_{d,+}}\right) + \exp\left(-\frac{t}{\tau_{d,-}}\right) \\
&= \exp(-t)\left[\exp\left(t\sqrt{\epsilon}\right) + \exp\left(-t\sqrt{\epsilon}\right)\right] \\
&\approx 2\,\exp(-t) \\
q_2(t) &= \exp\left(-\frac{t}{\tau_{d,+}}\right) - \exp\left(-\frac{t}{\tau_{d,-}}\right) \\
&= \exp(-t)\left[\exp\left(t\sqrt{\epsilon}\right) - \exp\left(-t\sqrt{\epsilon}\right)\right] \\
&\approx \exp(-t)\left[1 + t\sqrt{\epsilon} - 1 + t\sqrt{\epsilon}\right] \\
&= 2\,t\sqrt{\epsilon}\,\exp(-t)
\end{aligned}$$

The first superposition is just the solution we already had. The second superposition gives us a valid second solution. One can confirm that it is a valid solution for $Q = \frac{1}{2}$:

$$\left(\frac{d^2}{dt^2} + 2\,\frac{d}{dt} + 1\right) t\,\exp(-t) = (-2 + t + 2\,(1-t) + t)\,\exp(-t) = 0$$

So, we have that in the case $Q = \frac{1}{2}$, the generic solution is

$$q(t) = A\,\exp(-t) + B\,t\,\exp(-t)$$

For illustrations of the solutions, see Hand and Finch Figure 3.3. Note the distinct nature of the two solutions in all cases.

When it is desired to return to unscaled time for the above solutions, all occurrences of t should be replaced by $\omega\,t$ and all occurrences of ω' should be replaced by ω'/ω.

Energy Decay

The physical interpretation of Q is best seen in the underdamped case. There is a clear decay of the oscillation amplitude with time constant $\tau_d = 2\,Q$. Since the energy in the

oscillator is proportional to the square of the amplitude, the energy decays exponentially with time constant Q; the energy satisfies the differential equation

$$\frac{dE}{dt} = -\frac{E}{Q}$$

Recall that time has been rescaled so the angular frequency of oscillation is 1 and the period is 2π. The energy decay time is therefore $\frac{Q}{2\,\pi}$ periods of the oscillator (assuming $Q \gg \frac{1}{2}$ so we may ignore the frequency shift).

In the critically damped case, the energy decays twice as quickly as in the underdamped case. This is not a discontinous change, as the discussion in the previous paragraph assumed $Q \gg \frac{1}{2}$. As $Q \to \frac{1}{2}$, the frequency shift becomes significant and provides a smooth transition from the weakly underdamped case to the critically damped case.

In the overdamped case, the interpretation is less clear. The energy is not a well-defined quantity because the energy decays away in a time small compared to the original oscillation period.

3.1.4 The Driven Simple and Damped Harmonic Oscillator

General Considerations

So far we have considered an oscillator in isolation: initial conditions are imposed and then the system evolves, perhaps with damping. Frequently in reality we happen upon driven oscillators, oscillators that are exposed to a continuous driving force – *e.g.*, a driven LRC circuit. The oscillator's evolution is modified by the existence of the additional driving force.

Generically, the mathematical problem we now wish to consider is

$$\ddot{q} + \frac{\dot{q}}{Q} + q = F(t) \tag{3.7}$$

The equation remains linear but is now inhomogeneous due to the presence of the driving term on the right side. The presence of the driving term changes the character of the generic solution. It now has the form

$$q(t) = q_p(t) + q_h(t)$$

where $q_p(t)$ satisfies the driven differential equation and $q_h(t)$ is any solution to the homogeneous equation. These two terms are also referred to as the **particular** and **free** solutions for obvious reasons. We will see that, for the harmonic oscillator, these two terms take the form of the **steady-state** and **transient** response of the system to the external driving force. The free constants of the free solution must be set by some sort of boundary conditions.

The generic method for generating the particular solution is to find the **Green's function** for the equation, which is the particular solution when the driving force is a δ-function (impulse-like). The particular solution for arbitrary driving forces can be constructed using the Green's function. We will find the harmonic oscillator Green's function below.

Step Function Response

Prior to developing the harmonic oscillator Green's function, let's solve a simple case by guessing the solution so that we may build up some intuition. Consider an underdamped SHO drive by the step function driving force:

$$F(t) = F_0\,\theta(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}\,F_0 & t = 0 \\ F_0 & t > 0 \end{cases}$$

where $\theta(t)$ is the **step function** or **Heaviside function**: The value $\theta(0) = \frac{1}{2}$ is somewhat arbitrary since it occurs for an infinitesimally small time.

At $t \gg 0$, we expect that the constant F is exactly balanced by the restoring force at $q = F_0$ (the weird units – a position q and force F_0 being equal – arise because the differential equation is dimensionless due to rescaling by β). Since F is independent of time for $t > 0$, this constant solution must the steady-state response. So, our particular solution is

$$q_p(t > 0) = F_0$$

Our arbitrary underdamped free solution is of the form

$$q_h(t) = \exp\left(-\frac{t}{2Q}\right)\left[A\,\cos\omega' t + B\,\sin\omega' t\right]$$
$$\dot{q}_h(t) = -\frac{1}{2\,Q}\,q_h(t) + \exp\left(-\frac{t}{2Q}\right)\left[-\omega'\,A\,\sin\omega' t + \omega'\,B\,\cos\omega' t\right]$$

where Q and ω' are as defined in the last section. We assume that the oscillator was initially at rest, so $q(0^-) = \dot{q}(0^-) = 0$. The position obviously cannot change discontinuously, so $q(0^+) = 0$ also. To change the velocity discontinously would require a δ-function force (infinitely large for an infinitesimally short time), so the velocity is also continous, $\dot{q}(0^+) = 0$. Thus, our solution satisfies

$$0 = q(0^+) = q_p(0^+) + q_h(0^+) = F_0 + A$$
$$0 = \dot{q}(0^+) = \dot{q}_p(0^+) + \dot{q}_h(0^+) = 0 - \frac{A}{2\,Q} + \omega'\,B$$

The above two equations determine the constants A and B, so our solution is

$$q(t) = F_0\left\{1 - \exp\left(-\frac{t}{2\,Q}\right)\left[\cos\omega' t + \frac{1}{2\,Q\,\omega'}\,\sin\omega' t\right]\right\}$$

Hand and Finch Figure 3.5 illustrates this solution – the oscillator is driven to oscillate about the new equilibrium position $q = F_0$, with the oscillation damping out after being excited by the initial kick by the force. In the limit $Q \to \infty$, the oscillations do not damp but continue forever.

Delta-Function Response, or Green's Functions

The above solution was easy to find, but that was partially because the driving force was so simple and so determining the form of the particular solution was trivial. It may not always

be so easy. However, there is a generic technique. Any driving force can be decomposed into a series of delta functions [2]:

$$F(t) = \int_{-\infty}^{\infty} dt'\, F(t')\, \delta(t' - t) \tag{3.8}$$

Because the equation is linear, if we can find a solution for the special driving force $\delta(t'-t)$, then we can build up the full solution by linear combination. That is, if $G(t, t')$ satisfies the differential equation for a SHO driven by a delta-function impulse

$$\ddot{G} + \frac{\dot{G}}{Q} + G = \delta(t - t')$$

where the dots indicate derivatives with respect to t, and t' should be regarded as a parameter of the driving function, then the full solution can be calculated by

$$q(t) = \int_{-\infty}^{\infty} dt'\, F(t')\, G(t, t') \tag{3.9}$$

The function $G(t, t')$ is known as the **Green's function** for the differential equation under consideration. Note that G has units of inverse time because the delta function does; this provides the right units for q when the rescaling by β is undone.

Intuitively, one should think of the Green's function $G(t, t')$ as giving the motion at a time t if there was in impulsive force of unit amplitude at time t'. Since the differential equation is linear, we obviously can find the response to two separate impulses a and b at times t_a and t_b by calculating $q(t) = a\, G(t, t_a) + b\, G(t, t_b)$. For a force consisting of impulses F_i at times t_i, *i.e.*, $F(t) = \sum_i F_i \delta(t - t_i)$, the solution will be $q(t) = \sum_i F_i\, G(t, t_i)$. The extension to a continous force is trivial, giving Equations 3.8 and 3.9 above.

A note on units: in the above differential equation, everything is unitless because the time coordinate has been implicitly rescaled by $\beta = \omega^{-1}$; all the dots refer to $d/d\tau = d/d(\omega\, t)$ and the argument of the delta function should be τ, not t. If we eliminate τ in favor of t, we get

$$\omega^{-2}\, \ddot{G} + \frac{\dot{G}}{\omega\, Q} + G = \delta\left(\omega\, [t - t']\right) = \frac{1}{\omega}\, \delta(t - t')$$

where the δ function now has units of inverse time because its argument has units of time. Note that G remains unitless even when units are reinserted. The change comes in how G is applied. Recall that our forced unitless differential equation is

$$\ddot{q} + \frac{\dot{q}}{Q} + q = F(t)$$

Reinserting the time units and multiplying by mass gives

$$m\, \ddot{q} + \frac{m\, \omega}{Q}\, \dot{q} + m\, \omega^2\, q = m\, \omega^2\, F(t)$$

[2] The defining property of the delta function is $\int_I dt\, \delta(t - a)\, f(t) = f(a)$ where $f(t)$ is an arbitrary function and I is an interval in t containing a. The integral vanishes if I does not contain a. $\delta(t - a)$ looks like an impulse at $t = a$ of infinite height and infinitesimally small width. It has units of the inverse of its argument t so it cancels the dt in integrals. Here, since t is unitless, the delta-function is unitless; but, usually, t will be time so δ will have units of inverse time.

Our force $F(t)$ has the same units as q, length in this case. Multiplication of our $F(t)$ by $m\,\omega^2$ gives units of force. If $F'(t)$ is the dimensionful force, then we may rewrite Equation 3.9 as

$$q(t) = \frac{1}{m\,\omega^2}\int_{-\infty}^{\infty} d\tau'\, F'(\tau')\, G(\tau,\tau')$$

where the variable of integration is τ because our differential equation for G was in τ. Replacing $\tau = \omega\, t$, we obtain

$$q(t) = \frac{1}{m\,\omega}\int_{-\infty}^{\infty} dt'\, F'(t')\, G(t,t')$$

Now the units make sense: the integration gives force multiplied by time, which is mass times speed, then division by mass and frequency gives length.

The explicit method for finding the Green's function is to make use of the homogeneous solution and the properties of the δ-function. Away from $t = t'$, the Green's function satisfies the homogeneous differential equation. Causality requires $G(t,t') = 0$ for $t < t'$, which is a perfectly good solution of the differential equation. Continuity of position then implies $G(t',t') = 0$ too (*i.e.*, the position cannot change instantaneously, even with and infinite delta-function force). For $t > t'$, we can just take G to be the generic solution to the homogeneous equation. The coefficients A and B are obtained by integrating the differential equation for G (Equation 3.9) over a small interval around $t = t'$ and making careful use of the delta function's properties; essentially, this integration provides initial conditions for G that determine A and B:

$$\int_{t'-\epsilon}^{t'+\epsilon} dt\, \left[\ddot{G}(t,t') + \frac{\dot{G}(t,t')}{Q} + G(t,t')\right] = \int_{t'-\epsilon}^{t'+\epsilon} dt\, \delta(t-t')$$

$$\left[\dot{G}(t'+\epsilon,t') - \dot{G}(t'-\epsilon,t')\right] + \frac{1}{Q}\left[G(t'+\epsilon,t') - G(t'-\epsilon,t')\right] + 2\,\epsilon\, G(t',t') = 1$$

where we have explicitly done the integrations. Now, use the facts $\dot{G}(t'-\epsilon,t') = 0$ and $G(t'-\epsilon,t') = 0$ (which result from causality), giving

$$\dot{G}(t'+\epsilon,t') + \frac{1}{Q}\, G(t'+\epsilon,t') + 2\,\epsilon\, G(t',t') = 1$$

Now, Taylor-expand the second term to obtain

$$\dot{G}(t'+\epsilon,t') + \frac{1}{Q}\left[G(t',t') + \epsilon\,\dot{G}(t'^{+},t')\right] + 2\,\epsilon\, G(t',t') = 1$$

where $\dot{G}(t'^{+},t')$ is the derivative of G as $t \to 0$ from the positive side. This positive-side/negative-side distinction is necessary for $\dot{G}$ because it may change discontinuously at $t = t'$ (because the acceleration will be infinite due to the delta-function applied force.) Next, use the fact explained above that $G(t',t') = 0$ due to causality and continuity of position, yielding:

$$\dot{G}(t'+\epsilon,t') + \frac{\epsilon}{Q}\,\dot{G}(t'^{+},t') = 1$$

Finally, use the fact that $\dot{G}$, while it may change discontinuously at $t = t'$, must remain finite at all times – even the infinite acceleration provided by the delta-function impulse will integrate to only a step change in velocity. Thus, the second term goes to zero as ϵ goes to zero, giving

$$\dot{G}(t'^{+}, t') = 1$$

Thus, our initial conditions for G are $G(t', t') = 0$ and $\dot{G}(t'^{+}, t') = 1$. The $^{+}$ superscript simply means that the initial condition for the $t > t'$ solution will be that its initial velocity is 1, while for $t < t'$ it holds that $\dot{G} = 0$. The solution will have a velocity discontinuity at $t = t'$. With these and the known form of the solution of the homogeneous equation, we may determine the Green's function completely. Let's consider the cases separately:

- **Underdamped**: The generic underdamped solution is

$$q_h(t) = \exp\left(-\frac{t}{2Q}\right)\left[A\,\cos\omega' t + B\,\sin\omega' t\right]$$
$$\dot{q}_h(t) = -\frac{1}{2\,Q}\,q_h(t) + \exp\left(-\frac{t}{2Q}\right)\left[-\omega' A\,\sin\omega' t + \omega' B\,\cos\omega' t\right]$$

  We will replace t by $t - t'$ to set the time origin at t'. The initial conditions then require $A = 0$ and $\omega' B = 1$. Thus we have

$$G(t - t') \;=\; \frac{1}{\omega'}\,\exp\left(-\frac{t - t'}{2Q}\right)\,\sin\left[\omega'\,(t - t')\right]\,\theta(t - t') \tag{3.10}$$

  Note the use of the Heaviside function to impose causality. The generic solution for an arbitrary forcing function is therefore

$$q(t) = \int_{-\infty}^{\infty} dt'\,F(t')\,G(t - t') = \int_{-\infty}^{t} dt' F(t')\frac{1}{\omega'}\,\exp\left(-\frac{t - t'}{2Q}\right)\,\sin\left[\omega'\,(t - t')\right]$$

- **Overdamped**: The generic overdamped solution is

$$q_h(t) \;=\; A\,\exp\left(-\frac{t}{\tau_{d,+}}\right) + B\,\exp\left(-\frac{t}{\tau_{d,-}}\right)$$
$$\dot{q}_h(t) \;=\; -\frac{A}{\tau_{d,+}}\,\exp\left(-\frac{t}{\tau_{d,+}}\right) - \frac{B}{\tau_{d,-}}\,\exp\left(-\frac{t}{\tau_{d,-}}\right)$$

  The conditions on $G(t, t')$ are the same, which require

$$\begin{aligned} A + B &= 0 \\ -\frac{A}{\tau_{d,+}} - \frac{B}{\tau_{d,-}} &= 1 \end{aligned}$$

  The Green's function thus becomes

$$G(t - t') \;=\; \frac{\tau_{d,+}\,\tau_{d,-}}{\tau_{d,+} - \tau_{d,-}}\,\left[\exp\left(-\frac{t - t'}{\tau_{d,+}}\right) - \exp\left(-\frac{t - t'}{\tau_{d,-}}\right)\right]\,\theta(t - t') \tag{3.11}$$

  The generic solution for an arbitrary forcing function is

$$q(t) = \int_{-\infty}^{t} dt' F(t')\,\frac{\tau_{d,+}\,\tau_{d,-}}{\tau_{d,+} - \tau_{d,-}}\,\left[\exp\left(-\frac{t - t'}{\tau_{d,+}}\right) - \exp\left(-\frac{t - t'}{\tau_{d,-}}\right)\right]$$

- **Critically damped**: The generic critically damped solution is

$$q(t) = A\,\exp(-t) + B\,t\,\exp(-t)$$

Applying the boundary conditions is very simple, giving $A = 0$ and $B = 1$. So we find the Green's function is

$$G(t-t') \;=\; (t-t')\,\exp\big(-(t-t')\big)\,\theta(t-t') \tag{3.12}$$

and the generic solution is

$$q(t) = \int_{-\infty}^{t} dt' F(t')\,(t-t')\,\exp\big(-(t-t')\big)$$

Recall that ω' and t are unitless in the above; if one wants to rewrite with units, all occurrences of t should be replaced by $\omega\,t$ and all occurrences of ω' should be replaced by ω'/ω, where ω is the undamped natural frequency. Note that the latter also rescales the damping time constants $\tau_{d,\pm}$ to $\omega\,\tau_{d,\pm}$. In all cases, G remains unitless with these rescalings.

The technique we used to find the Green's function is generically applicable. Green's functions are ubiquitous in physics, providing the fundamental means by which to deal with the linear response of physical systems to stimuli. Learn how to use them!

3.1.5 Behavior when Driven Near Resonance

An important special topic that can be considered using the formalism of Green's functions is the response of an oscillator when driven at a frequency near its characteristic frequency. Such systems are also ubiquitous. Ever heard of the Tacoma Narrows Bridge disaster? You can find movies of it on the web.

Calculation of the Response Function

It would be straightword to calculate the temporal response of an oscillator to a sinusoidal driving force $F(t) = F_0\,\sin\omega_d t$ where ω_d is the drive frequency, not necessarily the same as the oscillator characteristic frequency ω_0. If the force turns on at $t = 0$, we would obtain a transient and steady-state term via the Green's function formalism

$$q(t) = q_p(t) + q_h(t) = \int_0^t dt'\;\sin\omega\,t'\,G(t-t')$$

For this discussion, we are not particularly interested in the transient term since it dies away after a few decay time constants. Linearity of the equation of motion guarantees that the steady-state solution can only contain sinuisoids of the same frequency as the driving force. Mathematically, this arises because we must obtain something on the left side of the equation of motion (the q terms) to match the sinusoidal driving force; with a linear differential equation, there is no way to generate a frequency ω from frequencies other than ω. Given this, we may simply assume that we are looking for a solution proportional to $e^{i\omega_d t}$. The equation of motion becomes

$$\mathcal{A}_c\left(-\omega_d^2 + \frac{i\,\omega_d}{Q} + 1\right) e^{i\omega_d t} = F_0\,e^{i\omega_d t}$$

where $\mathcal{A}_c$ is a to-be-determined complex coefficient. Solving for $\mathcal{A}_c$ gives

$$\mathcal{A}_c = \frac{F_0}{1 - \omega_d^2 + \frac{i\,\omega_d}{Q}}$$

This coefficient $\mathcal{A}_c$ contains all the information we need – the amplitude and the phase of the response relative to the driving function as a function of the driving frequency ω_d. Note that the response is entirely linear in the driving force F_0.

Amplitude Response and Stored Energy

Let's first calculate the amplitude response. The square of the amplitude response gives the stored energy. It is

$$E = \frac{1}{2}\,|q_p(t)|^2 = \frac{1}{2}\,\left|\mathcal{A}_c\, e^{i\omega_d t}\right|^2 = \frac{1}{2}\left|\frac{F_0}{1 - \omega_d^2 + \frac{i\,\omega_d}{Q}}\right|^2 = \frac{F_0^2}{2}\frac{1}{(1-\omega_d^2)^2 + \frac{\omega_d^2}{Q^2}}$$

where the phase terms $e^{i\omega_d t}e^{i\phi}$ have unity amplitude and thus no longer appear.[3] We can inquire as to whether there is a peak in the response and find its position by requiring $\frac{dE}{d\omega_d} = 0$. (The algebra can be done more easily by requiring $\frac{dE^{-1}}{d\omega_d} = 0$.) One obtains the condition

$$\begin{aligned} 2\left(1-\omega_r^2\right)(-2\,\omega_r) + \frac{2\,\omega_r}{Q^2} &= 0 \\ \omega_r = \sqrt{1 - \frac{1}{2\,Q^2}} \qquad & \omega_r = 0 \end{aligned} \tag{3.13}$$

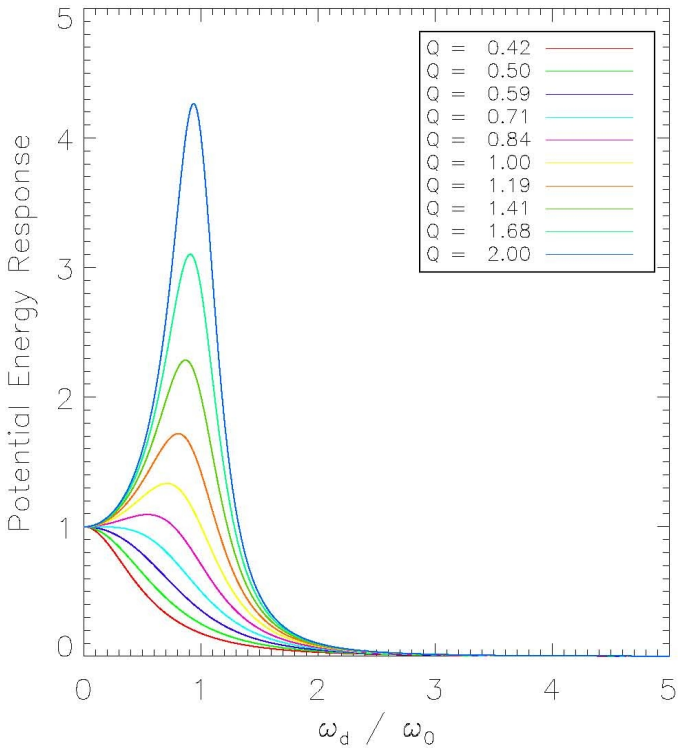
ω_r is known as the **resonant frequency** because the oscillator responds maximally at frequency ω_r. For comparison, we list side-by-side the three characteristic frequencies we have discussed so far (putting them back in dimensionful units):

$$\begin{aligned} \text{undamped characteristic}: \quad \omega_0 &= \sqrt{\frac{k}{m}} \\ \text{damped characteristic}: \quad \omega' &= \omega_0\sqrt{1 - \frac{1}{4\,Q^2}} < \omega_0 \\ \text{damped resonant}: \quad \omega_r &= \omega_0\sqrt{1 - \frac{1}{2\,Q^2}} < \omega' < \omega_0 \end{aligned}$$

There is a hierarchy of frequencies.

Note also that the nontrivial solution "disappears" for $Q \leq \frac{1}{\sqrt{2}}$, which is interesting: a slightly underdamped oscillator may not have a resonance, and certainly critically damped and overdamped oscillators have no resonant behavior. Curves of $E(\omega_d)$ for different Q are shown here:

[3] The reader may be concerned about the use of complex numbers; if we had done this with sines and cosines, would we not end up with $\cos^2(\omega_d t + \phi)$? Yes, but we only care about the peak displacement, when $\cos = 1$.



It is interesting to also calculate where the resonance in the *kinetic energy* occurs. This may be distinct from the amplitude resonance because the system is not conservative. The velocity is the time derivative of the position, so it acquires an extra factor of ω_d. The time-averaged kinetic energy is thus

$$T = \frac{1}{2}\,|\dot{q}_p(t)|^2 = \frac{1}{2}\,\left|i\,\omega_d\,\mathcal{A}_c\,e^{i\omega_d t}\right|^2 = \frac{1}{2}\left|\frac{i\,\omega_d\,F_0}{1-\omega_d^2+\frac{i\,\omega_d}{Q}}\right|^2 = \frac{F_0^2}{2}\,\frac{\omega_d^2}{(1-\omega_d^2)^2+\frac{\omega_d^2}{Q^2}}$$

After a little bit of algebra, one finds that the resonant frequency for the kinetic energy is just the original characteristic frequency, $\omega_T = 1$. This is actually not suprising. Because the power being absorbed by the oscillator is the product $F\,\dot{q}$, the largest power absorption occurs when the two factors are fully in phase. If one looks at the phase of the response (see the next section), one finds that the phase of the velocity equals the phase of the driving force when $\omega_d = \omega_T = 1$.

Let us specialize to the case where Q is large and thus $\omega_d \approx \omega_r \approx 1$ and the distinction between the three characteristic frequencies is lost. In this limit, we can make some approximations to simplify the stored energy expression:

$$\begin{aligned} 1-\omega_d^2 &= (1-\omega_d)(1+\omega_d) \approx 2(1-\omega_d) \\ \frac{\omega_d^2}{Q^2} &\approx \frac{1}{Q^2} \end{aligned}$$

We then obtain

$$E \approx \frac{F_0^2}{8}\,\frac{1}{(\omega_d-1)^2+\frac{1}{4\,Q^2}} \tag{3.14}$$

This function is known as the **Lorentzian** and is found associated with all kinds of resonant phenomena. A useful parameter is the **full-width at half maximum**, the separation between the two points where the energy drops to half of its maximum value. It is found by requiring

$$\begin{aligned} E\left(\omega_d = 1 \pm \frac{\Delta\omega}{2}\right) &= \frac{1}{2} E\left(\omega_d = 1\right) \\ \frac{F_0^2}{8} \frac{1}{(\frac{\Delta\omega}{2})^2 + \frac{1}{4\,Q^2}} &= \frac{F_0^2}{16} \frac{1}{\frac{1}{4\,Q^2}} \\ \Delta\omega &= \frac{1}{Q} \end{aligned} \tag{3.15}$$

Note that we have taken $\omega_r = 1$ here. Clearly, though, the idea is that Q characterizes the width of the resonance very simply. Also of interest is the value of the stored energy on resonance:

$$E \approx \frac{F_0^2}{8}\, 4\, Q^2 = \frac{1}{2}\, F_0^2\, Q^2 \tag{3.16}$$

Notice how the stored energy becomes infinite as $Q \to \infty$: if there is no damping, then the resonator cannot release the energy it receives from the drive force.

A note on units: The above calculation was done in terms of unitless frequency and time. Rescaling the frequencies is easy – just multiply all frequencies by the natural frequency ω_0. The energy is a bit more difficult. We calculated energy from $q^2/2$, so we need to multiply by $k = m\,\omega^2$ to get something with units of energy. F_0 has units of length in our differential equation; recall that, in connection with Green's functions, we converted to fully dimensional force by multiplication by $m\,\omega^2$. So, the fully dimensional version of the energy relation (using primes to indicate dimensional versions of quantities) is

$$E' = m\,\omega^2\, E \approx \frac{1}{2}\, m\,\omega^2 \left(\frac{F_0'}{m\,\omega^2}\right)^2 Q^2 = \frac{F_0'^2 Q^2}{2\,m\,\omega^2}$$

Phase Response

The other interesting aspect of the response function is the phase. This we find simply by calculating the angle in the complex plane of $\mathcal{A}_c$ since by definition the drive signal has zero phase. Going back to our formula for $\mathcal{A}_c$ and manipulating:

$$\begin{aligned} \mathcal{A}_c &= \frac{F_0}{1 - \omega_d^2 + \frac{i\,\omega_d}{Q}} \\ &= F_0\, \frac{(1 - \omega_d^2) - \frac{i\,\omega_d}{Q}}{\left(1 - \omega_d^2\right)^2 + \frac{\omega_d^2}{Q^2}} \end{aligned}$$

Now that the denominator is real, we can see the real and imaginary parts and obtain the phase trivially:

$$\tan\phi = -\frac{\frac{\omega_d}{Q}}{1 - \omega_d^2} \tag{3.17}$$

Let's consider the behavior of this function. As $\omega_d \to 0$, the right side vanishes and so $\phi \to 0$. As we increase ω_d, the quantity on the right side goes negative, implying that the phase angle heads into the fourth quadrant of the complex plane. At resonance, the denominator vanishes, giving $-\infty$ for the right side. This corresponds to $\phi = -\frac{\pi}{2}$. As ω_d increases beyond the resonance frequency, both the denominator and the numerator will be negative, indicating that ϕ moves into the third quadrant. Finally, as $\omega_d \to \infty$, the quantity on the right side approaches 0, indicating the phase angle is rotating through the third quadrant until it reaches $-\pi$. The amplitude of $\mathcal{A}_c$ is also changing through this entire process, starting at 0 at $\omega_d = 0$, going to its maximum value at $\omega_d = 1$, then returning back to zero as $\omega_d \to \infty$. There are excellent plots in Hand and Finch Section 3.9 illustrating the phase path.

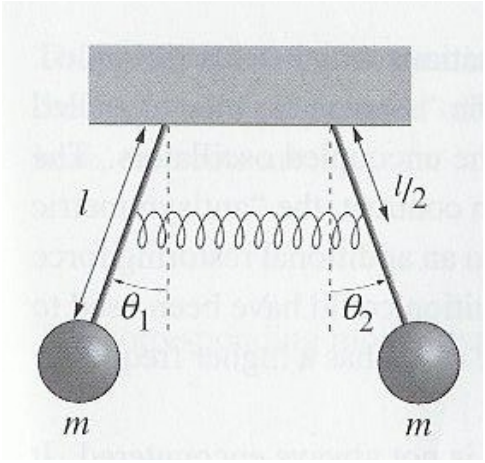
The sign of the phase indicates that the oscillator's phase lags that of the driving force. At very low drive frequency, the phase lag is small because the oscillator can "keep up" with the drive frequency. As the drive speeds up, the phase lag increases. The oscillator lags the drive by exactly one-quarter of a period on resonance, putting the drive in phase with the velocity and maximizing the stored kinetic energy. As $\omega_d \to \infty$, the oscillator becomes completely out of phase with the drive.

3.2 Coupled Simple Harmonic Oscillators

Coupled simple harmonic oscillators are physically important and seen everywhere in nature. The electromagnetic field at any point in space can be considered a simple harmonic oscillator, but Maxwell's Equations couple the oscillator at one point to the oscillators everywhere else in space. Coupled oscillations are critical in particle physics, seen in neutrino oscillations and in the particle-antiparticle mixing see in the $K_0 - \overline{K}_0$ and $B_0 - \overline{B}_0$ systems. They also serve to introduce the concept of eigenvalue/eigenvector systems, which we will use again when discussing rigid-body rotation. This section follows Hand and Finch Chapter 9, though does things in a different order and proves some things that may not be proven in Hand and Finch.

3.2.1 The Coupled Pendulum Example

Consider two pendula of the same mass and length whose shafts are attached to each other by a massless spring whose rest length is equal to the separation of the two pendula:



Let's set up and solve the equations of motion for this system in the limit of small displacements.

Equations of Motion

The kinetic energy of the two pendula is as usual

$$T = \frac{1}{2} m l^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 \right)$$

The gravitational potential energy is the usual form:

$$V_g = \frac{m g l}{2} \left(\theta_1^2 + \theta_2^2 \right)$$

The potential energy due to the spring coupling the two is

$$V_s = \frac{k}{2} \left(\frac{l}{2} \right)^2 (\theta_1 - \theta_2)^2$$

where $l/2$ comes in because that is the point where the spring is attached to the shafts. The assumption that the rest length of the spring equals the horizontal separation of the

two pendula makes the potential energy a simple function of θ_1 and θ_2; it could be more complicated, but the basic physics will stay the same. The Lagrangian therefore is

$$L = \frac{1}{2}\,m\,l^2\,\left(\dot\theta_1^2 + \dot\theta_2^2\right) - \frac{m\,g\,l}{2}\,\left(\theta_1^2 + \theta_2^2\right) - \frac{k}{2}\,\left(\frac{l}{2}\right)^2\,(\theta_2 - \theta_1)^2$$

The Euler-Lagrange equations give us

$$\begin{aligned}\ddot\theta_1 + \frac{g}{l}\,\theta_1 + \frac{k}{4\,m}\,(\theta_1 - \theta_2) &= 0\\ \ddot\theta_2 + \frac{g}{l}\,\theta_2 + \frac{k}{4\,m}\,(\theta_2 - \theta_1) &= 0\end{aligned}$$

As usual, we define the characteristic frequency $\omega_0^2 = \frac{g}{l}$. We also define a parameter $\eta = \frac{k/4\,m}{g/l}$ that characterizes the relative size of the spring term and the gravity term. The simplified equations are

$$\begin{aligned}\ddot\theta_1 + \omega_0^2\,\theta_1 + \omega_0^2\,\eta\,(\theta_1 - \theta_2) &= 0\\ \ddot\theta_2 + \omega_0^2\,\theta_2 + \omega_0^2\,\eta\,(\theta_2 - \theta_1) &= 0\end{aligned}$$

Normal Modes and Normal Coordinates

The equations are simplified by taking two linear combinations by adding and subtracting:

$$\begin{aligned}\frac{d^2}{dt^2}\,(\theta_1 + \theta_2) + \omega_0^2\,(\theta_1 + \theta_2) &= 0\\ \frac{d^2}{dt^2}\,(\theta_1 - \theta_2) + \omega_0^2\,(1 + 2\,\eta)\,(\theta_1 - \theta_2) &= 0\end{aligned}$$

If we write these in terms of the linear combinations $\theta_\pm = \theta_1 \pm \theta_2$ and define $\omega_+^2 = \omega_0^2$ and $\omega_-^2 = \omega_0^2\,(1 + 2\,\eta)$, we have two decoupled SHO eqauations:

$$\begin{aligned}\ddot\theta_+^2 + \omega_+^2\,\theta_+^2 &= 0\\ \ddot\theta_-^2 + \omega_-^2\,\theta_-^2 &= 0\end{aligned}$$

We can make the dynamics in the ω_- mode go away by taking the initial condition $\theta_1(0) = \theta_2(0)$. This will give $\theta_- = 0$ for all time, which implies $\theta_1 = \theta_2$ for all time. The two pendula oscillate together, so this is called the **symmetric mode**. Similarly, we can have trivial behavior in the ω_+ mode by starting with $\theta_1(0) = -\theta_2(0)$. This gives $\theta_1 = -\theta_2$ for all time, so the pendula oscillate completely opposite to each other; this is of course called the **antisymmetric mode**. The coordinates θ_+ and θ_- are called **normal coordinates** and the two modes are called **normal modes**. Clearly, the transformation of the system to the normal has yielded two uncoupled oscillators whose individual dynamics are simple.

If a mixed mode is excited at the start by beginning with a mixed initial condition, then both modes will be excited. There is no coupling between the modes, so they should evolve independently of each other, with the energy in each mode remaining constant.

Solution by Linear Algebra Methods

When the number of coordinates becomes larger than 2, solving the problem directly by algebraic methods becomes difficult. Let us consider a more generic method of solving the

equations of motion by linear algebra. We can write our original equations of motion in matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}+\omega_0^2\begin{pmatrix}1+\eta & -\eta\\ -\eta & 1+\eta\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

We **assume** there exists a solution to the motion in which both coordinates oscillate at the same frequency $e^{i\omega t}$, where we will determine ω if it exists. Each individual solution for ω will be called a mode. There may be degenerate modes – modes with the same ω – but that is a minor problem that can be dealt with.

Our solution is of the form

$$\begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix}=\begin{pmatrix}\Theta_1\\ \Theta_2\end{pmatrix}e^{i\omega t}$$

Define

$$\lambda^2\equiv\frac{\omega_2}{\omega_0^2}$$

so that our equation becomes

$$\begin{pmatrix}-\lambda+1+\eta & -\eta\\ -\eta & -\lambda+1+\eta\end{pmatrix}\begin{pmatrix}\Theta_1\\ \Theta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

This is just a linear algebra equation whose solution we want to find. You presumably know from your math classes that the solution vanishes unless the determinant of the matrix vanishes. The free parameter here is λ, so we want to see if there exist solutions λ for which the determinant vanishes. That is, solve for λ the equation

$$\left|\begin{pmatrix}-\lambda+1+\eta & -\eta\\ -\eta & -\lambda+1+\eta\end{pmatrix}\right|=0$$

where $|\ |$ is the determinant. This becomes a simple algebraic equation:

$$(-\lambda+1+\eta)^2+\eta^2=0$$

which is easily solved to find

$$\begin{aligned}\lambda_1&=1 & \lambda_2&=1+2\,\eta\\ \omega_1&=\omega_0 & \omega_2&=\omega_0\sqrt{1+2\,\eta}\end{aligned}$$

We've found the normal mode frequencies, but now we need to find the normal mode amplitudes. Since the determinant of the above matrix vanishes, the two linear equations are not independent and can be used to solved for the ratio Θ_1/Θ_2:

$$\frac{\Theta_1}{\Theta_2}=\frac{1+\eta-\lambda}{\eta}$$

which gives

$$\begin{aligned}\frac{\Theta_{(1),1}}{\Theta_{(1),2}}&=1 && \text{symmetric mode}\\ \frac{\Theta_{(1),1}}{\Theta_{(1),2}}&=-1 && \text{antisymmetric mode}\end{aligned}$$

We can only specify the ratio of amplitudes because the initial conditions set the overall normalization.

More generally, if there are more than two coupled coordinates, we simply have a larger matrix equation to solve with more normal modes to find. If there exist two modes with the same frequency, one will find that the system does not specify fully the coordinate ratios; there is freedom due to this degeneracy.

Eigenvalue-Eigenvector Interpretation

Now, of course, what we have just done is solve an eigenvalue-eigenvector problem. If we write the equation in full matrix form, with

$$\begin{aligned}
\vec{\theta} &= \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \\
\vec{0} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\mathbf{A} &= \begin{pmatrix} 1+\eta & -\eta \\ -\eta & 1+\eta \end{pmatrix} \\
\mathbf{I} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Then our original coupled equation of motion can be written

$$\frac{d^2}{dt^2}\,\vec{\theta} + \mathbf{A}\,\vec{\theta} = \vec{0}$$

We assume a solution of the form $\vec{\theta}(t) = \vec{\Theta}\, e^{i\omega t}$ and we obtain the equation

$$(-\lambda\,\mathbf{I} + \mathbf{A})\,\vec{\Theta}\, e^{i\omega t} = 0$$

which is solved by requiring

$$|-\lambda\,\mathbf{I} + \mathbf{A}| = 0$$

which is the eigenvalue problem. For a symmetric matrix $\mathbf{A}$, we are guaranteed that there is rotation matrix $\mathbf{R}$ that diagonalizes $\mathbf{A}$; *i.e.*, there exists a matrix $\mathbf{R}$ such that

$$\mathbf{R}^T\,\mathbf{A}\,\mathbf{R} = \lambda$$

where λ is a diagonal matrix whose elements are the squares of the normal mode frequencies of the system (normalized to ω_0^2). For our two dimensional case, we will have

$$\lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

For each eigenvalue λ_i, there is an eigenvector $\vec{\Theta}_i$ such that

$$\begin{aligned}
(-\lambda_i\,\mathbf{I} - \mathbf{A})\,\vec{\Theta}_i &= 0 \\
\mathbf{A}\,\vec{\Theta}_i &= \lambda_i\,\vec{\Theta}_i
\end{aligned}$$

or, returning to our original equation of motion,

$$\frac{d^2}{dt^2}\,\alpha\,\vec{\Theta}_i\,e^{i\omega_i t} + \mathbf{A}\,\alpha\,\vec{\Theta}_i\,e^{i\omega_i t} = 0$$

where α is the amplitude of the motion in the ith mode, to be determined by initial conditions. We will consider the problem more formally in the next section, proving that such systems always can be written in terms of eigenvalues and eigenvectors and finding explicit forms for the rotation matrix $\mathbf{R}$ and the eigenvectors $\vec{\Theta}_i$.

3.2.2 General Method of Solution

Definitions

Let's begin by making some generic symbolic definitions. We are going to assume that all oscillations are small; regardless of the form of the potential, we will assume that we will expand it around some equilibrium point. That is, if $\vec{\phi}$ is our position coordinate, with M components, then

$$V(\vec{\phi}) \approx V(\vec{\phi}_0) + \sum_{i,j} \frac{1}{2}\,\frac{\partial^2 V}{\partial\phi_i\,\partial\phi_j}\bigg|_{\vec{\phi}_0} (\phi_i - \phi_{0,i})\,(\phi_j - \phi_{0,j})$$

where $V(\vec{\phi}_0)$ is the value of the potential at the equilibrium position $\vec{\phi}_0$. Since $\vec{\phi}_0$ is an equilibrium position, it holds that $\frac{\partial V}{\partial\vec{\phi}}$ vanishes at $\vec{\phi}_0$, so that term is dropped. For convenience, we will shift the origin to $\vec{\phi}_0$ and drop the constant term so that V simplifies to

$$\begin{aligned} V(\vec{\phi}) &= \sum_{i,j} \frac{1}{2}\,\frac{\partial^2 V}{\partial\phi_i\,\partial\phi_j}\bigg|_0 \phi_i\,\phi_j \\ &\equiv \sum_{i,j} v_{ij}\,\phi_i\,\phi_j \\ &= \vec{\phi}^T\mathbf{v}\,\vec{\phi} \end{aligned}$$

where we have defined a matrix $\mathbf{v}$ to be the second partial derivatives of V and have written the expression in matrix notation. We note that our condition for stable equilibrium can now be written

$$\vec{\phi}^T\mathbf{v}\,\vec{\phi} > 0$$

for any displacement vector $\vec{\phi}$. If $\mathbf{v}$ meets this condition, it is called **positive definite**. $\mathbf{v}$ is also symmetric for any reasonable potential V whose mixed partial derivatives commute.

We may similarly expand the kinetic energy, except in this case we will expand about $\dot{\vec{\phi}} = 0$ because, obviously, the system cannot reside near a stable equilibrium if it has a net velocity. We define a similar matrix $\mathbf{t}$ and rewrite the kinetic energy T:

$$\begin{aligned} t_{ij} &= \frac{1}{2}\,\frac{\partial^2 T}{\partial\,\dot{\phi}_i\,\partial\,\dot{\phi}_j}\bigg|_{\dot{\phi}_i=0,\dot{\phi}_j=0} \\ T &= \sum_{i,j} t_{ij}\,\dot{\phi}_i\,\dot{\phi}_j \\ &= \dot{\vec{\phi}}^T\mathbf{t}\,\dot{\vec{\phi}} \end{aligned}$$

Since the kinetic energy is always positive, we know that the matrix $\mathbf{t}$ is positive definite. It may have vanishing components if there are nuisance coordinates for which there is no kinetic energy, but these can obviously be dropped because there is no momentum or dynamics in such a coordinate. $\mathbf{t}$ certainly can have no negative components. $\mathbf{t}$ will also be symmetric for the same "no-pathologies" reason as $\mathbf{v}$.

With these definitions, the Lagrangian is rewritten

$$L = \dot{\vec{\phi}}^T \mathbf{t}\, \dot{\vec{\phi}} - \vec{\phi}^T \mathbf{v}\, \vec{\phi}$$

We may find the equations of motion in the usual fashion. First, the partial derivatives are:

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\phi}_k} &= \sum_{i,j} t_{ij} \frac{\partial}{\partial \dot{\phi}_k} \left[\dot{\phi}_i\, \dot{\phi}_j\right] = \sum_{i,j} t_{ij} \left[\dot{\phi}_i\, \delta_{kj} + \delta_{ij}\, \dot{\phi}_j\right] = 2 \sum_j t_{kj}\, \dot{\phi}_j \\
\frac{\partial L}{\partial \phi_k} &= -\sum_{i,j} v_{ij} \frac{\partial}{\partial \phi_k} \left[\phi_i\, \phi_j\right] = -\sum_{i,j} v_{ij} \left[\phi_i\, \delta_{kj} + \delta_{ij}\, \phi_j\right] = -2 \sum_j v_{kj}\, \phi_j
\end{aligned}$$

The Euler-Lagrange equations are therefore

$$\begin{aligned}
\frac{d}{dt}\left(2 \sum_j t_{ij}\, \dot{\phi}_j\right) + 2 \sum_j v_{kj}\, \phi_j &= 0 \\
\sum_j t_{ij}\, \ddot{\phi}_j + \sum_j v_{kj}\, \phi_j &= 0
\end{aligned}$$

which can be rewritten in matrix form

$$\mathbf{t}\, \ddot{\vec{\phi}} + \mathbf{v}\, \vec{\phi} = 0$$

Finding the Normal Mode Frequencies

Now that we have a general way to write the equation of motion, let's look for a general way to solve it. Let's assume there is a solution of the form

$$\vec{\phi}(t) = \vec{\Phi}\, e^{i\omega\, t}$$

where $\vec{\Phi}$ is a constant vector that provides the relative displacements of the original coordinates for the solution with characteristic frequency ω. The ω are called the **normal modes** or **normal mode frequencies**. The vector $\vec{\Phi}$ is generically referred to as a **normal mode vector**. We will demonstrate later that $\vec{\Phi}$ is real.

Substitution of the above form into the equation of motion Equation 3.18 yields

$$\left[-\omega^2\, \mathbf{t} + \mathbf{v}\right] \vec{\Phi} = 0$$

Note that this is not a trivial eigenvalue-eigenvector problem because the matrix $\mathbf{t}$ need not be the identity matrix. In our simple example earlier, it was indeed the identity matrix, but that was a special case. Continuing onward, in order for the equation to have a nontrivial solution, the determinant of the matrix must as usual vanish:

$$\left|-\omega^2\, \mathbf{t} + \mathbf{v}\right| = 0$$

This equation is a polynomial of order M in ω^2. We obviously want to know how many solutions there are and whether they yield real oscillation frequencies. We can prove a number of properties about the solutions before continuing onward:

- **The solutions for ω^2 are always real.** We can see this using complex conjugation and recognizing that $\mathbf{v}$ and $\mathbf{t}$ are always real and symmetric. First, multiply our equation $\left[-\omega^2\,\mathbf{t}+\mathbf{v}\right]\vec{\Phi}=0$ on the left by $\vec{\Phi}^{*T}$ (where $\vec{\Phi}$ is some arbitrary solution to the equation) and also consider its complex conjugate, making explicit use of the fact that $\mathbf{v}$ and $\mathbf{t}$ are real:

$$\vec{\Phi}^{*T}\mathbf{v}\,\vec{\Phi}=\omega^2\,\vec{\Phi}^{*T}\mathbf{t}\,\vec{\Phi}\qquad\qquad\vec{\Phi}^{T}\mathbf{v}\,\vec{\Phi}^{*}=\omega^{*2}\,\vec{\Phi}^{T}\mathbf{t}\,\vec{\Phi}^{*}$$

  Notice that we have allowed ω^2 to be complex. Now, difference the two equations to find

$$0=\left(\omega^2-\omega^{*2}\right)\vec{\Phi}^{*T}\mathbf{t}\,\vec{\Phi}$$

  We noted earlier that $\mathbf{t}$ is positive definite, but remember that this fact was only stated for real $\vec{\Phi}$. We can see that it holds for complex $\vec{\Phi}$ also:

$$\vec{\Phi}^{*T}\mathbf{t}\,\vec{\Phi}=\vec{\Phi}_r^{R}\mathbf{t}\,\vec{\Phi}_r+i\,\vec{\Phi}_r^{T}\mathbf{t}\,\vec{\Phi}_i-i\,\vec{\Phi}_i^{T}\mathbf{t}\,\vec{\Phi}_r+\vec{\Phi}_i^{T}\mathbf{t}\,\vec{\Phi}_i$$

  The cross-terms vanish because $\mathbf{t}$ is symmetric:

$$i\,\vec{\Phi}_r^{T}\mathbf{t}\,\vec{\Phi}_i=i\sum_{j,k}\Phi_{r,j}\,t_{jk}\,\Phi_{i,k}=i\sum_{j,k}\Phi_{i,k}\,t_{kj}\,\Phi_{i,j}=i\,\vec{\Phi}_i^{T}\mathbf{t}\,\vec{\Phi}_r$$

  So we have

$$\vec{\Phi}^{*T}\mathbf{t}\,\vec{\Phi}=\vec{\Phi}_r^{T}\mathbf{t}\,\vec{\Phi}_r+\vec{\Phi}_i^{T}\mathbf{t}\,\vec{\Phi}_i$$

  Now, since all the multiplying vectors are real, positive definiteness of $\mathbf{t}$ implies that $\vec{\Phi}^{*T}\mathbf{t}\,\vec{\Phi}$ is positive. Therefore, the only solution to the above equation is $\omega^2=\omega^{*2}$ and thus ω^2 is real.

- **The solutions for ω^2 are not only real but are positive.** To see this, we require positive definiteness of $\mathbf{v}$ and $\mathbf{t}$. We multiply our equation of motion on the left by $\vec{\Phi}^{T}$:

$$\begin{aligned}-\omega^2\,\vec{\Phi}^{T}\mathbf{t}\,\vec{\Phi}+\vec{\Phi}^{T}\mathbf{v}\,\vec{\Phi}&=0\\ \omega^2=\frac{\vec{\Phi}^{T}\mathbf{v}\,\vec{\Phi}}{\vec{\Phi}^{T}\mathbf{t}\,\vec{\Phi}}&>0\end{aligned}$$

  Since both $\mathbf{v}$ and $\mathbf{t}$ are positive definite, the quantity on the right is positive definite. Recall that $\mathbf{v}$ positive definite was a result of assuming our expansion point is a stable equilibrium.

With the above results safely in hand, we can assume that the M solutions for ω^2 will indeed yield oscillating modes.

Finding the Normal Mode Vectors

Caveat: The following technique only works for **nondegenerate** modes. If we have two modes i and j with the same mode frequencies ω_i^2 and ω_j^2, then we must use special techniques to find the mode vectors for those two modes. The following technique works for any nondegenerate modes, even if there are degenerate modes in the problem.

To find the normal mode vectors, we return to our original equation of motion, this time using the normal mode frequencies we proved above exist. The solution for normal mode frequency i is

$$\vec{\phi}_i(t) = \vec{\Phi}_i \, e^{i\omega_i \, t}$$

We are not making any assumptions yet about whether $\vec{\Phi}_i$ is real or complex. Inserting this in our equation of motion, we have

$$\left[-\omega_i^2 \, \mathbf{t} + \mathbf{v}\right] \vec{\Phi}_i = 0$$

The unknowns are the M components of $\vec{\Phi}_i$. Since the equation is homogeneous – the right side is zero – the overall normalization of the mode vector is indeterminate and we should expect to obtain only $M-1$ independent equations from the above matrix equation. These $M-1$ equations will set the relative values of the components of $\vec{\Phi}_i$. We may now also see that the $\vec{\Phi}_i$ are real since they are the solution to a system of homogeneous linear equations with real coefficients. (This proof is simpler than the proof that the ω^2 are real because the equations determining the components of $\vec{\Phi}_i$ are linear, while ω^2 is determined by a Mth order polynomial in ω^2.)

We will solve the above equation using the **cofactors** of the matrix $-\omega^2 \, \mathbf{t} + \mathbf{v}$. Recall that cofactors arise in the calculation of the determinant of a matrix: the determinant of a matrix can be calculated by taking the sum over the product of any row or column and its cofactors, with sign flips between adjacent elements:

$$|\mathbf{a}| = \sum_j a_{ij} \, (-1)^{i+j} \, C(a_{ij})$$

or

$$|\mathbf{a}| = \sum_i a_{ij} \, (-1)^{i+j} \, C(a_{ij})$$

where $C(a_{ij})$ is the **cofactor** of the element a_{ij} and is given by calculating the determinant of the matrix one obtains by eliminating row i and column j from the matrix. Note the alternating -1. As an example, the cofactors of the elements of the top row of the matrix

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

are

$$C(a_{11}) = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad C(a_{12}) = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad C(a_{13}) = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

The quantity $|\mathbf{a}|$ can be viewed as the dot product of two vectors:

$$\begin{aligned} \vec{r}_i &= (a_{i1}, a_{i2}, \ldots, a_{iM}) \\ \vec{C}_i^{\,r} &= \left((-1)^{i+1} C(a_{i1}), (-1)^{i+2} C(a_{i2}), \ldots, (-1)^{i+M} C(a_{iM})\right) \end{aligned}$$

or the transpose version

$$\begin{aligned} \vec{c}_i &= (a_{1i}, a_{2i}, \ldots, a_{Mi}) \\ \vec{C}_i^{\,c} &= \left((-1)^{1+i} C(a_{1i}), (-1)^{2+i} C(a_{2i}), \ldots, (-1)^{M+i} C(a_{Mi})\right) \end{aligned}$$

Now that you are reminded of what a cofactor is, and we have introduced how the determinant can be written as a dot product of row or column vectors and their respective cofactor vectors, let us note a useful property of these dot products. For the specific $\mathbf{a}$ we are considering here, $\mathbf{a} = \omega_i^2 \mathbf{t} + \mathbf{v}$, we know that $|\mathbf{a}| = 0$: this was necessary to have nontrivial normal mode vectors. In terms of the row/column vectors and their cofactor vectors, this statement implies that $\vec{r}_i^T \vec{C}_i^r = 0$ or $\vec{c}_i^T \vec{C}_i^c = 0$; that is, any row or column of the matrix $\mathbf{a}$ and its cofactor vector are orthogonal. We shall use this orthogonality below.

Next, suppose we have $\mathbf{a}$ and we want to find $\vec{f}$ that solves $\mathbf{a}\vec{f} = 0$. This corresponds to the set of linear equations

$$\sum_j a_{ij} f_j = 0$$

for all rows i. Based on our work above, we can see that any of the row cofactor vectors $\vec{C}_i^r$ can be the vector $\vec{f}$ that would satisfy the ith equation of this set of equations.[4] We can see in fact that any of these cofactor vectors satisfies the full set of equations by realizing that $\sum a_{ij} \left(\vec{C}_k^r\right)_j$ for $k \neq i$ also must vanish: this quantity is the determinant of the matrix obtained by replacing row k of $\mathbf{a}$ with row i and calculating the determinant using the new row k (which is now identical to row i); since the determinant of a matrix with two equal rows vanishes, it holds that $\sum a_{ij} \left(\vec{C}_k^r\right)_j$ vanishes even when $k \neq i$.[5] One might be worried that we now have too many solutions: one could use the cofactor vector $\vec{C}_i^r$ for any row i, so we in principle have M different solutions when we should only have one! We need not worry – they **must** all be the same solution up to a normalization factor since the matrix equation can only have one unique solution (up to normalization). One finds in practice that the different cofactor vectors are always just multiples of one another.

So, to sum up: **the mode vector $\vec{\Phi}_i$ for mode i is found by solving the set of linear equations**

$$\left[-\omega_i^2 \mathbf{t} + \mathbf{v}\right] \vec{\Phi}_i = 0$$

[4] To be explicit, we are simply saying that $\sum_j a_{ij} (\vec{C}_i^r)_j$ for any i. This is true because it is simply the statement $\vec{r}_i^T \vec{C}_i^R = 0$, which we demonstrated was true in the previous paragraph.

[5] If this statement is not obvious, consider a system of linear equations viewed as a matrix equation. We are free to replace rows by linear combinations thereof (as long as we don't use the same linear combination twice). So, if we have two identical rows, we can replace them by their sum and difference. The difference row will be all zeros. The determinant of a matrix with one row identically zero is obviously zero (use that row for the cofactor expansion). For our 3×3 example above, if $|\mathbf{a}|$ vanishes, then we already know that

$$a_{11} (-1)^0 C(a_{11}) + a_{12} (-1)^1 C(a_{12}) + a_{13} (-1)^2 C(a_{13}) = 0$$

so, if we consider, for example, the product of the second row with the cofactor vector, we have

$$\begin{aligned}
& a_{21} (-1)^0 C(a_{11}) + a_{22} (-1)^1 C(a_{12}) + a_{23} (-1)^2 C(a_{13}) \\
&= a_{21} (-1)^0 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} (-1)^1 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} (-1)^2 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= \begin{vmatrix} a_{21} & a_{21} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= 0
\end{aligned}$$

The solution $\vec{\Phi}_i$ is given by the cofactor vector of any row of $-\omega_i^2\,\mathbf{t}+\mathbf{v}$.

Orthonormality

It is straightforward to demonstrate that the mode vectors satisfy the convenient **orthonormality** condition

$$\vec{\Phi}_i^T\mathbf{t}\,\vec{\Phi}_j=\delta_{ij}$$

We can prove this by techniques similar to those we used to prove the realness and positive definiteness of ω^2 and the realness of $\vec{\Phi}_i$. First, we have

$$\begin{aligned}0 &= \vec{\Phi}_j^T\mathbf{v}\,\vec{\Phi}_i-\vec{\Phi}_i^T\mathbf{v}\,\vec{\Phi}_j\\ &= \omega_i^2\,\vec{\Phi}_j^T\mathbf{t}\,\vec{\Phi}_i-\omega_j^2\,\vec{\Phi}_i^T\mathbf{t}\,\vec{\Phi}_j\\ &= \left(\omega_i^2-\omega_j^2\right)\vec{\Phi}_i^T\mathbf{t}\,\vec{\Phi}_j\end{aligned}$$

where we have used multiple times that $\vec{\Phi}_i^T\mathbf{a}\vec{\Phi}_j=\vec{\Phi}_j^T\mathbf{a}\vec{\Phi}_i$ when $\mathbf{a}$ is symmetric. So, as long as $\omega_i^2\neq\omega_j^2$, *i.e.*, as long as we have two nondegenerate modes, we are guaranteed of the orthogonality condition

$$\vec{\Phi}_i^T\mathbf{t}\,\vec{\Phi}_j=0\quad\text{ for } i\neq j$$

Note that this is not a "simple" orthogonality condition; it requires use of $\mathbf{t}$ as the metric matrix. To prove the final part of our relation, we need to calculate $\vec{\Phi}_i^T\mathbf{t}\,\vec{\Phi}_i$. But, as discussed, above, the normalization of the mode vectors is arbitrary because they solve a set of *homogeneous* linear equations. Therefore, we are free to pick the normalization, so we pick it to give the convenient relation $\vec{\Phi}_i^T\mathbf{t}\,\vec{\Phi}_i=1$, thus proving our orthonormality result.

Applying the Initial Conditions

The above orthonormality condition can be used to apply the initial conditions when we seek the solution to a particular initial value problem. By dimensionality arguments, our M normal modes with their M orthonormal mode vectors span the space of possible configurations of the M-dimensional system: an arbitrary state of the system at time t can be written as

$$\vec{\phi}_r(t)=\mathcal{R}\left[\vec{\phi}(t)\right]=\mathcal{R}\left[\sum_{i=1}^{M}A_i\,\vec{\Phi}_i\,e^{i\omega_i t}\right]$$

where the A_i are in general complex so that we may allow for relative phases between the different normal mode components. The velocity of the system can in general be written as

$$\dot{\vec{\phi}}_r(t)=\mathcal{R}\left[\dot{\vec{\phi}}(t)\right]=\mathcal{R}\left[\sum_{i=1}^{M}i\,\omega_i\,A_i\,\vec{\Phi}_i\,e^{i\omega_i t}\right]$$

To apply the initial conditions, consider the above equations at $t=0$:

$$\begin{aligned}\vec{\phi}_r(t=0) &= \mathcal{R}\left[\sum_{i=1}^{M}A_i\,\vec{\Phi}_i\right]=\sum_{i=1}^{M}\mathcal{R}\left[A_i\right]\vec{\Phi}_i\\ \dot{\vec{\phi}}_r(t=0) &= \mathcal{R}\left[\sum_{i=1}^{M}i\,\omega_i\,A_i\,\vec{\Phi}_i\right]=\sum_{i=1}^{M}\mathcal{I}\left[\omega_i\,A_i\right]\vec{\Phi}_i\end{aligned}$$

We can use our orthonormality conditions to invert the above to find the A_i:

$$\begin{aligned}\mathcal{R}\left[A_i\right] &= \vec{\Phi}_i^T \mathbf{t}\, \vec{\phi}_r(t=0) \\ \mathcal{I}\left[A_i\right] &= \frac{1}{\omega_i}\, \vec{\Phi}_i^T \mathbf{t}\, \dot{\vec{\phi}}_r(t=0)\end{aligned}$$

Preservation of Degrees of Freedom

We make note of an important point: the number of degrees of freedom in the problem is not changed by rewriting the system in terms of normal modes. As we have said above, when we have M position coordinates to begin with, we end up with $\mathbf{t}$ and $\mathbf{v}$ being $M \times M$ matrices and the eigenvalue equation being a Mth-order polynomial in ω^2. We therefore expect M normal mode frequencies and vectors. When we rewrite a solution $\vec{\phi}(t)$ in terms of the normal modes using the above expansion, we have M coefficients $\{A_n\}$. The $\{A_n\}$ are the new "coordinates" after transforming to normal modes. The freedom in initial conditions – the $2\,M$ values $\vec{\phi}(t=0)$ and $\dot{\vec{\phi}}(t=0)$ – are replaced by the $2\,M$ degrees of freedom in the real and imaginary parts of the $\{A_n\}$.

The Congruence Transformation, Diagonalization, and the Diagonalized Lagrangian and Hamiltonian

Our orthonormality condition can be reexpressed in the form

$$\mathbf{\Phi}^T \mathbf{t}\, \mathbf{\Phi} = \mathbf{I}$$

where $\mathbf{\Phi}$ defines the matrix whose columns are the normal mode vectors,

$$\mathbf{\Phi}_{ji} = \left[\vec{\Phi}_i\right]_j$$

The transformation of $\mathbf{t}$ obtained by application of $\mathbf{\Phi}$ in the above manner is called a **congruence transformation**.[6]

More interesting is the application of $\mathbf{\Phi}$ to $\mathbf{v}$. Recall that, by the equation of motion and orthonormality of the $\{\vec{\Phi}_i\}$,

$$\vec{\Phi}_i^T \left(\mathbf{v}\, \vec{\Phi}_j\right) = \vec{\Phi}_i^T \omega_j^2\, \mathbf{t}\, \vec{\Phi}_j = \omega_j^2\, \delta_{ij}$$

Thus, using $\mathbf{\Phi}$, we may write this in matrix form

$$\mathbf{\Phi}^T \mathbf{v}\, \mathbf{\Phi} = \Omega^2$$

where Ω is a matrix with

$$\Omega_{ij} = \omega_i\, \delta_{ij}$$

That is, the congruence transform diagonalizes $\mathbf{v}$ also.

[6] Some will recognize the transformation as something akin to the transformation one might apply to diagonalize a symmetric matrix in an eigenvalue-eigenvector problem. $\mathbf{\Phi}$ is **not** a rotation matrix in the strict sense because its columns are not orthogonal to each other; the orthornormality condition requires inclusion of the metric $\mathbf{t}$. If $\mathbf{\Phi}$ were indeed an orthogonal matrix (*i.e.*, having columns that are simply orthogonal), which would happen if $\mathbf{t}$ were a multiple of the identity matrix, then this would be called a **similarity transformation**.

We can prove a useful cyclicity property of the congruence transformation:

$$\mathbf{\Phi}^T\mathbf{t}\,\mathbf{\Phi} = \mathbf{\Phi}\,\mathbf{\Phi}^T\mathbf{t} = \mathbf{t}\,\mathbf{\Phi}\,\mathbf{\Phi}^T = \mathbf{I}$$

Note: this property only holds because of special properties of these matrices, it is not generically true. First, we note that the determinants of each of these matrices are nonzero; they must be, as $|\mathbf{A}\,\mathbf{B}| = |\mathbf{A}||\mathbf{B}|$, so if the product is equal the $\mathbf{I}$, none of the matrices may have a vanishing determinant. Let's write our starting relation in the form

$$\left(\mathbf{\Phi}^T\mathbf{t}\right)\mathbf{\Phi} = \mathbf{I}$$

That is, $\mathbf{\Phi}^T\mathbf{t}$ is the *left inverse* of $\mathbf{\Phi}$. For finite square matrices, if a left inverse exists, then that left inverse is the right inverse also because there is a unique formula for generating the inverse of a square matrix.[7] Therefore, we may also conclude

$$\mathbf{\Phi}\left(\mathbf{\Phi}^T\mathbf{t}\right) = \mathbf{I}$$

which is the second version. Finally, we obtain the third version by simply transposing the above:

$$\mathbf{t}\,\mathbf{\Phi}\,\mathbf{\Phi}^T = \mathbf{I}$$

Note that we used the fact that $\mathbf{t}$ is symmetric and that both $\mathbf{t}$ and $\mathbf{\Phi}$ had nonzero determinant. Why the above relation is useful can be seen by rewriting the kinetic and potential energy terms:

$$\begin{aligned} T &= \dot{\vec{\phi}}^T\mathbf{t}\,\dot{\vec{\phi}} \\ &= \dot{\vec{\phi}}^T\mathbf{t}\,\mathbf{\Phi}\,\mathbf{\Phi}^T\mathbf{t}\,\mathbf{\Phi}\,\mathbf{\Phi}^T\mathbf{t}\,\dot{\vec{\phi}} \\ &= \left[\mathbf{\Phi}^T\mathbf{t}\,\dot{\vec{\phi}}\right]^T\mathbf{I}\left[\mathbf{\Phi}^T\mathbf{t}\,\dot{\vec{\phi}}\right] \\ &= \dot{\vec{\psi}}^T\mathbf{I}\,\dot{\vec{\psi}} \end{aligned}$$

where the last step used $\mathbf{t}\,\mathbf{\Phi} = \mathbf{\Phi}^T\mathbf{t}$, which is easy to prove if you write it out by components, and we have defined

$$\vec{\psi}(t) = \mathbf{\Phi}^T\mathbf{t}\,\vec{\phi}(t)$$

which is a straightforward generalization of our formula for applying the initial conditions to obtain the normal mode expansion coefficients. The inverse transformation:

$$\vec{\phi}(t) = \mathbf{\Phi}\,\vec{\psi}(t)$$

which is shown by simply making use of the fact we proved above, that $\mathbf{\Phi}$ is the inverse of $\mathbf{\Phi}^T\mathbf{t}$.

Similarly,

$$\begin{aligned} V &= \vec{\phi}^T\mathbf{v}\,\vec{\phi} \\ &= \vec{\phi}^T\mathbf{t}\,\mathbf{\Phi}\,\mathbf{\Phi}^T\mathbf{v}\,\mathbf{\Phi}\,\mathbf{\Phi}^T\mathbf{t}\,\vec{\phi} \\ &= \left[\mathbf{\Phi}^T\mathbf{t}\,\vec{\phi}\right]^T\Omega^2\left[\mathbf{\Phi}^T\mathbf{t}\,\vec{\phi}\right] \\ &= \vec{\psi}^T\Omega^2\,\vec{\psi} \end{aligned}$$

[7] The formula is $\left[\mathbf{A}^{-1}\right]_{ij} = (-1)^{i+j}C(A_{ij})/|\mathbf{A}|$.

and the Lagrangian and Hamiltonian become

$$\begin{aligned} L &= \dot{\vec{\psi}}^T \mathbf{I}\, \dot{\vec{\psi}} - \vec{\psi}^T \Omega^2\, \vec{\psi} \\ H &= \dot{\vec{\psi}}^T \mathbf{I}\, \dot{\vec{\psi}} + \vec{\psi}^T \Omega^2\, \vec{\psi} \end{aligned}$$

Note that, because $\mathbf{I}$ and Ω^2 are diagonal, the dynamics in the different normal coordinates – *i.e.*, the different elements of $\vec{\psi}$ – are totally independent. The equations of motion one obtains are very simple:

$$\ddot{\vec{\psi}} + \Omega^2 \vec{\psi} = 0$$

or, if written row-by-row,

$$\ddot{\psi}_i + \omega_i^2 \psi_i = 0$$

Solving the equation of motion for each element is trivial, giving the time evolution

$$\psi_i(t) = \psi_i(t=0)\, e^{i\omega_i t}$$

or, in matrix form

$$\vec{\psi}(t) = e^{i\Omega t} \vec{\psi}(t=0)$$

where $e^{i\Omega t}$ can be calculate via power series expansion of the exponential. (Note that it had to be placed on the left because it is a matrix.) The initial conditions are applied via the relations found earlier, rewritten with new ψ symbols and the $\mathbf{\Phi}$ matrix:

$$\begin{aligned} \mathcal{R}\left[\vec{\psi}(t=0)\right] &= \mathbf{\Phi}^T \mathbf{t}\, \vec{\phi}_r(t=0) \\ \mathcal{I}\left[\vec{\psi}(t=0)\right] &= \Omega^{-1}\, \mathbf{\Phi}^T \mathbf{t}\, \dot{\vec{\phi}}_r(t=0) \end{aligned}$$

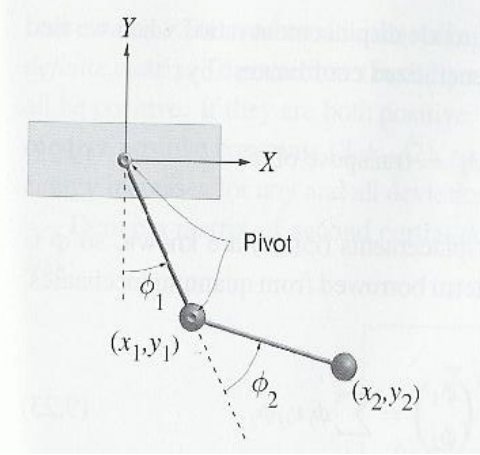
and we may obtain the values of the original coordinates at any time t by the inverse of the congruence transformation

$$\vec{\phi}_r(t) = \mathcal{R}\left[\vec{\phi}(t)\right] = \mathcal{R}\left[\Phi\, \vec{\psi}(t)\right]$$

So, overall, the congruence transformation by the mode vector matrix diagonalizes both $\mathbf{t}$ and $\mathbf{v}$, makes $\mathbf{t}$ the identity matrix, and provides a new set of coordinates in which the Lagrangian and Hamiltonian are quite simple and motion in the different normal coordinates are completely independent. This is clearly a grand simplification of the problem.

3.2.3 Examples and Applications

The Double Pendulum



We've seen the double pendulum before in our discussion of Hamilton's equations. We set it up somewhat differently in this case, as per the diagram. We will assume initial conditions $\phi_1(t=0)=0$, $\phi_2(t=0)=\alpha_0$, $\dot{\phi}_1(t=0)=0$, $\dot{\phi}_2(t=0)=0$. The kinetic and potential energies are

$$\begin{aligned} T &= \frac{1}{2}\,m\left(\dot{x}_1^2+\dot{y}_1^2+\dot{x}_2^2+\dot{y}_2^2\right) \\ V &= m\,g\,(y_1+y_2) \end{aligned}$$

Let's rewrite in convenient coordinates:

$$\begin{aligned} x_1 &= l\sin\phi_1 & y_1 &= -l\cos\phi_1 \\ x_2 &= x_1 + l\,\sin(\phi_1+\phi_2) & y_2 &= y_1 - l\,\cos(\phi_1+\phi_2) \end{aligned}$$

which provides for the kinetic and potential energies

$$\begin{aligned} T &= \frac{1}{2}\,m\,l^2\left(2\,\dot{\phi}_1^2+2\,\dot{\phi}_1\left(\dot{\phi}_1+\dot{\phi}_2\right)\cos\phi_2+\left(\dot{\phi}_1+\dot{\phi}_2\right)^2\right) \\ V &= -m\,g\,l\,(2\,\cos\phi_1+\cos(\phi_1+\phi_2)) \end{aligned}$$

The small angle approximation gives us

$$\begin{aligned} T &\approx m\,l^2\left(\dot{\phi}_1^2+\dot{\phi}_1\left(\dot{\phi}_1+\dot{\phi}_2\right)+\frac{1}{2}\left(\dot{\phi}_1+\dot{\phi}_2\right)^2\right) \\ V &\approx m\,g\,l\left(\phi_1^2+\frac{1}{2}\,(\phi_1+\phi_2)^2\right) \end{aligned}$$

where constant offset terms have been dropped. Clearly, our configuration vector is

$$\vec{\phi}(t)=\begin{pmatrix}\phi_1(t)\\ \phi_2(t)\end{pmatrix}$$

The kinetic energy and potential energy matrices are[8]

$$\mathbf{t} = \frac{m\,l^2}{2}\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad \mathbf{v} = \frac{m\,g\,l}{2}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$$

Finding the normal mode frequencies is simply a matter of solving the determinant equation $\left|-\omega^2\mathbf{t} + \mathbf{v}\right| = 0$. Clearly, everything will scale with $\frac{g}{l}$ since that is ω^2 in the simple single pendulum case, so let's define $\lambda = \omega^2/(g/l)$. We thus have

$$\begin{aligned} \frac{m\,g\,l}{2}\begin{vmatrix} 3-5\,\lambda & 1-2\,\lambda \\ 1-2\,\lambda & 1-\lambda \end{vmatrix} &= 0 \\ \frac{m\,g\,l}{2}\left(\lambda^2 - 4\,\lambda + 2\right) &= 0 \end{aligned}$$

The normal mode frequencies are therefore

$$\omega_1^2 = \left(2-\sqrt{2}\right)\frac{g}{l} \qquad \omega_2^2 = \left(2+\sqrt{2}\right)\frac{g}{l}$$

We have a low-frequency mode and a high-frequency mode. Let's find the mode amplitude ratios. For such a simple system, we can do it directly by solving the equation

$$\begin{aligned} \left[-\omega_i^2\,\mathbf{t} - \mathbf{v}\right]\vec{\Phi}_i &= 0 \\ \frac{m\,g\,l}{2}\begin{pmatrix} 3-5\,\lambda_i & 1-2\,\lambda_i \\ 1-2\,\lambda_i & 1-\lambda_i \end{pmatrix}\vec{\Phi}_i &= 0 \end{aligned}$$

which gives two equations

$$\begin{aligned} (3-5\,\lambda_i)\,\Phi_{i,1} + (1-2\,\lambda_i)\,\Phi_{i,2} &= 0 \\ (1-2\,\lambda_i)\,\Phi_{i,1} + (1-\lambda_i)\,\Phi_{i,2} &= 0 \end{aligned}$$

We are guaranteed by our previous derivations that the above two equations are redundant when one of the λ_i is plugged in. So we choose to use the second, simpler equation. Recall also that the normalization is not specified, so we choose a convenient way of writing the normal mode vector and leave the normalization as an undetermined coefficient, giving

$$\vec{\Phi}_i = \alpha\begin{pmatrix} -\frac{1-\lambda_i}{1-2\,\lambda_i} \\ 1 \end{pmatrix}$$

which give for the two cases

$$\vec{\Phi}_1 = \alpha_1\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} \qquad \vec{\Phi}_2 = \alpha_2\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$$

It's worth a try at working out the algebra that yielded the above to remind yourself how to rationalize fractions that have a radical in the denominator. It may also be amusing to see that the two different equations yield the same result. Let's also do it using the cofactor method to see we get the same result (we use the top row of the $-\omega_i^2\mathbf{t} - \mathbf{v}$ matrix):

$$\vec{\Phi}_i = \alpha'\begin{pmatrix} 1-\lambda_i \\ -(1-2\,\lambda_i) \end{pmatrix} = \alpha\begin{pmatrix} -\frac{1-\lambda_i}{1-2\,\lambda_i} \\ 1 \end{pmatrix}$$

[8] Remember that the diagonal terms are relative to 1/2 the second-order partial derivatives while the off-diagonal terms are not! A seemingly inconsequential fact, but the polynomial you end up with can be much worse if the matrices you started with are incorrect.

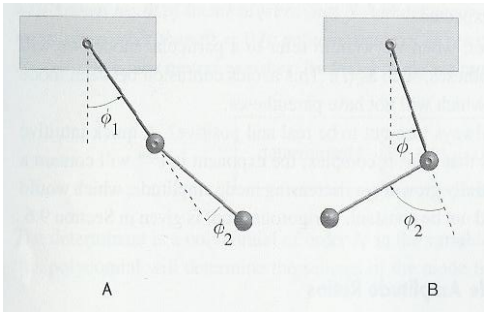
which will yield the same result. If we decide to use the orthonormalization convention, we require

$$\begin{aligned} 1 &= \vec{\Phi}_i^T \mathbf{t}\, \vec{\Phi}_i \\ &= \alpha^2 \begin{pmatrix} -\frac{1-\lambda_i}{1-2\,\lambda_i} & 1 \end{pmatrix} \frac{m\,l^2}{2} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1-\lambda_i}{1-2\,\lambda_i} \\ 1 \end{pmatrix} \end{aligned}$$

which you can obviously expand out and solve. The result is

$$\begin{aligned} \vec{\Phi}_1 &= \left(m\,l^2\left(10+7\sqrt{2}\right)\right)^{-1/2} \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} \\ \vec{\Phi}_2 &= \left(m\,l^2\left(10-7\sqrt{2}\right)\right)^{-1/2} \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} \end{aligned}$$

Note how the low-frequency mode's mode vector has a large ratio of the displacement of the top pendulum to the bottom pendulum – the bottom pendulum mostly follows the top pendulum, exceeding its swing by a small amount (because ϕ_2 is not perfectly locked to ϕ_1). The high-frequency mode has ϕ_1 and ϕ_2 having opposite signs – they oscillate antisymmetrically – and has a large amplitude for ϕ_2, it swings more than twice as much as ϕ_1. This is illustrated here:



Analytical Mechanics

Check orthonormality of the mode vectors:

$$\begin{aligned} \vec{\Phi}_i\, \mathbf{t}\, \vec{\Phi}_j &= \frac{\frac{m\,l^2}{2}\,\frac{1}{m\,l^2}}{\sqrt{\left(10\pm 7\sqrt{2}\right)\left(10\pm 7\sqrt{2}\right)}} \begin{pmatrix} 1\pm\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1\pm\sqrt{2} \\ 1 \end{pmatrix} \\ &= \frac{\begin{pmatrix} 5\left(1\pm\sqrt{2}\right)+2 & 2\left(1\pm\sqrt{2}\right)+1 \end{pmatrix} \begin{pmatrix} 1\pm\sqrt{2} \\ 1 \end{pmatrix}}{2\sqrt{\left(10\pm 7\sqrt{2}\right)\left(10\pm 7\sqrt{2}\right)}} \\ &= \frac{5\left(1\pm\sqrt{2}\right)\left(1\pm\sqrt{2}\right)+2\left(1\pm\sqrt{2}\right)+2\left(1\pm\sqrt{2}\right)+1}{2\sqrt{\left(10\pm 7\sqrt{2}\right)\left(10\pm 7\sqrt{2}\right)}} \end{aligned}$$

Now, specialize to one case or the other. First, for $i=j$ the signs are the same:

$$\vec{\Phi}_i\, \mathbf{t}\, \vec{\Phi}_i = \frac{5+10\pm 10\sqrt{2}+2\pm 2\sqrt{2}+2\pm 2\sqrt{2}+1}{2\left(10\pm 7\sqrt{2}\right)} = 1$$

and for $i \neq j$:

$$\vec{\Phi}_i \, \mathbf{t} \, \vec{\Phi}_j = \frac{5(1-2) + 2 \pm 2\sqrt{2} + 2 \mp 2\sqrt{2} + 1}{2\sqrt{(100-98)}} = 0$$

Let's apply the initial conditions. We will need $\mathbf{\Phi}^T \mathbf{t}$, so let's calculate it:

$$\mathbf{\Phi}^T = \begin{pmatrix} \vec{\Phi}_1 \\ \vec{\Phi}_2 \end{pmatrix} = \frac{1}{\sqrt{m\,l^2}} \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{10+7\sqrt{2}}} & \frac{1}{\sqrt{10+7\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{10-7\sqrt{2}}} & \frac{1}{\sqrt{10-7\sqrt{2}}} \end{pmatrix}$$

which gives

$$\begin{aligned} \mathbf{\Phi}^T \mathbf{t} &= \frac{\sqrt{m\,l^2}}{2} \begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{10+7\sqrt{2}}} & \frac{1}{\sqrt{10+7\sqrt{2}}} \\ \frac{1-\sqrt{2}}{\sqrt{10-7\sqrt{2}}} & \frac{1}{\sqrt{10-7\sqrt{2}}} \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \frac{\sqrt{m\,l^2}}{2} \begin{pmatrix} \frac{7+5\sqrt{2}}{\sqrt{10+7\sqrt{2}}} & \frac{3+2\sqrt{2}}{\sqrt{10+7\sqrt{2}}} \\ \frac{7-5\sqrt{2}}{\sqrt{10-7\sqrt{2}}} & \frac{3-2\sqrt{2}}{\sqrt{10-7\sqrt{2}}} \end{pmatrix} \end{aligned}$$

The mode coefficients are given by

$$\begin{aligned} \mathcal{R}\left[\vec{\psi}(t=0)\right] &= \mathbf{\Phi}^T \mathbf{t} \, \vec{\phi}_r(t=0) \\ &= \frac{\sqrt{m\,l^2}}{2} \begin{pmatrix} \frac{7+5\sqrt{2}}{\sqrt{10+7\sqrt{2}}} & \frac{3+2\sqrt{2}}{\sqrt{10+7\sqrt{2}}} \\ \frac{7-5\sqrt{2}}{\sqrt{10-7\sqrt{2}}} & \frac{3-2\sqrt{2}}{\sqrt{10-7\sqrt{2}}} \end{pmatrix} \begin{pmatrix} 0 \\ \alpha_0 \end{pmatrix} \\ &= \frac{\sqrt{m\,l^2}}{2} \begin{pmatrix} \frac{3+2\sqrt{2}}{\sqrt{10+7\sqrt{2}}} \\ \frac{3-2\sqrt{2}}{\sqrt{10-7\sqrt{2}}} \end{pmatrix} \alpha_0 \\ \mathcal{I}\left[\vec{\psi}(t=0)\right] &= \Omega^{-1} \mathbf{\Phi}^T \mathbf{t} \, \dot{\vec{\phi}}_r(t=0) \\ &= \frac{\sqrt{m\,l^2}}{2} \sqrt{\frac{g}{l}} \begin{pmatrix} 2-\sqrt{2} & 0 \\ 0 & 2+\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{7+5\sqrt{2}}{\sqrt{10+7\sqrt{2}}} & \frac{3+2\sqrt{2}}{\sqrt{10+7\sqrt{2}}} \\ \frac{7-5\sqrt{2}}{\sqrt{10-7\sqrt{2}}} & \frac{3-2\sqrt{2}}{\sqrt{10-7\sqrt{2}}} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

So, finally writing out the completely explicit solution:

$$\begin{aligned} \vec{\psi}(t) &= e^{i\Omega t} \vec{\psi}(t=0) \\ &= \exp\left[i \sqrt{\frac{g}{l}} \begin{pmatrix} 2-\sqrt{2} & 0 \\ 0 & 2+\sqrt{2} \end{pmatrix} t \right] \alpha_0 \, \frac{\sqrt{m\,l^2}}{2} \begin{pmatrix} \frac{3+2\sqrt{2}}{\sqrt{10+7\sqrt{2}}} \\ \frac{3-2\sqrt{2}}{\sqrt{10-7\sqrt{2}}} \end{pmatrix} \\ &= \alpha_0 \, \frac{\sqrt{m\,l^2}}{2} \begin{pmatrix} \frac{3+2\sqrt{2}}{\sqrt{10+7\sqrt{2}}} \, e^{i\sqrt{\frac{g}{l}}(2-\sqrt{2})t} \\ \frac{3-2\sqrt{2}}{\sqrt{10-7\sqrt{2}}} \, e^{i\sqrt{\frac{g}{l}}(2+\sqrt{2})t} \end{pmatrix} \end{aligned}$$

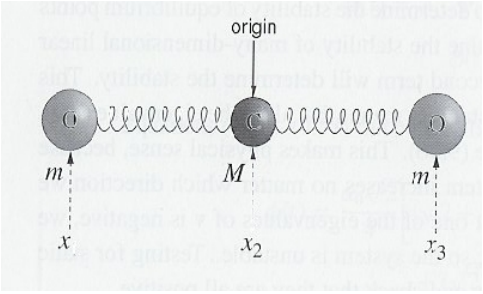
We may then recover $\vec{\phi}_r(t)$:

$$\begin{aligned}
\vec{\phi}_r(t) &= \mathcal{R}\left[\vec{\phi}(t)\right] = \mathcal{R}\left[\mathbf{\Phi}\,\vec{\psi}(t)\right] \\
&= \mathcal{R}\left\{\frac{1}{\sqrt{m\,l^2}}\begin{pmatrix} \frac{1+\sqrt{2}}{\sqrt{10+7\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{10-7\sqrt{2}}} \\ \frac{1}{\sqrt{10+7\sqrt{2}}} & \frac{1}{\sqrt{10-7\sqrt{2}}} \end{pmatrix} \alpha_0 \frac{\sqrt{m\,l^2}}{2} \begin{pmatrix} \frac{3+2\sqrt{2}}{\sqrt{10+7\sqrt{2}}}\, e^{i\sqrt{\frac{g}{l}}(2-\sqrt{2})t} \\ \frac{3-2\sqrt{2}}{\sqrt{10-7\sqrt{2}}}\, e^{i\sqrt{\frac{g}{l}}(2+\sqrt{2})t} \end{pmatrix}\right\} \\
&= \frac{\alpha_0}{2}\,\mathcal{R}\left\{\tfrac{3+2\sqrt{2}}{10+7\sqrt{2}}\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} e^{i\sqrt{\frac{g}{l}}(2-\sqrt{2})t} + \tfrac{3-2\sqrt{2}}{10-7\sqrt{2}}\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix} e^{i\sqrt{\frac{g}{l}}(2+\sqrt{2})t}\right\} \\
&= \frac{\alpha_0}{2\sqrt{2}}\left\{(\sqrt{2}-1)\begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix}\cos\!\left[t\sqrt{\tfrac{g}{l}}(2-\sqrt{2})\right] + (\sqrt{2}+1)\begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}\cos\!\left[t\sqrt{\tfrac{g}{l}}(2+\sqrt{2})\right]\right\}
\end{aligned}$$

As far as I can tell, Equation 9.60 of Hand and Finch is just utterly wrong – it looks like they just made some algebraic errors and/or typos in going from 9.59 to 9.60 – and so does not match the above.

The Symmetric Linear Triatomic Molecule

The symmetric linear triatomic molecule can be modelled simply as three masses connected by two springs, as illustrated here:



The molecule is assumed to be symmetric about the center atom and with equilibrium distance l between the outer atoms and the center atom.

In addition to providing an example of normal mode decomposition, this example serves to illustrate a case where one of the modes has zero frequency; *i.e.*, is simply a translation. This kind of mode violates our assumptions to some extent (we assumed that there was an equilibrium point in the coordinates), indicating we started out in the wrong coordinate system.

The kinetic and potential energies are

$$\begin{aligned}
T &= \frac{m}{2}\left(\dot{x}_1^2 + r\,\dot{x}_2^2 + \dot{x}_3^2\right) \\
V &= \frac{k}{2}\left[(x_2 - x_1 - l)^2 + (x_3 - x_2 - l)^2\right]
\end{aligned}$$

where $r = \frac{M}{m}$. Let us redefine the coordinates as

$$y_1 \equiv x_1 + l \qquad y_2 \equiv x_2 \qquad y_3 \equiv x_3 - l$$

Then the kinetic and potential energies are

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{y}_1^2 + r\,\dot{y}_2^2 + \dot{y}_3^2\right) \\ V &= \frac{k}{2}\left[(y_1 - y_2)^2 + (y_3 - y_2)^2\right] \end{aligned}$$

where we have chosen to flip the sign inside the square to make the expressions symmetric. Now, the reader will immediately realize that we can get rid of one degree of freedom by transforming to center-of-mass coordinates. The transformation is as follows:

$$z_1 \equiv x_1 - X + l \qquad z_2 \equiv x_2 - X \qquad z_3 \equiv x_3 - X - l \qquad X = \frac{m\,x_1 + M\,x_2 + m\,x_3}{M + 2\,m}$$

We have four coordinates variables now, but of course we only have three degrees of freedom, so let's get rid of one by using the constraint that, in the z coordinate system, the center of mass is at the origin:

$$\begin{aligned} m\,(z_1 - l) + M\,z_2 + m\,(z_3 + l) &= 0 \\ m\,z_1 + M\,z_2 + m\,z_3 &= 0 \end{aligned}$$

which changes the second equation to

$$-\frac{m}{M}\,(z_1 + z_3) = x_2 - X$$

Now, we must rewrite the x coordinates in terms of the X, z_1, and z_2 coordinates so we can rewrite the Lagrangian. We find

$$x_1 = X + z_1 - l \qquad x_2 = X - \frac{m}{M}\,(z_1 + z_3) \qquad x_3 = X + z_3 + l$$

We don't need to work through the rewrite of the kinetic energy directly; referring back to Equation 1.24, we can simply write

$$\begin{aligned} T &= \frac{M + 2\,m}{2}\,\dot{X}^2 + \frac{m}{2}\,\left(\dot{z}_1^2 + \dot{z}_3^2\right) + \frac{M}{2}\,\dot{z}_2^2 \\ &= \frac{M + 2\,m}{2}\,\dot{X}^2 + \frac{m}{2}\,\left(\dot{z}_1^2 + \dot{z}_3^2\right) + \frac{m^2}{2\,M}\,(\dot{z}_1 + \dot{z}_3)^2 \end{aligned}$$

The potential energy can be rewritten ($s = \frac{m}{M}$)

$$\begin{aligned} V &= \frac{k}{2}\left[\left(\frac{M + m}{M}\,z_1 + \frac{m}{M}\,z_3\right)^2 + \left(\frac{M + m}{M}\,z_3 + \frac{m}{M}\,z_1\right)^2\right] \\ &= \frac{k}{2}\left\{\left[(1 + s)^2 + s^2\right]\,z_1^2 + \left[(1 + s)^2 + s^2\right]\,z_3^2 + 4\,(1 + s)\,s\,z_1\,z_3\right\} \end{aligned}$$

So, clearly, there is no dynamics in the X coordinate and the z_1 and z_3 coordinates are clearly coupled. Let us carry through both analyses and see how they compare. First, calculate the $\mathbf{t}$ and $\mathbf{v}$ matrices:

$$\mathbf{t} = \frac{m}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{v} = \frac{k}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$\mathbf{t} = \frac{m}{2}\begin{pmatrix} 1 + s & s \\ s & 1 + s \end{pmatrix} \qquad \mathbf{v} = \frac{k}{2}\begin{pmatrix} (1 + s)^2 + s^2 & 2\,(1 + s)\,s \\ 2\,(1 + s)\,s & (1 + s)^2 + s^2 \end{pmatrix}$$

Define $\omega_0^2 \equiv \frac{k}{m}$ and $\lambda \equiv \frac{\omega^2}{\omega_0^2}$. The determinant equation then becomes

$$\begin{aligned} \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-r\,\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} &= 0 \\ \begin{vmatrix} (1+s)^2+s^2-\lambda\,(1+s) & 2\,(1+s)\,s-\lambda\,s \\ 2\,(1+s)\,s-\lambda\,s & (1+s)^2+s^2-\lambda\,(1+s) \end{vmatrix} &= 0 \end{aligned}$$

which give the equations

$$\begin{aligned} \lambda\left(r\,\lambda^2-2\,(r+1)\,\lambda+(r+2)\right) &= 0 \\ \Big[(1+s)^2+s^2-\lambda\,(1+s)\Big]^2-\left[2\,(1+s)\,s-\lambda\,s\right]^2 &= 0 \end{aligned}$$

One can easily show that the roots of the first equation are

$$\lambda_0 = 0 \qquad \lambda_1 = 1 \qquad \lambda_2 = 1+\frac{2}{r}$$

It is also easy to see that the roots of the second equation are (remember, $a^2 = b^2 \rightarrow a = \pm b$, which is how one gets two solutions):

$$\lambda_1 = 1 \qquad \lambda_2 = 1+2\,s$$

Since $s = \frac{1}{r}$, the two nonzero solutions are the same. The λ_0 solution in the first version is our indication that we had more coordinates than dynamical degrees of freedom. Another way to look at it is that the first mode is simple translational motion, no oscillation, though it is not obvious that one should draw that conclusion since our entire formalism using the $\mathbf{t}$ and $\mathbf{v}$ matrices assumed we were expanding about an equilibrium point with no net velocity. Writing in physical units, we obtain

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{k}{m}}\sqrt{1+2\,\frac{m}{M}}$$

Let's calculate the normal mode vectors using the cofactors:

$$\begin{aligned} \vec{\Phi}_i &= \begin{pmatrix} \begin{vmatrix} 2-r\,\lambda_i & -1 \\ -1 & 1-\lambda_i \end{vmatrix} \\ -\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda_i \end{vmatrix} \\ \begin{vmatrix} -1 & 2-r\,\lambda_i \\ 0 & -1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} (2-r\,\lambda_i)\,(1-\lambda_i)-1 \\ 1-\lambda_i \\ 1 \end{pmatrix} \\ \vec{\Phi}_i' &= \begin{pmatrix} (1+s)^2+s^2-\lambda\,(1+s) \\ \lambda\,s-2\,(1+s)\,s \end{pmatrix} \end{aligned}$$

where we have designated the second version of $\vec{\Phi}_i$ with a $'$ because, even though the frequencies are the same, the mode vectors will be different because we started out with

two different coordinates systems. For the specific modes, we have

$$\begin{aligned}\vec{\Phi}_0 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \vec{\Phi}_1 &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ \vec{\Phi}_2 &= \begin{pmatrix} 1 \\ -2\frac{m}{M} \\ 1 \end{pmatrix}\end{aligned}$$

The zeroth mode is clearly simple translation – all three atoms move with the same amplitude and phase. In the first mode, the two outer atoms move exactly opposite to each other and the center atom does not move at all. In the second mode, the two outer atoms move together and the center atom moves opposite them. The ratios of the motions depend on the relative masses. If we instead write using the $\prime$ system, we have

$$\begin{aligned}\vec{\Phi}_1' &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \vec{\Phi}_2' &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\end{aligned}$$

Since the two coordinates in the $\prime$ system are the motions of the outer atoms relative to the center of mass, it is sensible that the first mode is simple antisymmetric motion in these two coordinates. The second mode is not simple translation, though. The two outer atoms move together – as seen in the original coordinate system – but, because the center of mass must stay stationary in the center-of-mass frame, the center atom must obviously move opposite to the outer atoms. The amplitude can be found by going back to our relation for z_2 in terms of z_1 and z_3 (which comes from requiring the center of mass be fixed in the center-of-mass system):

$$z_2 = -\frac{m}{M}(z_1 + z_3) = -2\frac{m}{M}$$

which is as we found in the original coordinate system.

More examples: See Thornton, Chapter 12.

3.2.4 Degeneracy

Our discussions so far have assumed that the roots ω^2 to the discriminant equation are all unique. Our proof of orthogonality of different normal mode vectors required this. But there are some cases when two modes have the same normal mode frequency. Such modes are termed **degenerate**. It is still possible to find normal mode vectors; the degeneracy simply implies there is freedom in the choice.

General Considerations

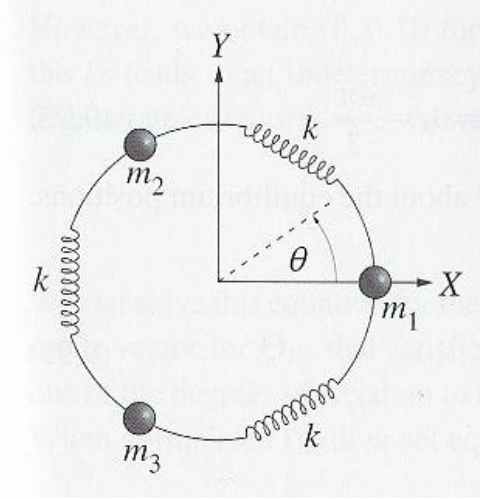
What happens to the usual procedure when degeneracies arise? The fundamental problem that arises is that Equation 3.18 will no longer provide $M - 1$ independent equations,

but $M - n - 1$ independent equations where n is the number of degenerate normal mode frequencies. The way to see this is to suppose that this was not true, that the normal mode vectors for the degenerate frequencies are fully determined. Let these be the first n modes, so the normal mode vectors are labeled $\vec{\Phi}_i$, $i = 1, \ldots, n$. Because these mode vectors all correspond to the same normal mode frequencies – *i.e.*, $\mathbf{v}\,\vec{\Phi}_i = \omega_i^2\,\mathbf{t}\,\vec{\Phi}_i$ for $i = 1 \ldots, n$, then any linear combination of them also satisfies this equality. If that is true, then there must be n undetermined degrees of freedom in the original equations determining $\vec{\Phi}_i$. Hence there will only be $M - n - 1$ independent equations for $\vec{\Phi}_i$.

In this case, you are free to specify the undetermined degrees of freedom. Certainly, you can use this freedom to force your choices to be orthonormal. Even with this constraint, you still have a fair amount of freedom. In picking the ith normal mode vector, you have $n - i + 1$ degrees of freedom, one of which is the normalization. Thus, only the nth normal mode vector is completely determined up to length.

Example: Masses Coupled by Springs on a Circle

To illustrate degeneracy, consider the example of three identical masses constrained to move on a circle and coupled along the arc of the circle by three identical springs, as illustrated below. (We could have done this with the masses unconstrained, but the problem becomes two-dimensional and the algebra becomes more difficult without providing further illumination.).



The kinetic and potential energies are

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right) \\ V &= \frac{k}{2}R^2\left[\left(\theta_2 - \theta_1 - \frac{2\,\pi}{3}\right)^2 + \left(\theta_3 - \theta_2 - \frac{2\,\pi}{3}\right)^2 + \left(\theta_1 - \theta_3 - \frac{2\,\pi}{3}\right)^2\right] \end{aligned}$$

Again, define new coordinates that express the offsets from the equilibrium positions,

$$\delta_1 = \theta_1 \qquad \delta_2 = \theta_2 - \frac{2\,\pi}{3} \qquad \delta_3 = \theta_3 + \frac{2\,\pi}{3}$$

which allows us to rewrite the kinetic and potential energies as

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{\delta}_1^2 + \dot{\delta}_2^2 + \dot{\delta}_3^2\right) \\ V &= \frac{k}{2} R^2 \left[(\delta_2 - \delta_1)^2 + (\delta_3 - \delta_2)^2 + (\delta_1 - \delta_3)^2\right] \end{aligned}$$

The kinetic and potential energy matrices become

$$\mathbf{t} = \frac{m}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{v} = \frac{k\,R^2}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

The determinant equation is

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

with $\lambda = \frac{\omega^2}{k\,R^2/m}$. The solutions are

$$\lambda_0 = 0 \qquad \lambda_{1,2} = 3$$

The zeroth mode is just rotation of the entire system without oscillation (again, it could have been eliminated by appropriate coordinate choice). The first and second modes are degenerate. If we try to find the normal mode vectors via the usual cofactor vectors, we find

$$\vec{\Phi}_i = \alpha \begin{pmatrix} \lambda_i^2 - 4\,\lambda_i + 3 \\ 3 - \lambda_i \\ 3 - \lambda_i \end{pmatrix}$$

We get all three elements equal for $\lambda = 0$ and we get all three elements vanishing for $\lambda = 3$. Correctly normalized, the $\lambda = 0$ mode vector is

$$\vec{\Phi}_0 = \sqrt{\frac{2}{3\,m}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

So now we are free to choose the two degenerate mode vectors as we like. One obvious choice based on our solution for the very similar triatomic linear molecule is

$$\vec{\Phi}_1 = \frac{1}{\sqrt{m}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

The above normal mode vector satisfies the normalization condition $\vec{\Phi}_1^T \mathbf{t}\,\vec{\Phi}_1 = 1$ Now, if we want a vector $\vec{\Phi}_2$ that is orthogonal to $\vec{\Phi}_2$, $\vec{\Phi}_1$ and properly normalized, we have the following equations:

$$\vec{\Phi}_2 = a\begin{pmatrix} 1 \\ \beta \\ \gamma \end{pmatrix} \qquad \vec{\Phi}_0^T \mathbf{t}\,\vec{\Phi}_2 = 0 \qquad \vec{\Phi}_1^T \mathbf{t}\,\vec{\Phi}_2 = 0 \qquad \vec{\Phi}_2^T \mathbf{t}\,\vec{\Phi}_2 = 1$$

Writing these out explicitly give

$$1+\beta+\gamma=0 \qquad \beta-\gamma=0 \qquad a^2\,\frac{m}{2}\,\left(1+\beta^2+\gamma^2\right)=1$$

which yields

$$\vec{\Phi}_2=\frac{1}{\sqrt{3\,m}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

3.3 Waves

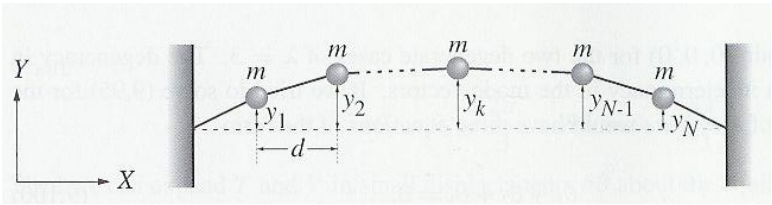
This section introduces one-dimensional waves, beginning by calculating the normal modes of a loaded string. We then take the limit of a continous string, which introduces the general wave equation. Solutions are discussed, energy transport, phase velocity vs. group velocity, etc. The initial material on the loaded string is found in Hand and Finch Section 9.7, but the rest of the section follows Thornton Chapter 13. You will only be responsible for material presented here.

3.3.1 The Loaded String

The normal modes of a system consisting of M equally spaced point masses connected by a massless string is presented.

The Problem

Consider a set of M point masses of mass m spaced at intervals d and connected by a massless string with tension τ, with the two ends of the string fixed. Assume the all displacements $\{y_p\}$ of the masses are perpendicular to the string and small.



The distance between two adjacent masses is

$$l = \sqrt{d^2 + (y_{p+1} - y_p)^2} \approx d\left(1 + \frac{1}{2}\frac{(y_{p+1} - y_p)^2}{d^2}\right)$$

so the change in the distance from the equilibrium separation d is

$$\Delta l = l - d \approx \frac{1}{2\,d}\,(y_{p+1} - y_p)^2$$

The potential energy stored in the link between masses p and $p+1$ is the product of the force and the displacement:

$$V_p = F\,\Delta\,l = \frac{\tau}{2\,d}\,(y_{p+1} - y_p)^2$$

The total potential energy is the sum over the V_p; note that, by considering the potential energy as stored in the link, we have counted a link when we have counted its left member. We have to be careful not to double count. We have

$$V = \frac{\tau}{2\,d}\left(y_1^2 + y_M^2\right) + \sum_{p=1}^{M-1}\frac{\tau}{2\,d}\,(y_{p+1} - y_p)^2$$

We include explicit terms for the leftmost and rightmost links because one end of each link is fixed and so must be treated specially. The kinetic energy is obviously

$$T = \sum_{p=1}^{M}\frac{m}{2}\,\dot{y}_p^2$$

The Solution

The kinetic and potential energy matrices are therefore

$$\mathbf{t} = \frac{m}{2}\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \mathbf{v} = \frac{\tau}{2\,d}\begin{pmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \cdots \\ 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The matrix $-\omega^2\mathbf{t} + \mathbf{v}$ is

$$-\omega^2\mathbf{t} + \mathbf{v} = \begin{pmatrix} -\omega^2\frac{m}{2} + \frac{\tau}{d} & -\frac{\tau}{2\,d} & 0 & \cdots \\ -\frac{\tau}{2\,d} & -\omega^2\frac{m}{2} + \frac{\tau}{d} & -\frac{\tau}{2\,d} & \cdots \\ 0 & -\frac{\tau}{2\,d} & -\omega^2\frac{m}{2} + \frac{\tau}{d} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Taking the determinant of this matrix explicitly is not a reasonable thing to do if M is large. We seek a general solution anyways, and taking the determinant directly is bound to be M-specific. The way to find a solution is to make use of the band-diagonal nature of the matrix and use this to return to the linear algebra equation from which the determinant conditions is derived – Equation 3.18 – which tells us

$$\left[-\omega^2\mathbf{t} + \mathbf{v}\right]\vec{\Phi} = 0$$

If we consider a particular row of the equation, we find

$$-\omega^2\frac{m}{2}\,\Phi_p + \frac{\tau}{d}\,\Phi_p - \frac{\tau}{2\,d}\,(\Phi_{p+1} + \Phi_{p-1}) = 0 \tag{3.18}$$

Note that this is just Newton's second law for the mass p up to a factor of 2.[9] Earlier, at Equation 3.18 we had demonstrated that the $\vec{\Phi}$ are completely real. However, this does not prevent us, for the sake of convenience, of assuming a complex solution and taking the real part at the end. We will in fact assume that $\vec{\Phi}$ is of the form

$$\Phi_p = \mathcal{R}\left[e^{ip\gamma - i\delta}\right]$$

where we have dropped any normalizing coefficient for now. The phase δ is necessary to avoid assuming that the real part of the solution is $\cos k\,\gamma$ since we have no justification for such an assumption. That this form is correct is, as usual, not obvious from the start but

[9] The left-most term is the acceleration of the mass p (the ω^2 factor arises from the two time derivatives). The other three terms can be rewritten

$$\frac{1}{2}\,\tau\left[\frac{1}{d}\,(\Phi_p - \Phi_{p+1}) + \frac{1}{d}\,(\Phi_p - \Phi_{p-1})\right] \tag{3.19}$$

The quantities $\frac{1}{d}\,(\Phi_p - \Phi_{p\pm1})$ give the tangent of the angle between the rope and the horizontal at mass p, which is approximately the sine of that angle for small vertical displacements; this gives the component of the tension along the y axis, which is the restoring force pulling the mass p back to the equilibrium position. When new kinds of systems are encountered, such as with additional objects attached to the rope, come back to this force equation to understand how to add them in.

must be confirmed. It is of course motivated by the physical expectation of a sinusuoidal wave motion of the masses on the string. Inserting the solution in Equation 3.18, we find

$$\left[-\omega^2\frac{m}{2}+\frac{\tau}{d}-\frac{\tau}{2\,d}\left(e^{i\gamma}+e^{-i\gamma}\right)\right]e^{ip\gamma-i\delta}=0$$

We obtain the equation

$$\omega^2=\frac{4\,\tau}{m\,d}\sin^2\frac{\gamma}{2}$$

This does not yet provide normal mode frequencies because γ is undetermined. Let us apply the boundary conditions that the amplitude must vanish at $p=0$ and $p=M+1$.[10] Note that these are **boundary conditions**, not **initial conditions**; initial conditions should not have any effect on the mode structure, just how a particular solution evolves with time. Boundary conditions, on the other hand, are physical constraints that affect all possible solutions. The boundary conditions imply for the solution that

$$\begin{aligned}\mathcal{R}\left[e^{i(0)\gamma-i\delta}\right]&=0 & \mathcal{R}\left[e^{i(M+1)\gamma-i\delta}\right]&=0\\ \cos\delta&=0 & \cos\left[(M+1)\,\gamma-\delta\right]&=0\\ \delta&=\frac{\pi}{2}\end{aligned}$$

which we can combined to find

$$\sin\left[(M+1)\,\gamma\right]=0\quad\Longrightarrow\quad\gamma=\frac{n\,\pi}{M+1}$$

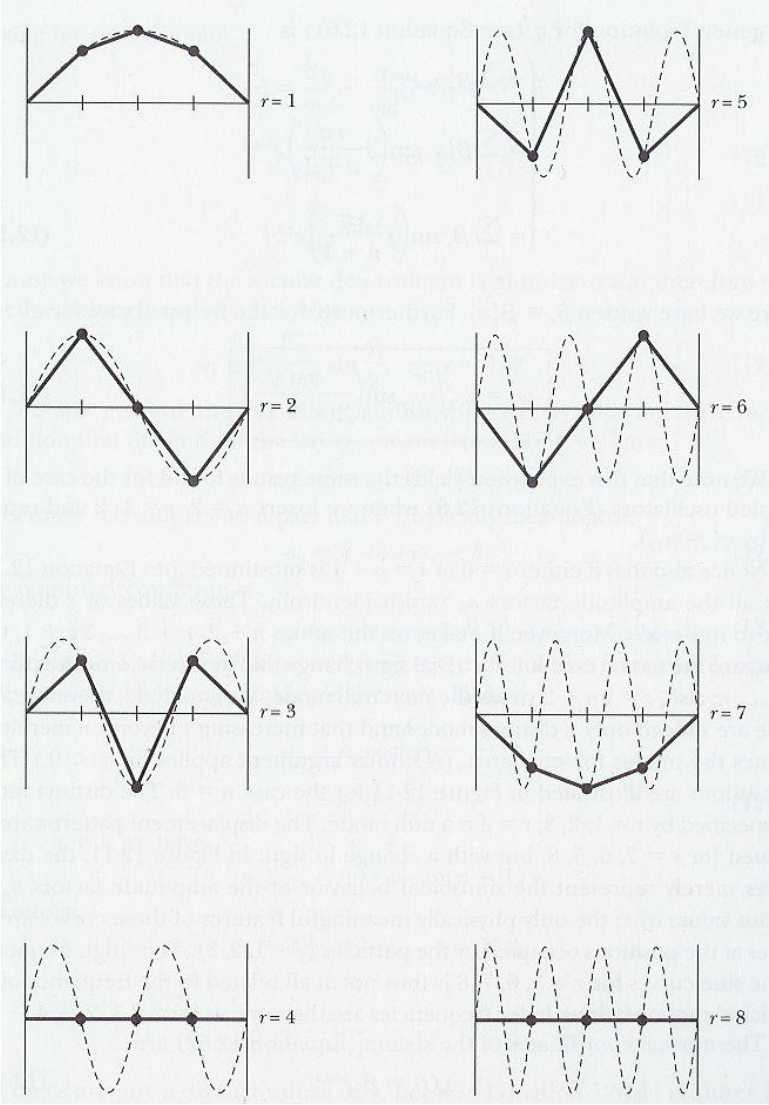
The n is an undetermined integer. We will see below that, though n could of course take on any integer value, only the values $n=1,\ldots,M$ yield physically unique solutions. Using the above, we find

$$\begin{aligned}\omega_n&=2\sqrt{\frac{\tau}{m\,d}}\sin\left(\frac{1}{2}\frac{n\,\pi}{M+1}\right)\\ \Phi_{n,p}&=\alpha\mathcal{R}\left[e^{ip\gamma-i\delta}\right]\\ &=\alpha\cos\left(\frac{n\,p\,\pi}{M+1}-\frac{\pi}{2}\right)\\ &=\alpha\sin\left(\frac{n\,p\,\pi}{M+1}\right)\end{aligned}$$

where $\Phi_{n,p}$ is the solution corresponding to ω_n at the pth point along the string. α is a yet-to-be-determined normalization.

Now, since sin is periodic, we can restrict the range of n to only those that will give unique physical solutions. The values $n<0$ and $n\geq 2(M+1)$ are redundant. The values $n=M+2,\ldots,2\,M+1$ are also redundant because they yield, up to an overall -1 sign the same function of p as the values $n=M-1,M-2,\ldots,1$. The -1 should be absorbed into the initial conditions. And, finally $n=M+1$ gives the same dependence on p as $n=0$, both of which give trivial solutions $y_{n,p}=0$ for all p. So there are really only M independent dynamical normal modes, $n=1,\ldots,M$. This is as we would expect because there are only M displacements to measure. The modes for a $M=3$ system are shown below, including not just the unique modes $n=1,2,3$ but also the redundant modes $n=4,\ldots,8$.

[10] Strictly speaking, Φ_0 and Φ_{M+1} are not defined; think of them instead as $\lim_{d\to 0}\Phi_1$ and $\lim_{d\to 0}\Phi_M$



Normalization of the Modes

Let's fix the normalization α. We are going to do something nonstandard but physically reasonable. The kinetic energy matrix $\mathbf{t}$ is just a multiple of the identity matrix. Therefore, there seems to be no reason to carry around the $\frac{m}{2}$ in the normalization. So, instead of requiring $\vec{\Phi}_n^T\,\mathbf{t}\,\vec{\Phi}_n = 1$, we simply require $\vec{\Phi}_n^T\vec{\Phi}_n = 1$. This requires

$$\alpha^2 \sum_{p=1}^{M} \sin^2\left(\frac{n\,p\,\pi}{M+1}\right) = 1$$

for any n. A trigonometric identity is available,[11]

$$\sum_{p=1}^{M} \sin^2\left(\frac{n\,p\,\pi}{M+1}\right) = \frac{M+1}{2}$$

[11] One might think this is easy to prove. It turns out not to be.

which implies $\alpha^2 = \frac{2}{M+1}$. So, our full normal mode solutions are

$$\omega_n = 2\sqrt{\frac{\tau}{m\,d}}\left|\sin\left(\frac{1}{2}\frac{n\,\pi}{M+1}\right)\right| \tag{3.20}$$

$$\Phi_{n,p} = \sqrt{\frac{2}{M+1}}\sin\left(\frac{n\,p\,\pi}{M+1}\right) \tag{3.21}$$

Applying Initial Conditions

We may use the standard formalism to apply the initial conditions. We will write the normal mode vectors $\Phi_{n,p}$ with vector notation $\vec{\Phi}_n$; these are M-component vectors representing the displacement at the M positions on the string. This now mimics our normal mode notation. As in that case, the generic solution is

$$\vec{y}_r(t) = \mathcal{R}\left[\vec{y}(t)\right] = \mathcal{R}\left[\sum_{n=1}^{M} A_n\,\vec{\Phi}_n\,e^{i\omega_n t}\right]$$

The $_r$ subscript indicates "real part", it is not an index. We again allow the A_n to be complex so that the different normal modes may be excited with different phases. Obtaining the mode coefficients from the initial conditions:

$$\begin{aligned}\mathcal{R}\left[A_n\right] &= \vec{\Phi}_n^T\,\vec{y}_r(t=0)\\ \mathcal{I}\left[A_n\right] &= \frac{1}{\omega_n}\,\vec{\Phi}_n^T\,\dot{\vec{y}}_r(t=0)\end{aligned}$$

where we have replaced $\vec{\Phi}_n^T\mathbf{t}$ with $\vec{\Phi}_n^T$ because of our normalization convention that replaces $\mathbf{t}$ with $\mathbf{I}$. Writing the above out explicitly using the p index gives

$$\mathcal{R}\left[A_n\right] = \sqrt{\frac{2}{M+1}}\sum_{p=1}^{M}\sin\left(\frac{n\,p\,\pi}{M+1}\right)y_{r,p}(t=0) \tag{3.22}$$

$$\mathcal{I}\left[A_n\right] = \frac{1}{\omega_n}\sqrt{\frac{2}{M+1}}\sum_{p=1}^{M}\sin\left(\frac{n\,p\,\pi}{M+1}\right)\dot{y}_{r,p}(t=0)$$

Remember that the $_r$ subscript on y stands for "real", it is not an index.

3.3.2 The Continuous String

We take the continuous limit of the loaded string case.

The Continous String Limit

Let us consider the loaded string solution in the limit of $d \to 0$, $M \to \infty$. Begin with Equations 3.21 and rewrite using $L = d\,(M+1)$ as the length of the string, $\Lambda = \frac{m}{d}$ as the string linear mass density, and $x = p\,d$ as the coordinate along the string:

$$\begin{aligned}\omega_n &= \frac{2}{d}\sqrt{\frac{\tau}{\Lambda}}\left|\sin\left(\frac{1}{2}\frac{n\,\pi\,d}{L}\right)\right| \to \frac{n\,\pi}{L}\sqrt{\frac{\tau}{\Lambda}}\\ \Phi_{n,p} &= \sqrt{\frac{2\,d}{L}}\sin\left(\frac{n\,p\,\pi\,d}{L}\right)\end{aligned}$$

The normalization convention is nonsensical as it is because the coefficient vanishes in the limit $d \to 0$. But let's reconsider how we arrived at the convention:

$$\begin{aligned} 1 &= \sum_{p=1}^{M} \Phi_{n,p}^2 \\ &= \sum_{p=1}^{M} \frac{2\,d}{L} \sin^2\left(\frac{n\,p\,\pi}{M+1}\right) \end{aligned}$$

Since d is the step size in x, we can take d to be the differential dx, which then converts the sum to an integral as $d \to 0$:

$$1 = \int_0^L dx\, \frac{2}{L} \sin^2\left(\frac{n\,\pi\,x}{L}\right)$$

With the new normalization condition, the normalization factor becomes the sensible $\sqrt{\frac{2}{L}}$. So our full solution is

$$\omega_n = \frac{n\,\pi}{L}\sqrt{\frac{\tau}{\Lambda}} \qquad\qquad \Phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\,\pi\,x}{L}\right) \tag{3.23}$$

where x now takes on the role of the index p. The "vector" nature of Φ_n corresponded to the index p, so we no longer write Φ_n as a vector. Initial conditions are applied via relations analogous to those found for the discrete case above. The general solution has expansion

$$y_r(x,t) = \mathcal{R}\left[\sum_{n=1}^{\infty} A_n\,\Phi_n(x)\,e^{i\omega_n t}\right]$$

where the limit of the sum is now ∞ because $M \to \infty$. The analogous relations for obtaining the expansion coefficients are (using the conversion $\sum_{p=1}^{M} d = \int_0^L dx$):

$$\begin{aligned} \mathcal{R}\,[A_n] &= \sqrt{\frac{2}{L}} \int_0^L dx\, \sin\left(\frac{n\,\pi\,x}{L}\right)\, y_r(x, t=0) \\ \mathcal{I}\,[A_n] &= \frac{1}{\omega_n}\sqrt{\frac{2}{L}} \int_0^L dx\, \sin\left(\frac{n\,\pi\,x}{L}\right)\, \dot{y}_r(x, t=0) \end{aligned} \tag{3.24}$$

Remember that the $_r$ subscript on y stands for "real", it is not an index. Note how the position index p has been completely replaced by the continuous variable x, and the vector dot products are replaced by the integral over x of a product.

Correspondence to Normal Mode Formalism

Though we have solved the problem using other techniques, it is useful to show how our solutions correspond to the normal mode formalism. Recall our normal mode formalism, where we transformed to normal mode coordinates $\vec{\psi}(t) = \mathbf{\Phi}^T\,\mathbf{t}\,\vec{\phi}(t)$ with $\mathbf{\Phi}$ being the matrix of normal mode vectors $\vec{\Phi}_n$, $\vec{\phi}(t)$ being the vector of position coordinates, and $\vec{\psi}(t)$ being the vector of coefficients of the normal modes. For the continous string case, we have

the correspondence

$$\begin{aligned}
p &\longleftrightarrow x \\
\phi_p(t) &\longleftrightarrow y(x,t) \\
\mathbf{t} &\longleftrightarrow \frac{\Lambda}{2} \\
\mathbf{\Phi}_{pn} = \Phi_{n,p} &\longleftrightarrow \sqrt{\frac{2}{\Lambda}}\,\Phi_n(x) = \sqrt{\frac{2}{\Lambda}}\,\sqrt{\frac{2}{L}}\,\sin\Big(\frac{n\,\pi\,x}{L}\Big) \\
\psi_n(t=0) &\longleftrightarrow \sqrt{\frac{\Lambda}{2}}\,A_n \\
\psi_n(t) &\longleftrightarrow \sqrt{\frac{\Lambda}{2}}\,\mathcal{R}\left[A_n\,e^{i\omega_n t}\right] \\
\dot{\psi}_n(t) &\longleftrightarrow \sqrt{\frac{\Lambda}{2}}\,\mathcal{R}\left[i\,\omega_n\,A_n\,e^{i\omega_n t}\right]
\end{aligned}$$

Recall that Λ is the linear mass density, the limit of m/d. The $\sqrt{\frac{\Lambda}{2}}$ factors arise because we chose to unit-normalize the normal-mode vectors rather than normalize them using the $\mathbf{t}$ matrix. Recall that, for a finite number of position coordinates $p = 1 \ldots M$, $\vec{\psi}(t)$ was the transformation of $\vec{\phi}(t)$ from position coordinates to normal modes; $\psi_n(t)$ describes the time evolution of the coefficient of the nth normal mode, while $\phi_p(t)$ describes the time evolution of the pth position coordinate. Here we of course go from the discrete index p to the continuous index x to label the y position coordinates.

It is interesting to count up degrees of freedom. We have a continuous variable x to label the degrees of freedom in the y direction. This is an uncountably infinite number of degrees of freedom; for the non-mathematicians, this simply means that the number of degrees of freedom is of the same order as the number of real numbers, rational and irrational. On the other hand, the number of independent normal modes is countably infinite because they are labeled by the integer n; that is, the number of modes is of the same order as the number of integers, which, while infinite, is less infinite than the number of rational and irrational numbers. But, we expect our normal mode formalism to preserve the number of degrees of freedom. What has happened? Boundary conditions. They add more information to the problem, restricting the freedom in the normal modes. If we had an infinitely long string with no boundary conditions imposed, n could take on noninteger rational and irrational values, matching the number of degrees of freedom in the position coordinates.

The Diagonalized Hamiltonian

From our generic discussion of normal modes, the Hamiltonian can be rewritten in terms of normal modes as

$$H = \dot{\vec{\psi}}^T \mathbf{I}\,\dot{\vec{\psi}} + \vec{\psi}^T \Omega^2\,\vec{\psi}$$

So, with the above correspondences, the Hamiltonian becomes

$$\begin{aligned} H &= \sum_{n=1}^{\infty} \frac{\Lambda}{2} \left\{ \mathcal{R}\left[i\,\omega_n\, A_n\, e^{i\omega_n t} \right] \right\}^2 + \sum_{n=1}^{\infty} \frac{\Lambda}{2}\, \omega_n^2 \left\{ \mathcal{R}\left[A_n\, e^{i\omega_n t} \right] \right\}^2 \\ &= \frac{\Lambda}{2} \sum_{n=1}^{\infty} \omega_n^2 \left\{ \left(\mathcal{R}[A_n] \right)^2 + \left(\mathcal{I}[A_n] \right)^2 \right\} \left\{ \cos^2(\omega_n t) + \sin^2(\omega_n t) \right\} \\ &= \frac{\Lambda}{2} \sum_{n=1}^{\infty} \omega_n^2 \left| A_n \right|^2 \end{aligned}$$

Recall that the A_n have units of length$^{\frac{3}{2}}$, ω_n has units of inverse time, and Λ has units of mass per unit length, so the resulting quantity does indeed have units of energy.

3.3.3 The Wave Equation

We derive the wave equation for the loaded string case and then consider it more generally. We study solutions of the equation.

Deriving the Wave Equation

If we consider our solution Equation 3.23, we quickly see that there is a relationship between the time dependence and the spatial dependence. Two derivatives of a sine or a cosine return the original function, which lead us to consider

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\, y_n(x,t) &= -\left(\frac{n\,\pi}{L} \right)^2 y_n(x,t) \\ \frac{\partial^2}{\partial t^2}\, y_n(x,t) &= -\left(\frac{n\,\pi}{L} \right)^2 \frac{\tau}{\Lambda}\, y_n(x,t) \end{aligned}$$

We thus obtain the **wave equation**:

$$\frac{\partial^2}{\partial t^2}\, y_n(x,t) = \frac{\tau}{\Lambda} \frac{\partial^2}{\partial x^2}\, y_n(x,t)$$

Since we have proven that the wave equation holds for every normal mode with the same coefficient $\frac{\tau}{\Lambda}$, it holds for any linear combination thereof and thus for any continuous string solution. But let's derive it directly also.

First, let's work from the equation of motion. We have Equation 3.18, but let's take it a step backward before the assumption of harmonic ($e^{i\omega t}$) time dependence is made. We have

$$\begin{aligned} \frac{m}{2} \frac{d^2}{dt^2}\, y_p(t) + \frac{\tau}{d}\, y_p(t) - \frac{\tau}{2\,d} \left(y_{p+1}(t) + y_{p-1}(t) \right) &= 0 \\ \frac{m}{2} \frac{d^2}{dt^2}\, y_p(t) + \frac{\tau}{2\,d} \left[y_p(t) - y_{p-1}(t) \right] + \frac{\tau}{2\,d} \left[y_p(t) - y_{p+1}(t) \right] &= 0 \end{aligned}$$

Recall that these equations are essentially just Newton's second law applied to the mass p, as was discussed in a footnote in Section 3.3.1. Now, the finite difference $y_p(t) - y_{p-1}(t)$ becomes $d\,\frac{\partial}{\partial x}\, y(x + \frac{d}{2}, t)$ in the limit $d \to 0$. So we have

$$\frac{m}{2} \frac{\partial^2}{\partial t^2}\, y(x,t) + \frac{\tau}{2} \left[\frac{\partial}{\partial x}\, y\left(x - \frac{d}{2}, t \right) - \frac{\partial}{\partial x}\, y\left(x + \frac{d}{2}, t \right) \right] = 0$$

The difference of two partial derivatives evaluated at points separated by d is a second partial derivative (have to insert a sign)

$$\frac{m}{2}\frac{\partial^2}{\partial t^2}\,y(x,t) - \frac{\tau}{2}\,d\,\frac{\partial^2}{\partial x^2}\,y\,(x,t) \;=\; 0$$

Now, $\Lambda = \frac{m}{d}$ is the string linear mass density, so we have

$$\frac{\partial^2}{\partial t^2}\,y(x,t) - \frac{\tau}{\Lambda}\,\frac{\partial^2}{\partial x^2}\,y\,(x,t) \;=\; 0 \tag{3.25}$$

So we find the wave equation can be derived directly from the equation of motion without assuming harmonic time dependence.

Functional Form of Solutions

The key to seeing the general form of solutions to the wave equation is to recognize that $\frac{\tau}{\Lambda}$ has dimensions of a squared velocity. Thus, with $v = \sqrt{\frac{\tau}{\Lambda}}$, we obtain

$$\frac{\partial^2}{\partial t^2}\,y(x,t) - v^2\,\frac{\partial^2}{\partial x^2}\,y\,(x,t) \;=\; 0$$

We thus see that the functional dependence of a solution $y(x,t)$ on x and t must be the same up to a factor v. It is therefore natural to consider the solutions to be functions of the variables

$$\xi \equiv x + v\,t$$
$$\eta \equiv x - v\,t$$

Note that we are not constraining y in any fashion, since the linear combinations $\xi + \eta$ and $\frac{1}{v}\,(\xi - \eta)$ return x and t. Using the chain, rule, we may write the partial derivatives in terms of ξ and η:

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial \xi}\,\frac{\partial \xi}{\partial x} + \frac{\partial y}{\partial \eta}\,\frac{\partial \eta}{\partial x} = \frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}$$

The second derivative is therefore

$$\begin{aligned}\frac{\partial^2 y}{\partial x^2} &= \frac{\partial}{\partial \xi}\left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial y}{\partial \xi} + \frac{\partial y}{\partial \eta}\right)\frac{\partial \eta}{\partial x} \\ &= \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \eta^2} + 2\,\frac{\partial^2 y}{\partial \xi\,\partial \eta}\end{aligned}$$

Similarly, we may show

$$\begin{aligned}\frac{1}{v}\frac{\partial y}{\partial t} &= \frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial \eta} \\ \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} &= \frac{\partial^2 y}{\partial \xi^2} + \frac{\partial^2 y}{\partial \eta^2} - 2\,\frac{\partial^2 y}{\partial \xi\,\partial \eta}\end{aligned}$$

If the wave equation is to hold, the mixed second partial derivative must vanish. Thus, $y(x,t)$ must be of the form

$$y(x,t) = f(\xi) + g(\eta) = f(x + v\,t) + g(x - v\,t)$$

That is, the wave equation puts a constraint on the functional form of y that requires it to be a sum of separate functions of ξ and η. This is a nontrivial constraint. It tells us that we have "propagating" solutions: to maintain fixed ξ or η, the solution must advance in x at $\pm\, v\, t$. The solution $f(x + v\, t)$ propagates to the left and the solution $g(x - v\, t)$ propagates to the right.

Separation of Variables for the Wave Equation

Let us consider a generic solution by **separation of variables**. We assume the solution is of the form

$$y(x,t) = \phi(t)\,\psi(x)$$

Plugging into the wave equation, we find

$$\frac{1}{\phi}\,\frac{d^2}{dt^2}\,\phi(t) = \frac{v^2}{\psi}\,\frac{d^2}{dx^2}\,\psi(x)$$

where we have divided by $y(x,t) = \phi\,\psi$ under the assumption that the solution is not identically zero. (Appropriate limits may be taken if the solutions vanish at particular values of x and/or t.) Since the two sides are dependent on different variables – the variables have been **separated** – they both must equal a constant, which we will call $-\omega^2$ so that the definition of ω will come out consistent with our previous solution. That is, we now have two equations:

$$\frac{d^2}{dt^2}\,\phi(t) + \omega^2\phi(t) = 0 \qquad \frac{d^2}{dx^2}\,\psi(x) + k^2\psi(x) = 0 \tag{3.26}$$

where $k^2 = \frac{\omega^2}{v^2}$. We know the solutions to these equations are of the form

$$\psi(x) = A\,e^{ikx} + B\,e^{-ikx} \qquad \phi(t) = C\,e^{i\omega t} + D\,e^{-i\omega t}$$

where all four coefficients may be complex (but must have certain relations among them to obtain real solutions in the end). No boundary conditions have been applied, so ω and all four constants for each value of ω remain free. The generic solution is therefore

$$y(x,t) = \sum_r \left[a_{r++}\,e^{i(k_r x+\omega_r t)} + a_{r+-}\,e^{i(k_r x-\omega_r t)} + a_{r-+}\,e^{-i(k_r x-\omega_r t)} + a_{r--}\,e^{-i(k_r x+\omega_r t)}\right]$$

The coefficients are complex and have 8 degrees of freedom for each r. To ensure real solutions in the end, we require that $\mathcal{I}[y(x,t)]$ vanish at all times. Since each mode has different time dependence, each mode's imaginary part must vanish separately. So we have the condition

$$\begin{aligned} 0 &= \mathcal{I}\left[a_{r++}\,e^{i(k_r x+\omega_r t)} + a_{r+-}\,e^{i(k_r x-\omega_r t)} + a_{r-+}\,e^{-i(k_r x-\omega_r t)} + a_{r--}\,e^{-i(k_r x+\omega_r t)}\right] \\ &= \mathcal{R}\left[a_{r++} - a_{r--}\right]\sin(k_1 x + \omega_r t) + \mathcal{R}\left[a_{r+-} - a_{r-+}\right]\sin(k_1 x - \omega_r t) \\ &\quad + \mathcal{I}\left[a_{r++} + a_{r--}\right]\cos(k_1 x + \omega_r t) + \mathcal{I}\left[a_{r+-} + a_{r-+}\right]\cos(k_1 x - \omega_r t) \end{aligned}$$

The different spatial and temporal dependences of the four terms imply that each coefficient must vanish separately. This implies

$$a_{r--} = a_{r++}^* \qquad a_{r-+} = a_{r+-}^*$$

i.e., the two modes with the same propagation direction have complex conjugate coefficients. The coeffcients remain complex, but the solution is now simplified to

$$\begin{aligned} y(x,t) &= \sum_r \left[a_{r<}\, e^{i(k_r x+\omega_r t)} + a_{r>}\, e^{i(k_r x-\omega_r t)} + a^*_{r>}\, e^{-i(k_r x-\omega_r t)} + a^*_{r<}\, e^{-i(k_r x+\omega_r t)} \right] \\ &= 2\,\mathcal{R}\left\{ \sum_r \left[a_{r<}\, e^{i(k_r x+\omega_r t)} + a_{r>}\, e^{i(k_r x-\omega_r t)} \right] \right\} \end{aligned}$$

where we have relabeled the coefficients with $_>$ and $_<$ to indicate right- and left-going waves. The second step is mathematically rigorous, not just justified by saying that our function must be real in the end. (It reflects the fact that there is no physical difference between $\omega < 0$ and $\omega > 0$, while there is such a difference for k.) There now remain four degrees of freedom. Boundary conditions will reduce the number of degrees of freedom further.

k is the **wavenumber** and clearly $k = \frac{2\pi}{\lambda}$ where λ is the spatial wavelength (distinct from Λ, the mass density). The modes can be rewritten to make clear the relative roles of the wavenumber and velocity:

$$e^{i(k_r x \pm \omega_r t)} = e^{ik_r(x \pm v\, t)}$$

Note that, unless the velocity has explicit dependence on ω, the propagation velocity v is the same for all frequencies. k_r is the quantity that determines how the spatial and temporal dependence varies between modes.

Solutions for Particular Boundary Conditions

Further detailing of solutions requires application of boundary conditions.[12] We consider some examples.

- **Standing Waves**: This is the type of solution we originally found. Our boundary condition is to require that the solution vanishes identically at $x = 0$ and $x = L$ for all time. As usual, the condition separates by modes, giving

$$0 = \mathcal{R}\left\{ \left[a_{r<} + a^*_{r>} \right] e^{i\omega_r t} \right\} \qquad 0 = \mathcal{R}\left\{ \left[a_{r<}\, e^{ik_r L} + a^*_{r>}\, e^{-ik_r L} \right] e^{i\omega_r t} \right\}$$

  Writing the above out gives

$$\begin{aligned} 0 &= \mathcal{R}\left\{ a_{r<} + a^*_{r>} \right\} \cos\omega_r t - \mathcal{I}\left\{ a_{r<} + a^*_{r>} \right\} \sin\omega_r t \\ 0 &= \mathcal{R}\left\{ a_{r<}\, e^{ik_r L} + a^*_{r>}\, e^{-ik_r L} \right\} \cos\omega_r t - \mathcal{I}\left\{ a_{r<}\, e^{ik_r L} + a^*_{r>}\, e^{-ik_r L} \right\} \sin\omega_r t \end{aligned}$$

  Since these conditions must hold for all time, the coefficients of the cos and the sin terms vanish separately, which imply

$$a_{r<} = -a^*_{r>} \qquad a_{r>}\, e^{ik_r L} = -a^*_{r<}\, e^{-ik_r L}$$

[12] As a matter of cultural interest, we note that quantization of energies and momenta in quantum mechanics is identical to the way quantization of wave frequencies and wavevectors will arise in the following examples. Quantization in quantum mechanics is entirely the result of a) treating particles as complex waves and b) imposing boundary conditions on those waves.

Given the left relation, the right relation can hold only if $\mathcal{I}[e^{ik_rL}]$ vanishes, which implies $k_rL = r\,\pi$ with r an integer. Thus our solution is of the form

$$k_rL = r\,\pi \quad \Longrightarrow \quad \omega_r = k_r\,v = \frac{r\,\pi}{L}\,v = \frac{r\,\pi}{L}\sqrt{\frac{\tau}{\Lambda}} \tag{3.27}$$

$$y(x,t) = \mathcal{R}\left\{\sum_r \left[a_r\,e^{i(k_rx+\omega_rt)} - a_r^*\,e^{i(k_rx-\omega_rt)}\right]\right\}$$

where we write a_r for $a_{r<}$ now that there is only one coefficient to worry about. (We have also dropped the 2 in front of the expression for $y(x,t)$; it is just an overall normalization that can be absorbed into the a_r.) We recover the quantization condition on the mode frequencies that we found earlier. This leaves two degrees of freedom $\mathcal{R}[a_r]$ and $\mathcal{I}[a_r]$ to be determined by initial conditions. We may rewrite to obtain a form more similar to what we obtained earlier:

$$\begin{aligned} y(x,t) &= 2\sum_r \Big[\mathcal{R}(a_r)\,\cos(k_rx+\omega_rt) - \mathcal{I}(a_r)\,\sin(k_rx+\omega_rt) \\ &\qquad\qquad - \mathcal{R}(a_r)\,\cos(k_rx-\omega_rt) - \mathcal{I}(a_r)\,\sin(k_rx-\omega_rt)\Big] \\ &= \sum_r \left[4\,\mathcal{R}(a_r)\,\sin(k_rx)\,\sin(\omega_rt) - 4\,\mathcal{I}(a_r)\,\sin(k_rx)\,\cos(\omega_rt)\right] \\ &= \sum_r \sin(k_rx)\left[\gamma_r\cos(\omega_rt) + \delta_r\sin(\omega_rt)\right] \end{aligned}$$

where γ_r and δ_r are both *real* and are determined by the initial conditions. This may look a bit different from our earlier result for the normal modes of a continuous string with the same boundary conditions, but we can see that they are the same. Recall from Equations 3.23 and 3.24 our solution was

$$y(x,t) = \mathcal{R}\left[\sum_{n=1}^{\infty} A_n\,\Phi_n(x)\,e^{i\omega_nt}\right] \qquad \Phi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi\,x}{L}\right) \qquad \omega_n = \frac{n\,\pi}{L}\sqrt{\frac{\tau}{\Lambda}}$$

Rewriting, we have

$$y(x,t) = \sum_{n=1}^{\infty}\left\{\mathcal{R}(A_n)\sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi\,x}{L}\right)\cos\omega_n\,t - \mathcal{I}(A_n)\sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi\,x}{L}\right)\sin\omega_n\,t\right\}$$

clearly providing a direct correspondence between the two cases.

- **Traveling Wave Incident on Interface**: Two strings of differing densities Λ_1 and Λ_2 are joined at $x = 0$. A continuous wave train of frequency ω is incident on the interface. What is the solution and what are the ratios of the amplitudes of the reflected and transmitted waves to the incident wave amplitude? What are the transmitted and reflected power fractions?

  We write the solution in the two regions as

$$\begin{aligned} y_{x<0}(x,t) &= \mathcal{R}\left\{a_>\,e^{i(k_1x-\omega_1t)} + a_<\,e^{i(k_1x+\omega_1t)}\right\} \\ y_{x>0}(x,t) &= \mathcal{R}\left\{b_>\,e^{i(k_2x-\omega_2t)}\right\} \end{aligned}$$

We have made only one assumption so far, which is that, because the incoming wave is rightward propagating, we need only include rightward-propagating modes in the transmitted wave; obviously, it would violate causality to generate a leftward-propagating wave on the right side of the interface if the incoming wave is incident from the left side. The phase and amplitude of the incoming wave are set by initial conditions we have not yet specified, so we will not try to determine $a_>$. Thus, we have four degrees of freedom to determine now, and two more to be set by initial conditions.

The time dependence of the second term in the $x < 0$ solution appears to have no match on the right side, but one must remember that we will take the real part. To remove this apparent contradiction, we may rewrite the above as

$$\begin{aligned} y_{x<0}(x,t) &= \mathcal{R}\left\{a_> \, e^{i(k_1 x - \omega_1 t)} + a_<^* \, e^{-i(k_1 x + \omega_1 t)}\right\} \\ y_{x>0}(x,t) &= \mathcal{R}\left\{b_> \, e^{i(k_2 x - \omega_2 t)}\right\} \end{aligned}$$

which is valid because the real part of a complex number and of its complex conjugate are the same.

The boundary condition we need to apply in this case is to require continuity of the wave function and its first spatial derivative at the interface. The first spatial derivative must be continuous in order to have a finite value of the second spatial derivative at the interface; an infinite value would imply by the wave equation that $\frac{\partial^2 y}{\partial t^2}$ is also infinite at the interface, which requires infinite force. So we have

$$\begin{aligned} \mathcal{R}\left\{(a_> + a_<^*)\, e^{-i\,\omega_1 t}\right\} &= \mathcal{R}\left\{b_> \, e^{-i\omega_2 t}\right\} \\ \mathcal{R}\left\{i\,k_1\,(a_> - a_<^*)\, e^{-i\omega_1 t}\right\} &= \mathcal{R}\left\{i\,k_2\, b_> \, e^{-i\omega_2 t}\right\} \end{aligned}$$

If we do not want a trivial solution, we must have $\omega_1 = \omega_2$ in order for the conditions to be satisfied at all time. Using this fact, and writing out the above explicitly gives

$$\begin{aligned} \mathcal{R}\left\{a_> + a_<^*\right\}\cos\omega_1 t + \mathcal{I}\left\{a_> + a_<^*\right\}\sin\omega_1 t &= \mathcal{R}\left\{b_>\right\}\cos\omega_1 t + \mathcal{I}\left\{b_>\right\}\sin\omega_1 t \\ -\mathcal{I}\left\{a_> - a_<^*\right\}k_1\cos\omega_1 t + \mathcal{R}\left\{a_> - a_<^*\right\}k_1\sin\omega_1 t &= -\mathcal{I}\left\{b_>\right\}k_2\cos\omega_1 t + \mathcal{R}\left\{b_>\right\}k_2\sin\omega_1 t \end{aligned}$$

The coefficients of the cos and sin terms must be separately equal for the equations to be satisfied at all times. So we have

$$a_> + a_<^* = b_> \qquad k_1\,(a_> - a_<^*) = k_2\, b_>$$

The solutions for $a_<$ and $b_>$ are therefore

$$\frac{a_<^*}{a_>} = \frac{k_1 - k_2}{k_1 + k_2} \qquad \frac{b_>}{a_>} = \frac{2\,k_1}{k_1 + k_2}$$

Up to π, the phases of $a_<$ and $b_>$ are the same as the phase of $a_>$. We will in general let $a_>$ be real because an overall phase factor has no physical consequences; if we do this, then $a_<$ and $b_>$ are also real. The sign of $a_<$ relative to $a_>$ depends on the relative sizes of the wavevectors, which is set by the different velocities because the frequencies are identical (recall, $k = \frac{\omega}{v}$). $a_<$ has the same sign as $a_>$ when $k_1 > k_2$, which occurs when $v_1 < v_2$ or, more fundamentally, $\Lambda_1 > \Lambda_2$ (assuming continuous tension to ensure finite acceleration of the interface). The sign of $b_>$ is always positive.

The reader will note that we have not obtained any kind of quantization of ω or k – no conditions have been placed on ω. This is simply because our boundary conditions are not restrictive enough. ω may now take on any real number value, implying that the number of possible modes is now uncountably infinite. The number of modes in this case is indeed equal to the number of position coordinates x at which the string may oscillate. This is in contrast to the countably infinite number of modes we obtained in the standing wave problem because of the more constraining boundary conditions.

To determine the transmitted and reflected power fractions, we first recall from earlier that the energy of a system that has been decomposed into normal modes may be written as follows

$$E = \frac{\Lambda}{2} \sum_n \omega_n^2 \left|A_n\right|^2$$

We have two modifications in this case: 1) n may take on any real number value, and the sum over n should really become an integral, because no quantization condition has arisen for ω; and 2), our standing wave normalization convention does not work because $L = \infty$, so we will likely have to normalize on some fixed standard length (*e.g.*, 1 m), and, similarly, E will be infinite, so we should restrict ourselves to consider the energy density $\mathcal{E}$, the energy per unit length. Suffice it to say that the expression for energy density will look like

$$\mathcal{E} \propto \frac{\Lambda}{2} \int_0^\infty dn\, \omega^2(n) \left|A(n)\right|^2$$

We have rewritten n as a continuous variable and ω and A as continuous functions of n to reflect the lack of quantization. Since the as-yet-undetermined constant of proportionality will cancel in calculating power ratios, we will not discuss it yet. The power passing any given point is, on purely dimensional grounds, the product of energy density and wave velocity. This definition of course also makes sense physically: if $\mathcal{E}$ is the energy density (energy per unit length) then a length l of the wave contains energy $\mathcal{E}\, l$ and it takes l/v seconds for that length to pass through any given point. Thus, the power is $\mathcal{E}\, l/(l/v) = \mathcal{E}\, v$. So, considering a single mode, the power in that mode is

$$P(n) = \mathcal{E}(n)\, v \propto \frac{\Lambda}{2}\, \omega^2(n) \left|A(n)\right|^2 v$$

ω is the same between the two rope sections and so will cancel for any given mode, but $v = \sqrt{\tau/\Lambda}$ and $A(n)$ will differ for modes on different sides of the interface. Therefore, the reflected energy fraction is

$$R = \frac{\Lambda_1 \left|a_<\right|^2 v_1}{\Lambda_1 \left|a_>\right|^2 v_1} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

We obtain the transmitted energy fraction by using the expression for $P(n)$:

$$\begin{aligned} T &= \frac{\Lambda_2 \left|b_>\right|^2 v_2}{\Lambda_1 \left|a_>\right|^2 v_1} = \frac{\Lambda_2}{\Lambda_1}\, \frac{4\, k_1^2}{\left(k_1 + k_2\right)^2} \sqrt{\frac{\Lambda_1}{\Lambda_2}} = \frac{k_2}{k_1}\, \frac{4\, k_1^2}{\left(k_1 + k_2\right)^2} \\ &= \frac{4\, k_1\, k_2}{\left(k_1 + k_2\right)^2} \end{aligned}$$

It is critical to remember the factor $\Lambda\, v$ when comparing modes on the two sides of the interface. We can check that energy is conserved by calculating $1 - R$:

$$1 - R = \frac{4\, k_1\, k_2}{(k_1 + k_2)^2} = \frac{k_2}{k_1} \frac{|b_>|^2}{|a_>|^2} = T$$

as expected (calculating $T = 1 - R$ is a perfectly valid alternative to using the power expression). Physically, the results are as we expect: all the energy is transmitted when the wavevectors k_1 and k_2 are matched, which occurs when the velocities and hence the mass densities in the two sections of the string are matched.

3.3.4 Phase Velocity, Group Velocity, and Wave Packets

We consider the difference between phase velocity and group velocity for dispersive media and introduce wave packets.

Phase Velocity

As we have already hinted at, the velocity parameter v is the speed at which the wave propagates. We may demonstrate this explicitly by considering a point of fixed phase ϕ for a particular mode propagating in a specific direction:

$$y(x,t) = A\, e^{i(kx - \omega t)} \qquad \phi = kx - \omega t$$

If we require the phase to be constant as x and t vary, we find

$$0 = d\phi = p\, dx - \omega\, dt \quad \Longrightarrow \quad V = \frac{dx}{dt} = \frac{\omega}{k} = v$$

So the velocity parameter is indeed the speed at which the wave propagates. For the loaded and continous strings, we have

$$\text{loaded}: \quad V = \frac{\omega}{k} = \frac{2\sqrt{\frac{\tau\, d}{m}}\, \left|\sin\left(\frac{k\, d}{2}\right)\right|}{\frac{k\, d}{2}}$$

$$\text{continuous}: \quad V = \frac{\omega}{k} = v = \sqrt{\frac{\tau}{\Lambda}}$$

(We have made the obvious correspondence $k = \frac{n\,\pi}{(M+1)\, d}$.) For the loaded string, the velocity depends on the frequency, while for the continuous string it does not. The loaded string is termed a **dispersive** medium for reasons we will see later. In the loaded string, dispersion arises because the properties of the mode depend on the ratio of the wavelength to the spacing of the masses. As this spacing goes to zero in the continous string limit, the dependence vanishes.

Group Velocity and Wave Packets

If we are in a dispersive medium and the phase velocity is a function of frequency, at what speed does the wave truly propagate? There is a fundamental difficulty in asking this question because we have so far only considered waves that are infinite in spatial and temporal extent. Isolating the energy of the wave to a localized position and time is not possible. To do so, we must consider **wave packet** solutions.

Our general solution for a rightgoing wave on a continous string is

$$\begin{aligned} y(x,t) &= \sum_{r=1}^{\infty} \left[a_{r>} e^{i(k_r x - \omega_r t)} + a_{r>}^* e^{-i(k_r x - \omega_r t)} \right] \\ &= \mathcal{R} \left[\sum_{r=1}^{\infty} \alpha_{r>} e^{i(\omega_r t - k_r x)} \right] \end{aligned}$$

where we have defined $\alpha_{r>} = 2\, a_{r>}^*$ for $r > 0$. Now, if we let $L \to \infty$, we may take the sum on discrete modes over to an integral:

$$y(x,t) = \mathcal{R} \left[\int_0^{\infty} dk\, \alpha_>(k)\, e^{i(\omega(k)t - kx)} \right]$$

where $dk\, \alpha_>(k_r) = \alpha_{r>}$. dk can be taken to be the mode spacing in k units, $\frac{\pi}{L}$, which goes to zero as $L \to \infty$. We can allow for left-going waves by allowing the integral to extend from $k = -\infty$ to $k = 0$; we ought to drop the $>$ subscript, giving

$$y(x,t) = \mathcal{R} \left[\int_{-\infty}^{\infty} dk\, \alpha(k)\, e^{i(\omega(k)t - kx)} \right]$$

We have written ω as a function of k because k is now effectively the mode index. $\alpha(k)$ is known as the **spectral distribution** or the **Fourier transform** of the wave solution $y(x,t)$. Now, let us consider a wave with spectral distribution that drops to zero outside some interval $(k_0 - \Delta k, k_0 + \Delta k)$. That is, consider

$$y(x,t) = \mathcal{R} \left[\int_{k_0 - \Delta k}^{k_0 + \Delta k} dk\, \alpha(k)\, e^{i(\omega(k)t - kx)} \right]$$

The above definition should be considered "schematic"; a spectral distribution $\alpha(k)$ that does not vanish identically outside some interval but does drop off quickly will have similar characteristics to what we will derive below.

Since we are working in a small interval around k_0, we may Taylor expand $\omega(k)$:

$$\omega(k) \approx \omega(k_0) + \left. \frac{d\omega}{dk} \right|_{k=k_0} (k - k_0) \equiv \omega_0 + \omega_0'(k - k_0)$$

The argument of the exponential becomes

$$\begin{aligned} \omega(k)t - kx &\approx (\omega_0 t - k_0 x) + \omega_0'(k - k_0)t - (k - k_0)x \\ &= (\omega_0 t - k_0 x) + (k - k_0)(\omega_0' t - x) \end{aligned}$$

The wave solution becomes

$$\begin{aligned} y(x,t) &= \mathcal{R} \left\{ e^{i(\omega_0 t - k_0 x)} \int_{k_0 - \Delta k}^{k_0 + \Delta k} dk\, \alpha(k)\, e^{i(k - k_0)(\omega_0' t - x)} \right\} \\ &= \mathcal{R} \left\{ e^{i(\omega_0 t - k_0 x)} \int_{-\Delta k}^{\Delta k} d\tilde{k}\, \alpha(k_0 + \tilde{k})\, e^{i(\tilde{k}\omega_0' t - \tilde{k}x)} \right\} \end{aligned}$$

where we have brought the term independent of k outside the integral, changed variables from the absolute k to the offset from k_0, and brought the term independent of k outside

the integral. The exponential out front is quickly oscillating while the exponential in the integral is slowly oscillating in comparison because $|\tilde{k}| < \Delta k \ll k_0$. The exponential term in front thus has no effect on the the localization of the wave function; it advances in phase with time, but the amplitude is independent of position. It is the integral term that provides the spatial dependence of the amplitude. The spatial dependence is the wave packet. At what speed does this wave packet propagate? We get coherent propagation of the amplitude function if the phases of all the spectral components stay lined up; *i.e.*, when $\tilde{k}\,\omega_0' \, t - \tilde{k}\, x$ is fixed. Taking the differential:

$$\omega_0' \, dt - dx = 0 \qquad \Longrightarrow \qquad \frac{dx}{dt} = \omega_0' = \left.\frac{d\omega}{dk}\right|_{k=k_0}$$

Thus, the amplitude function propagates with speed $v_g = \left.\frac{d\omega}{dk}\right|_{k=k_0}$, which is termed the **group velocity**. With the group velocity, we may rewrite the wave solution as

$$y(x,t) \;\; = \;\; \mathcal{R}\left\{ e^{i(\omega_0 t - k_0 x)} \int_{-\Delta k}^{\Delta k} d\tilde{k}\, \alpha(k_0 + \tilde{k})\, e^{i\tilde{k}(v_g t - x)} \right\} \tag{3.28}$$

It is instructive to write v_g as a derivative of v with respect to ω to show how the dispersion of the medium comes in:

$$v_g^{-1} \;\; = \;\; \left.\frac{dk}{d\omega}\right|_{k=k_0} = \left.\frac{d}{d\omega}\left(\frac{\omega}{v}\right)\right|_0 = \left.\frac{1}{v}\right|_0 - \left.\frac{\omega}{v^2}\frac{dv}{d\omega}\right|_0$$

or

$$v_g = \frac{v_0}{1 - \left.\frac{\omega_0}{v_0}\frac{dv}{d\omega}\right|_0}$$

where $_0$ subscripts indicate evaluation at $k = k_0$, $\omega = \omega(k_0)$, $v = v(\omega_0)$. Thus, we find the group velocity is related to the variation of the phase velocity with frequency. For nondispersive media, the derivative term in the denominator vanishes and $v_g = v_0 = v$ as one expects. Because the group velocity describes the speed of motion of the amplitude function, the group velocity is the speed at which energy, momentum, and information propagate.

Examples of dispersion in reality abound. You will see in your electromagnetism course how the dispersion of a medium is related to the real part of the dielectric constant. The dielectric constant is determined by the polarizability of the medium. When the incoming light wave photon energy approaches that of quantum mechanical transitions (atomic, molecular, etc.), the polarizability of the medium changes quickly due to the possibility of resonant absorption and emission. Thus, one gets extreme dispersion in the vicinity of such frequencies. The dispersion of the ionosphere as the frequency approaches the plasma frequency explains why AM radio can be accessible at very large distances – the dispersion of the ionosphere also causes refractive bending (Snell's law) that results in reflection of the waves back to earth.

Chapter 4

Central Force Motion and Scattering

The problem of the motion of two bodies interacting via a central force is an important application of Lagrangian dynamics and the conservation theorems we have learned about. Central forces describe a large variety of classical systems, ranging from gravitationally interacting celestial bodies to electrostatic and nuclear interactions of fundamental particles. The central force problem provides one of the few exactly solvable problems in mechanics. And central forces underly most scattering phenomena, again ranging from gravitational to electrostatics to nuclear.

This chapter is rather short, covering the material in Chapter 4 of Hand and Finch with some additional material from other sources (Thornton, Goldstein).

4.1 The Generic Central Force Problem

We first discuss the central force problem in general terms, considering arbitrary radially dependent potential energy functions.

4.1.1 The Equation of Motion

Review of Central Forces

In our discussion of Newton's third law in Section 1.3, we defined a **central force** as one that satisfies the strong form of Newton's third law. That is, given two particles a and b, the force exerted by particle a on b is equal and opposite to that exerted by particle b on particle a, and, moreover, the force depends only on the separation of the two particles and points along the vector between the two particles. Mathematically, this means

$$\vec{f}_{ab} = -\vec{f}_{ba} \qquad \vec{f}_{ab} = f_{ab}(r_{ab})\,\hat{r}_{ab}$$
$$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b \qquad r_{ab} = |\vec{r}_{ab}| \qquad \hat{r}_{ab} = \frac{\vec{r}_{ab}}{r_{ab}}$$

Properties of an Isolated Two-Body Central-Force System

Let us now consider an isolated two-body system interacting via a conservative central force. There are no other forces acting on the bodies.

In Section 1.3, we explored the concepts of force, momentum, and energy for systems for particles. We may abstract from that discussion the following facts about our isolated two-body system:

- Since no external forces act on the system, Newton's second law for systems of particles (Equation 1.17) tells us that the total linear momentum is conserved:

$$0 = \frac{d}{dt}\,\vec{P} = \frac{d}{dt}\left(m_a\dot{\vec{r}}_a + m_b\dot{\vec{r}}_b\right)$$

  Since $\vec{P}$ is constant, the velocity of the center-of-mass $\vec{R} = \vec{P}/M$ is fixed and thus the center-of-mass system is inertial. We may therefore assume, without loss of generality, that $\vec{r}_a$ and $\vec{r}_b$ are coordinates in the center of mass system, where $\vec{P}$ vanishes and the center of mass is at the origin:

$$0 = \vec{R} = m_a\vec{r}_a + m_b\vec{r}_b$$
$$0 = \vec{P} = m_a\dot{\vec{r}}_a + m_b\dot{\vec{r}}_b$$

  This eliminates three of the six degrees of freedom in the problem. The difference coordinate $\vec{r}_{ab}$ is now

$$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b = \vec{r}_a\left(1 + \frac{m_a}{m_b}\right) = -\vec{r}_b\left(1 + \frac{m_b}{m_a}\right)$$

  Defining the **reduced mass**

$$\mu \equiv \frac{m_a\,m_b}{m_a + m_b}$$

gives us

$$\vec{r}_a = \frac{\mu}{m_a}\,\vec{r}_{ab}$$
$$\vec{r}_b = -\frac{\mu}{m_b}\,\vec{r}_{ab}$$

We shall see that the dynamics of the two-particle system will be reduced to that of a single particle with mass μ moving in the potential $U(r_{ab})$. We may consider two simple limits immediately:

- In the limit $m_b \gg m_a$, we have $\mu \to m_a$, $\vec{r}_b \to 0$, and $\vec{r}_{ab} \to \vec{r}_a$. That is, the center of mass is fixed on the heavier mass and the motion is entirely of the smaller mass.
- In the limit $m_a = m_b = m$, we have $\mu = \frac{m}{2}$, $\vec{r}_a = \frac{\vec{r}_{ab}}{2}$ and $\vec{r}_b = -\frac{\vec{r}_{ab}}{2}$. In this case, the motion of the two particles is completely symmetric about the center of mass.

- Since there are no external forces, there are no external torques either. There are no internal torques because the forces are central ($\vec{r}_{ab} \times \vec{f}_{ab} = 0$ because $\vec{f}_{ab}$ points along $\vec{r}_{ab}$). Therefore, according to Equation 1.23, the total angular momentum of the system is conserved. Since the system's center-of-mass has been taken to be at rest at the origin, the angular momentum consists only of the internal angular momentum due to motion of the two particles about the center of mass. This angular momentum is

$$\vec{L} = m_a\,\vec{r}_a \times \dot{\vec{r}}_a + m_b\,\vec{r}_b \times \dot{\vec{r}}_b$$

Let's rewrite this in terms of $\vec{r}_{ab}$:

$$\vec{L} = \mu\,\vec{r}_{ab} \times \dot{\vec{r}}_a - \mu\,\vec{r}_{ab} \times \dot{\vec{r}}_b = \vec{r}_{ab} \times \mu\dot{\vec{r}}_{ab} = \vec{r}_{ab} \times \vec{p}_{ab}$$

The two-body system begins to look like a single particle of mass μ and coordinate $\vec{r}_{ab}$.

- The kinetic and potential energies of the system are

$$T = \frac{1}{2}\,m_a\,\dot{\vec{r}}_a^{\,2} + \frac{1}{2}\,m_b\,\dot{\vec{r}}_b^{\,2} = \frac{1}{2}\,\mu^2\,\dot{\vec{r}}_{ab}^{\,2}\left(\frac{1}{m_a} + \frac{1}{m_b}\right) = \frac{1}{2}\,\mu\,\dot{\vec{r}}_{ab}^{\,2}$$
$$U = U(r_{ab})$$

The Lagrangian is

$$L = \frac{1}{2}\,\mu\,\dot{\vec{r}}_{ab}^{\,2} - U(r_{ab})$$

The Lagrangian is identical to that of a single particle system with mass μ and coordinate $\vec{r}_{ab}$.

- Since $\vec{L}$ is conserved, we know the motion is restricted to the plane defined by $\vec{r}_{ab}$ and $\vec{p}_{ab}$. Let this plane define a spherical polar coordinate system $(r_{ab}, \theta_{ab}, \phi_{ab})$, where ϕ_{ab} is the azimuthal angle of the plane and θ_{ab} is the polar angle of the position vector $\vec{r}_{ab}$ relative to the z-axis in the plane. Rewriting L in this system gives

$$L = \frac{1}{2}\,\mu\left(\dot{r}_{ab}^2 + r_{ab}^2\,\dot{\theta}_{ab}^2 + r_{ab}^2 \sin^2\theta_{ab}\,\dot{\phi}_{ab}^2\right) - U(r_{ab})$$

We may choose $\phi_{ab} = 0$ without loss of generality. The angular momentum vector points out of this plane (in the y direction) and the motion remains in this plane at all time by conservation of angular momentum. However, though we know $\phi_{ab} = 0$ and $\dot{\phi}_{ab} = 0$ will therefore hold, these facts arise from the dynamics and so we should not substitute these values into the Lagrangian. They will come out of the Euler-Lagrange equations.

Dynamics of an Isolated Two-Body Central-Force System

Now, let's explore the dynamics using the Lagrangian. Conservation of $\vec{L}$ already leads us to expect that one of the Euler-Lagrange equations will be trivial. Explicitly, we have (dropping the $_{ab}$ subscripts now):

$$\begin{aligned} \mu\,\ddot{r} &= -\frac{dU}{dr} + \mu\, r\,\dot{\theta}^2 + \mu\, r\,\sin^2\theta\,\dot{\phi}^2 \\ \frac{d}{dt}\left[\mu\, r^2\dot{\theta}\right] &= \mu\, r^2 \sin\theta\,\cos\theta\,\dot{\phi}^2 \\ \frac{d}{dt}\left[\mu\, r^2 \sin^2\theta\,\dot{\phi}\right] &= 0 \end{aligned}$$

We make the general point that, when writing the Euler-Lagrange equations for a multi-dimensional system, it is a good idea to start with the time derivatives on the left side unexpanded until the rest has been simplified because they are just the time derivatives of the canonical momenta and some of them may end up being conserved if the right side of the corresponding equation vanishes, either explicitly or by appropriate choice of initial conditions. We see in the above case that the ϕ equation of motion tells us $l_\phi =$ constant, which would have become very unobvious if the derivative had been expanded into its three terms.

For initial conditions, we take $\vec{r}$ and $\vec{p}$ to be in the plane $\phi = 0$.[1] That $\vec{p}$ is in the plane $\phi = 0$ also implies $\dot{\phi} = 0$ initially. With these initial conditions, the ϕ equation of motion implies that $\phi = 0$ for all time.[2] Thus, $l_\phi = \mu\, r^2 \sin^2\theta\,\dot{\phi} = 0$ for all time. The θ equation then tells us we have $l_\theta = \mu\, r^2\,\dot{\theta} =$ constant. The angular momentum vector has length $|\vec{L}| = l_\theta$ and points perpendicular to the plane $\phi = 0$ in which the motion occurs: $\vec{L}$ is along the y axis. The r equation of motion simplifies to

$$\mu\,\ddot{r} = -\frac{dU}{dr} + \frac{l_\theta^2}{\mu\, r^3} \tag{4.1}$$

The equation of motion is now that of a single particle in one dimension with the effective potential function

$$U_{eff}(r) = U(r) + \frac{l_\theta^2}{2\,\mu\, r^2}$$

We acquire a new "centrifugal potential" that arises due to conservation of angular momentum. It is a repulsive potential, reflecting the fact that, with l_θ constant, the kinetic energy

[1] As stated above, since $\vec{r}$ and $\vec{p}$ define a plane, we are free to orient the coordinate system to make that plane the $\phi = 0$ plane.

[2] One can see this explicitly be expanding out the derivative on the left side of the ϕ equation and noting that the terms without $\ddot{\phi}$ have $\dot{\phi}$ and thus vanish at $t = 0$. So $\ddot{\phi}$ vanishes at $t = 0$, which implies that $\dot{\phi}$ does not deviate from its initial vanishing value.

must increase as r^{-2} if r decreases – more energy is needed to go to small radii. We may use the effective potential to write an effective one-dimensional Lagrangian:

$$L_{1D} = \frac{1}{2}\,\mu\,\dot{r}^2 - \frac{l_\theta^2}{2\,\mu\,r^2} - U(r)$$

Note that the 1D Lagrangian is not obtained simply by rewriting the 3D Lagrangian using $\phi = 0$, $\dot{\phi} = 0$ and $\mu\,r^2\dot{\theta} = l_\theta$; one would have gotten the wrong sign for the centrifugal term. This difficulty occurs because the Lagrangian formalism assumes independent variations of the different coordinates. Instead, we must simply start with the effective potential.

The effective total energy is

$$E = \frac{1}{2}\,\mu\,\dot{r}^2 + \frac{l_\theta^2}{2\,\mu\,r^2} + U(r)$$

which is conserved because the effective potential is conservative. This effective total energy turns out to be equal to the true total energy because the θ kinetic energy is included via the l_θ term.

Qualitative Dynamics

It is instructive to consider the shape of the effective potential energy function and its implications for the motion of the system. The effective potential consists of two terms. The first is the true central force potential. The second is a "centrifugal" term: it is a repulsive potential arising from angular momentum conservation, which requires the kinetic energy to increase as r is reduced[3]. The relative sizes of the two terms determine if and where the effective potential is attractive and repulsive. The shape of the effective potential and the total energy of the system determine whether the orbits are unbounded, bounded, or bounded and circular, and whether bounded orbits are periodic (closed). To be clear: bounded and unbounded refers to whether there is an upper limit on r or not; open and closed refer to whether the orbit repeats itself after some period. All unbounded orbits are open, but not all bounded orbits are closed.

The effective potential is:

$$U_{eff}(r) = U(r) + \frac{l_\theta^2}{2\,\mu\,r^2}$$

Consider different cases for the shape of $U(r)$. Some of these are illustrated in the Figure 4.1.

- **Repulsive Potentials**: If $U(r)$ has no attractive regions (no regions with positive slope), then both terms are repulsive at all r and all orbits are unbounded and open. This occurs, for example, for the Coulomb force between particles of like charge.
- **Small r Behavior**: The small r behavior of the effective potential determines whether r is bounded below or whether there are "small r" bounded orbits with r bounded above.
  - If $U(r)$ is attractive and either converges to a finite value at $r = 0$ or approaches negative infinity as a power law shallower than r^{-2} near the origin, then the centrifugal term dominates near the origin and r is always bounded below. This

[3] Because the angular kinetic energy is $\frac{l_\theta^2}{2\mu\,r^2}$

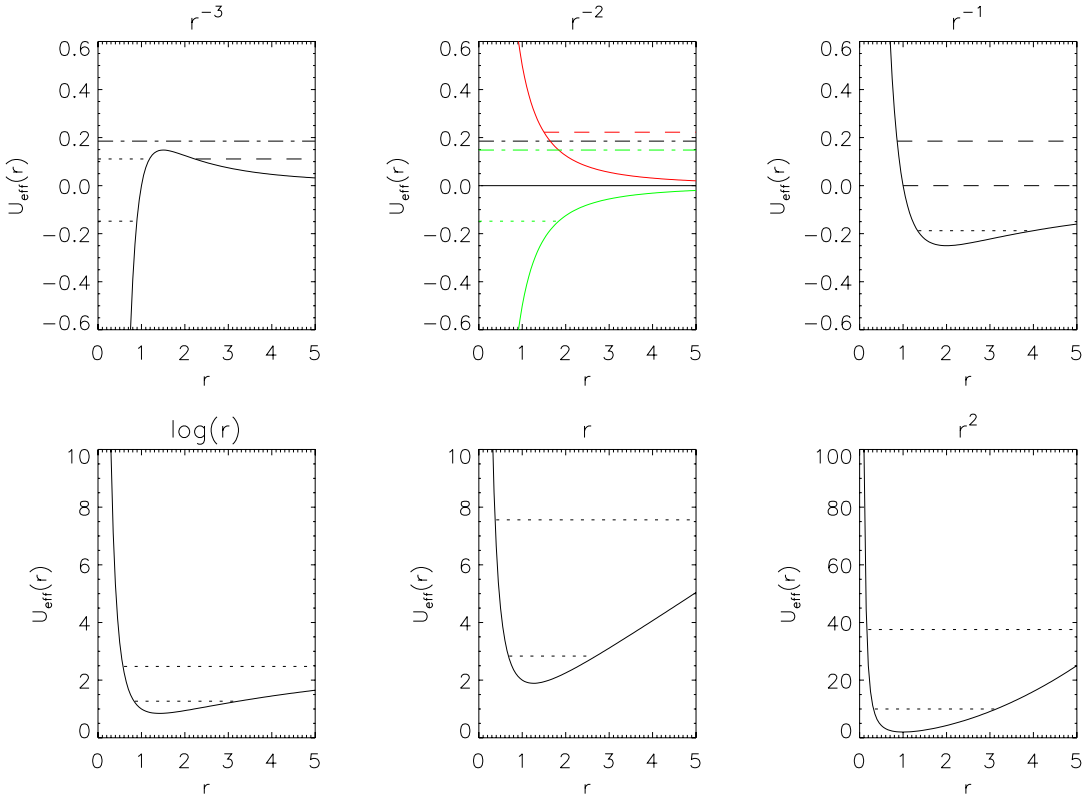


Figure 4.1: Effective potential for various true potentials. The potentials are $U(r) = r^n$ or $U(r) = \log r$ with unity coefficient except for the r^{-2} case. For all cases except r^{-2}, the behavior is qualitatively independent of the factor $\frac{l_\theta^2}{2\mu}$, so it has been set to 1. For the r^{-2} potential we show the three cases $\frac{l_\theta^2}{2\mu} = 0.5, 1, 1.5$ (green, black, red). The locus of r values filled by example bound orbits are indicated by dotted lines, by example orbits bounded below but unbounded above by dashed lines, and by completely unbounded example orbits by dash-dot lines, respectively (color-coded for the r^{-2} potential).

is illustrated for r^{-1}, r, and r^2 (with $\frac{l_\theta^2}{2\mu} = 1$) in Figure 4.1. Physically, the centrifugal barrier wins against the attractive potential near the origin. Only in the case $l_\theta = 0$ (no initial θ velocity) can r reach zero (because the centrifugal term vanishes). Whether the orbit as a whole is bounded depends completely on the large-r behavior of the potential and possibly the total energy.

– Conversely, if $U(r)$ is attractive and approaches negative infinity faster than r^{-2} near the origin, then the potential is attractive near the origin and repulsive further out. This is illustrated by the r^{-3} plot in Figure 4.1. For some initial conditions, bound orbits for r inside the maximum of the effective potential are obtained; when this occurs, the potential energy term has won over the centrifugal term. Such orbits are illustrated by the dotted lines in the the r^{-3} plot. These orbits are obtained only if the total energy E is less than the maximum value of the effective potential E_{max} and if the initial conditions place r inside the location of that maximum r_0. The maximum value of the radius for such orbits r_{max} will satisfy

$r_{max} < r_0$. The kinetic energy will go formally to infinity as the radius passes through zero, but this occurs instantaneously. When the total energy $E < E_{max}$ but initial conditions place $r > r_0$, then the orbit is bounded below (at some radius $r_{min} > r_0$) and not above. Such an orbit is indicated by the dashed line in the r^{-3} plot. If $E > E_{max}$, then the orbit is neither bounded above nor below. Such an orbit is indicated by the dash-dot line in the r^{-3} plot.

- If $U(r)$ goes as r^{-2} at small radii, then the behavior is sensitive to the relative size of the coefficient in $U(r)$ and $\frac{l_\theta^2}{2\mu}$. The plot above shows the three possible cases: the cases $\frac{l_\theta^2}{2\mu}$ less than, equal to, and greater than the coefficient of the power law give attractive, vanishing, and repulsive effective potentials as $r \to 0$. The orbits will be bounded below in the repulsive case only (dashed line). The orbits can be bounded above in the attractive case only, and one obtains a bounded orbit only for initial conditions with $E < 0$ (dotted line). For the attractive or vanishing potential, orbits with $E \geq 0$ are completely unbounded (dash-dot line).

- **Large r Behavior**: The large r behavior of the effective potential determines whether r is bounded above at large r, which is a necessary condition for bounded orbits.[4] Whether the orbit is closed is another question – see below.

  - If $U(r)$ is attractive but a steeper power law than r^{-2} at large r, then the centrifugal term dominates at large r and the overall potential is repulsive there. If the system's initial position is in this repulsive region, then r is bounded below and unbounded above; the system as a whole is unbounded, the orbits are open. This is illustrated by the dashed line in the r^{-3} power law case. In such cases, the centrifugal term is always too large for the potential to overcome it. If $E > E_{max}$, then the centrifugal barrier can be overcome and the orbit is bounded neither above nor below (dash-dot line).
  - The r^{-2} power law case is again sensitive to the relative value of the coeffficient in $U(r)$ and $\frac{l_\theta^2}{2\mu}$, with the effective potential being repulsive when the centrifugal term coefficient is larger. In the repulsive case, r is bounded below, unbounded above, and the orbits are open (dashed line). In the vanishing or negative effective potential case, r is unbounded below and above for $E \geq 0$ and the orbits are open (dash-dot lines).
  - For power laws between r^{-2} and r^0, the potential energy term always dominates at large r and the centrifugal barrier becomes negligible. But the potential energy approaches zero from below as $r \to \infty$, so r is bounded above and bound orbits are obtained only if $E < 0$; r is unbounded above (but always bounded below) and open orbits are found otherwise.
  - For $U(r) \propto \log r$ and positive exponent power laws, the potential energy not only wins at large r but goes to ∞; hence r is always bounded above and all orbits are bounded.

- Circular orbits are obtained when there is a point in the effective potential where the gradient (effective force) vanishes. For potentials steeper than r^{-2}, there is one

[4] As we discussed above, for sufficiently strongly attractive potentials (steeper than r^{-2} at the origin), orbits can be bounded above at small r. Since such potentials remain steeper than r^{-2} for all r, there are no orbits bounded at large r for such potentials. Boundedness at large r is only a consideration for attractive potentials whose large r behavior is shallower than r^{-2}.

unstable circular orbit. For potentials shallower than r^{-2} and for log or positive power law potentials, there is one stable circular orbit.

- Of course, if the true potential is a more complicated function of r than simple power law or log functions, there may be additional bound orbits and circular orbits.
- Whether an orbit is periodic (= closed = the orbit repeats on itself after some time) is a nontrivial problem. Clearly, all bound orbits show periodic motion in r as a function of time. But periodicity in space – periodicity of $r(\theta)$ – is a more complicated problem. All circular orbits are periodic in space because r is constant. More generally, periodicity of $r(\theta)$ requires that the time periods for radial motion and angular motion be **commensurate** – their ratio must be a rational fraction. It is not obvious what the relation is between the form of the potential and whether this condition is satisfied. **Bertrand's theorem** tells us that bound orbits are closed only if the potential in the vicinity of the bound orbits follows r^{-1} or r^2. Details on Bertrand's theorem can be found in Goldstein, Section 3.6.

When orbits are bounded, the two turning points are called the **apsides** or **apsidal distances**. When orbits are not closed, the apsides precess in (r, θ) space. The angle between two consecutive apsides is called the **apsidal angle**. Precession of the apsides occurs whenever the conditions of Bertrand's theorem are not satisfied, including small perturbations of Bertrand's theorem potentials by non-Bertrand's theorem terms.

4.1.2 Formal Implications of the Equations of Motion

Kepler's Second Law

The equation of motion for the θ coordinate gives us angular momentum conservation. One implication of angular momentum conservation is Kepler's second law, which states that "Equal areas are swept out in equal times." In a small time interval dt, the vector $\vec{r}$ sweeps out an area

$$dA = \frac{1}{2}\, r\, (r\, d\theta) = \frac{1}{2}\, r^2 \dot{\theta}\, dt = \frac{1}{2}\, \frac{l_\theta}{\mu}\, dt \tag{4.2}$$

The first form of the expression calculates the area of the infinitesimal triangle of height r and base $r\, d\theta$. Finally, since l_θ is constant, we have that dA/dt is constant and Kepler's second law is proven.

Note that Kepler's second law holds for any central force, not just gravity. It is a result of angular momentum conservation.

The Formal Solution to the Equations of Motion

We have obtained two equations of motion

$$\begin{aligned} \mu\, \ddot{r} &= -\frac{d}{dr}\, U_{eff}(r, l_\theta) \\ \frac{d}{dt}\left[\mu\, r^2 \dot{\theta}\right] &= 0 \end{aligned}$$

Let's attempt to integrate these two equations. Integrating the r equation must obviously yield an equation for $\dot{r}$. But we already know what that equation will be, by energy

conservation:

$$\dot{r} = \pm \left[\frac{2}{\mu} \left(E - U(r) \right) - \frac{l_\theta^2}{\mu^2 \, r^2} \right]^{1/2}$$

Integration of the θ equation gives

$$\mu \, r^2 \dot{\theta} = l_\theta$$

We can obviously integrate each of these equations again, at least formally:

$$t = \pm \int_{r(0)}^{r} dr' \left[\frac{2}{\mu} \left(E - U(r) \right) - \frac{l_\theta^2}{\mu^2 \, r'^2} \right]^{-1/2} \tag{4.3}$$

$$\theta - \theta(0) = \frac{l_\theta}{\mu} \int_0^t \frac{dt'}{\left[r(t') \right]^2} \tag{4.4}$$

Depending on the form of $U(r)$, it may in some cases be possible to perform the first integral explicitly, which may then allow the second integral to be done.

We can eliminate t to obtain a relation between θ and r; just return to the differential version of the first equation and substitute for dt in the second to obtain

$$\theta(r) - \theta(0) = \pm l_\theta \int_{r(0)}^{r} \frac{dr'}{\left[r(t) \right]^2} \left[2 \, \mu \left(E - U(r) \right) - \frac{l_\theta^2}{r'^2} \right]^{-1/2} \tag{4.5}$$

The above function is in general not analytically integrable. For $U(r) \propto r^n$, the integral becomes a standard elliptic integral for some values of n. For $n = 2$, -1, and -2, the integral yields sinusoidal functions.

A Differential Relation between r and θ

While we have formally eliminated t and obtained an integral relation between θ and r, the fact that the integral is not in general analytic limits its usefulness. It may be more useful to have a differential, rather than integral, relation between r and θ. We need to eliminate $\frac{d}{dt}$ from our original differential equations. The θ equations tells us

$$\begin{aligned} dt &= \frac{\mu \, r^2}{l_\theta} \, d\theta \\ \frac{d}{dt} &= \frac{l_\theta}{\mu \, r^2} \frac{d}{d\theta} \end{aligned}$$

Our equation of motion for r can thus be rewritten:

$$\begin{aligned} \mu \, \frac{l_\theta}{\mu \, r^2} \frac{d}{d\theta} \left[\frac{l_\theta}{\mu \, r^2} \frac{dr}{d\theta} \right] - \frac{l_\theta^2}{\mu \, r^3} &= F(r) \\ \frac{d}{d\theta} \left[\frac{1}{r^2} \frac{dr}{d\theta} \right] - \frac{1}{r} &= \frac{\mu \, r^2}{l_\theta^2} \, F(r) \\ -\frac{d}{d\theta} \left[\frac{d}{d\theta} \frac{1}{r} \right] - \frac{1}{r} &= \frac{\mu \, r^2}{l_\theta^2} \, F(r) \\ \frac{d^2}{d\theta^2} \left(\frac{1}{r} \right) + \frac{1}{r} &= -\frac{\mu \, r^2}{l_\theta^2} \, F(r) \end{aligned} \tag{4.6}$$

We now have a differential equation with θ as the independent variable and r as the dependent variable. This equation may be useful for obtaining the shapes of the orbits, and would have obvious applications for numerical calculation of the orbits. This equation is frequently written via a change of variables to $u = \frac{1}{r}$ in the form

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l_\theta^2\, u^2}\, F\left(\frac{1}{u}\right)$$

The constant of the motion, the energy, can be rewritten in terms of u and θ alone (*i.e.*, eliminate explicit time derivatives):

$$\begin{aligned} E &= \frac{1}{2}\,\mu\,\dot{r}^2 + \frac{l_\theta^2}{2\,\mu\,r^2} + U(r) \\ &= \frac{l_\theta^2}{2\,\mu}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{l_\theta^2}{2\,\mu\,r^2} + U(r) \\ E &= \frac{l_\theta^2}{2\,\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + U\left(\frac{1}{u}\right) \end{aligned} \tag{4.7}$$

4.2 The Special Case of Gravity – The Kepler Problem

We now specialize to gravity, which allows us to fix the form of the central-force potential energy function. We solve the equation of motion and study the various solutions. We use essentially the same techniques as Hand and Finch Section 4.5, 4.6, and 4.7 to obtain the solutions, though in a different order. All the notation is consistent, with the equivalency $k = G\,\mu\,M$ and the replacement of ϕ in Hand and Finch by θ here.

4.2.1 The Shape of Solutions of the Kepler Problem

The General Solution

In our generic study of central forces, we obtained the differential relation Equation 4.6

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} \;=\; -\frac{\mu\, r^2}{l_\theta^2}\, F(r)$$

For the gravitational force, we have $F(r) = -G\,\frac{m_a\, m_b}{r^2}$, so the above reduces to

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} \;=\; \frac{G\,\mu^2\,(m_a + m_b)}{l_\theta^2}$$

Rewriting using $u = \frac{1}{r}$, we have

$$\frac{d^2u}{d\theta^2} + u \;=\; \frac{G\,\mu^2 M}{l_\theta^2}$$

where $M = m_a + m_b = m_a\, m_b/\mu$. This is now a simple harmonic oscillator equation with a constant driving force. We know from our study of oscillations that the solution is the sum of the generic solution to the homogeneous equation and a particular solution to the inhomogeneous equation:

$$u(\theta) = A\cos(\theta - \theta_0) + \frac{G\,\mu^2 M}{l_\theta^2}$$

We can relate the coefficient A in the solution to the constants of the motion, the energy and angular momentum, by using Equation 4.7:

$$\begin{aligned}
E &= \frac{l_\theta^2}{2\mu}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + U\left(\frac{1}{u}\right) \\
&= \frac{l_\theta^2}{2\mu}\left[A^2\sin^2(\theta-\theta_0) + A^2\cos^2(\theta-\theta_0) + 2\,A\,\cos(\theta-\theta_0)\,\frac{G\,\mu^2\,M}{l_\theta^2} + \left(\frac{G\,\mu^2 M}{l_\theta^2}\right)^2\right] \\
&\quad -G\,\mu\,M\left(A\cos(\theta-\theta_0) + \frac{G\,\mu^2\,M}{l_\theta^2}\right) \\
&= \frac{l_\theta^2\,A^2}{2\mu} + A\,G\,\mu\,M\,\cos(\theta-\theta_0) + \frac{G^2\mu^3M^2}{2\,l_\theta^2} - G\,\mu\,M\,A\cos(\theta-\theta_0) - \frac{G^2\mu^3M^2}{l_\theta^2} \\
&= \frac{l_\theta^2\,A^2}{2\,\mu} - \frac{G^2\mu^3M^2}{2\,l_\theta^2} \\
&= \frac{l_\theta^2}{2\,\mu}\left[A^2 - \left(\frac{G\,\mu^2 M}{l_\theta^2}\right)^2\right]
\end{aligned}$$

Let us rewrite the orbit in terms of r instead of u, and also drop the offset phase θ_0 (it is not important for what follows):

$$\frac{p}{r} = 1 + \epsilon \cos\theta \qquad \text{with} \qquad p = \frac{l_\theta^2}{G\,\mu^2 M} \qquad \epsilon = p\,A$$
$$\implies \quad p = r + \epsilon\, r\, \cos\theta \tag{4.8}$$

We take ϵ to be nonnegative – a sign flip in ϵ can be removed by a coordinate system rotation.[5] We will see below the significance of p – it is the radius at which the effective potential is minimized, and it defines the energy scale of the solutions. If we rewrite in cartesian coordinates, with $x = r\,\cos\theta$ and $y = r\,\sin\theta$, we have

$$p = \sqrt{x^2 + y^2} + \epsilon\, x$$
$$(p - \epsilon\, x)^2 = x^2 + y^2$$
$$\left(1 - \epsilon^2\right) x^2 + 2\,\epsilon\, p\, x + y^2 - p^2 = 0$$

In terms of p and ϵ, the total energy is

$$E = \frac{l_\theta^2}{2\,\mu}\left[\frac{\epsilon^2}{p^2} - \frac{1}{p^2}\right] \tag{4.9}$$
$$= \frac{G^2\,\mu^3\,M^2}{2\,l_\theta^2}\left(\epsilon^2 - 1\right)$$
$$= \frac{G\,\mu\,M}{2\,p}\left(\epsilon^2 - 1\right) \tag{4.10}$$

Let's rewrite in a more obvious form: complete the square on x to obtain

$$\left(1 - \epsilon^2\right)\left(x + \frac{\epsilon\, p}{1 - \epsilon^2}\right) + y^2 = \frac{p^2}{1 - \epsilon^2}$$
$$\frac{\left(x - x_c\right)^2}{a^2} \pm \frac{y^2}{b^2} = 1 \tag{4.11}$$

which is the equation for a conic section with

$$x_c = -\frac{\epsilon\, p}{1 - \epsilon^2} \qquad a = \frac{p}{1 - \epsilon^2} \qquad b = \frac{p}{\sqrt{\pm\left(1 - \epsilon^2\right)}} \qquad f = x_c \pm \epsilon\, a = 0 \text{ and } 2\,x_c \tag{4.12}$$

where f denotes the x coordinates of the foci of the conic section. Recall that the center of mass of the system is at the origin, so one of the foci coincides with the center of mass. The $\pm$ sign is picked depending on the sign of $1 - \epsilon^2$ to ensure that b is real. The turning points of the motion are given by the maximum and minimum values of r. Our polar form for the orbit, Equation 4.8, provides the easiest means to obtain these: they are at $\cos\theta = \pm 1$. They are therefore

$$r_1 = \frac{p}{1 + \epsilon} \qquad r_2 = \frac{p}{1 - \epsilon}$$
$$x_1 = \frac{p}{1 + \epsilon} \qquad x_2 = -\frac{p}{1 - \epsilon}$$
$$y_1 = 0 \qquad y_2 = 0$$

[5] Because $-\cos(\theta) = \cos(\pi - \theta) = \cos(\theta - \pi)$, a sign flip in ϵ is equivalent to rotating the coordinate system by π in the θ direction.

where $x_{1,2} = r_{1,2}\cos\theta$, so x picks up a sign, and $y_{1,2} = r_{1,2}\sin\theta$, so y vanishes in both cases. The energy is now

$$E = \frac{G\mu M}{2p}\left(\epsilon^2 - 1\right) = -\frac{G\mu M}{2a} \tag{4.13}$$

The sign and magnitude of the energy thus scales inversely as the semimajor axis a.

So, what we have is the equation of a conic section, being a circle, ellipse, or hyperbola depending on the sign of $1-\epsilon^2$. Recall that we started out with the center of mass at the origin. The "center" of the conic section $(x_c, 0)$ is therefore displaced from the center of mass unless $\epsilon = 0$ (circular solution), but the center of mass coincides with one of the foci of the conic section. The energy takes on the sign of $1-\epsilon^2$. From our qualitative discussion of central force orbits, we know that $E = 0$ is the dividing line between bound and unbound orbits. The implication then is that bound orbits with $E < 0$ have positive a and $\epsilon^2 < 1$, while unbound orbits with $E > 0$ have negative a and $\epsilon^2 > 1$. The dividing case is $E = 0$, $a = \infty$, $\epsilon = 1$. The conic section formula is formally undefined there, but if we go back to before we completed the square, we see that $\epsilon^2 = 1$ causes the x^2 term to vanish leaving us with the equation of a parabola (x a quadratic function of y).

Detailed Study of the Different Solutions

Let's study these various solutions in some detail. First, let's obtain a dimensionless parameterization of the solutions. The shape of the effective potential is set by l_θ and μ. The effective potential is minimized when the effective force vanishes. Using Equation 4.1, we obtain

$$\frac{G\mu M}{r^2} = \frac{l_\theta^2}{\mu r^3} \quad \Longrightarrow \quad r = \frac{l_\theta^2}{G\mu^2 M} = p$$

The value of the effective potential at this point, which gives the minimum physically allowed value of the total energy, is

$$\begin{aligned} E_{min} = U_{eff}(r=p) &= \frac{l_\theta^2}{2\mu p^2} - \frac{G\mu M}{p} = -\frac{l_\theta^2}{2\mu p^2} \\ &= -\frac{1}{2}\frac{G\mu M}{p} \equiv -E_{scale} \end{aligned}$$

where we have defined a scale energy that is the absolute value of the minimum energy. Referring back to our equation for E in terms of ϵ and p, Equation 4.10, we see that

$$E = E_{scale}\left(\epsilon^2 - 1\right) \tag{4.14}$$

With E_{min} and E_{scale} in hand, let's consider the various cases. Examples are illustrated in Figure 4.2.

- $E/E_{scale} < -1$: not physically allowed
- $E/E_{scale} = -1$: Equation 4.14 tells us that $E = -E_{scale}$ corresponds to $\epsilon^2 = 0$. Since the eccentricity vanishes, the solution from Equation 4.8 is $p = r$ for all θ; *i.e.*, the orbit is a circle. This is as one would expect from the effective potential – there is no radial force if the solution is at the minimum of the effective potential. The conic section solution is elliptical (because $\epsilon^2 < 1$) and the semimajor axes are equal $a = b = p$ as one would expect for a circle.

- $-1 < E/E_{scale} < 0$: Because the energy remains negative, Equation 4.14 implies that $0 < \epsilon^2 < 1$ and the conic section solution is an ellipse. As the energy increases, the eccentricity of the ellipse increases. Remember that the center of mass coincides with one of the foci of the ellipse. The center of the ellipse x_c and the second focus $2\,x_c$ move off to $-\infty$ as $\epsilon \to 1$.
- $E/E_{scale} = 0$: For this solution, Equation 4.14 tells us $\epsilon^2 = 1$. Our derivation of the conic section form fails here because the coefficient of the x^2 term vanishes, but we can return to the Cartesian form of the solution to find

$$\begin{aligned} 2\,p\,x + y^2 - p^2 &= 0 \\ x &= \frac{p}{2} - \frac{y^2}{2\,p} \end{aligned}$$

  This is a parabola whose vertex is at $\frac{p}{2}$, whose focus is at the origin, and whose directrix is at p. Recall that the directrix is the line perpendicular to the axis of the parabola such that the parabola consists of the set of points equidistant from the focus and the directrix. The system is just barely not bound, with the radial kinetic energy and velocity approaching zero as $r \to \infty$ because the total energy vanishes and the effective potential energy approaches zero as $r \to \infty$.
- $E/E_{scale} > 0$: For this solution, Equation 4.14 gives $\epsilon^2 > 1$. The conic section is a hyperbola. A hyperbola has two branches. Because the polar form of the solution, Equation 4.8, implies a one-to-one relationship between r and θ, only one branch of the hyperbola can be a valid solution. Intuitively, based on continuously transforming the eccentricity, we expect this to be the left branch. We can see this explicitly as follows. The left and right branches are distinguished by the fact that the former has regions with $x < 0$, while the latter does not. In order to have negative values of x, θ must be allowed to go outside the range $(-\pi/2, +\pi/2)$. A restriction on θ is placed by the requirement that r be positive, which translates to the requirement $\cos\theta \geq -\frac{1}{\epsilon}$. Since $\epsilon^2 > 1$ (and ϵ is taken to always be positive), this defines a maximum value of $|\theta|$ that is between $\pi/2$ and π. Hence, x is allowed to be negative, and so the left branch solution is the appropriate one. For the hyperbolic solution, there is only one turning point, which is the x_1 turning point thanks to our choice of the left branch. Let's consider the evolution of the solution with ϵ. One focus of the hyperbola always remains at the origin. The "center" of the hyperbola, x_c, starts out at $+\infty$ and moves in toward the origin as ϵ gets larger, with $x_c \to 0$ in the limit $\epsilon \to \infty$. Thus, the hyperbola continuously transforms from a parabolic-like orbit to a straight vertical line, with the turning point x_1 moving closer to the origin as ϵ increases. These solutions are definitely not bound. They in fact have excess kinetic energy so that, as $r \to \infty$, the radial kinetic energy (and hence radial velocity) remains nonzero.

A Note on Repulsive Potentials

While we have so far considered only attractive potentials, it is straightforward to translate the above solution to the case of repulsive potentials. We will see that, for a given energy E, we obtain the same hyperbola as for the attractive potential, but we must choose the right branch, not the left branch. Hand and Finch obtain the repulsive potential solutions by working the derivation through again with the necessary minus sign. But we can obtain the solution more easily.

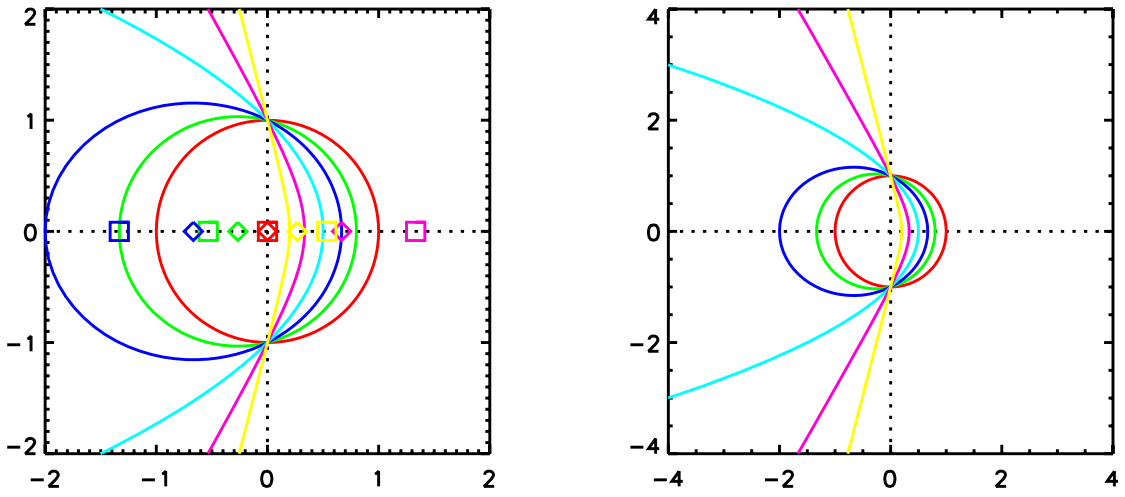


Figure 4.2: Example Keplerian orbits. The left and right figures have identical orbits; only the axis range is different. All these orbits have $\frac{l_\theta^2}{2\mu} = 1$ and $p = 1$, so they have the same angular momentum (same centrifugal barrier) but different total energies. The scale factor for the energy $\frac{l_\theta^2}{2\,\mu\,p}$ is therefore also 1, so $E_{scale} = 1$ and the various orbits have energy $E = \epsilon^2 - 1$. The legend is, in order of increasing eccentricity: $\epsilon = 0$ (red), $\epsilon = 0.25$ (green), $\epsilon = 0.5$ (blue), $\epsilon = 1$ (cyan), $\epsilon = 2$ (magenta), $\epsilon = 4$ (yellow). The center of each orbit (x_c) is shown by the diamond of the same color, and the second focus by the squares. The first focus of all orbits is at the origin. The second branch of the hyperbolic orbits is not shown.

The repulsive potential solution can be obtained from the attractive potential solution by simply taking $G\,\mu\,M < 0$ and making use of the physical fact that only $E > 0$ is allowed for a repulsive potential. Let's step through the derivation with these changes. First, the solution for u becomes

$$u(\theta) = A\cos\theta - \frac{|G\,\mu^2 M|}{l_\theta^2}$$

Since $u \geq 0$ is required, the solution must have $A > 0$ and is only valid for some range of θ. Keeping our original definition of p (which now implies $p < 0$), our polar form of the solution is

$$\frac{p}{r} = 1 + \epsilon\,\cos\theta$$

Since $p < 0$ and $A > 0$, we have $\epsilon = p\,A < 0$ also. Since $p < 0$, the solution is valid only when the right side is **less than** zero. (Originally, our requirement was the right side be greater than zero because both p and r were positive.) The region of validity is given by $\cos\theta \geq -\frac{1}{\epsilon} = \frac{1}{|\epsilon|}$. For there to be any region of validity, we must have $|\epsilon| > 1$, which implies only hyperbolic solutions will be allowed. Furthermore, the region of validity has $\cos\theta > 0$, so we must use the right branch of the hyperbolic solution we find.

The conversion from polar to cartesian coordinates goes as before, as does the completion of the square. So the hyperbolic solution is still valid, as are the formulae for x_c, a, b, and f. x_c has the same sign as in the attractive hyperbolic case (since the sign flips in ϵ and p

cancel each other), a and b change sign ($a > 0$, $b < 0$ for all repulsive solutions). Because x_c does not change sign, the foci are in the same place as the attractive hyperbolic case. The sign flips in a and b do not affect the shape of the hyperbola since only a^2 and b^2 enter in the conic section formula. Thus, we have the exact same hyperbola as in the attractive case, except that the restriction on θ implies we must now take the right branch, not the left branch. This also means that the turning point is now x_2, not x_1.

The energy expressions, Equation 4.10 and 4.13, hold without change. The starting point for the energy equation, Equation 4.9, is insensitive to the sign of ϵ and p. Equation 4.10 does not change meaning because the sign flips in $G\,\mu\,M$ and p cancel each other, so $|\epsilon| > 1$ still gives $E > 0$ for all repulsive potential solutions. Equation 4.13 also keeps its same sign because both $G\,\mu\,M$ and a change sign.

Intuitively, the change from the left branch to the right branch reflects the fact that an attractive potential turns the trajectory inward toward the center of force while a repulsive potential turns it outward.

Summary of Quantities

The various quantities involved in Keplerian orbits are summarized in Table 4.1.

4.2.2 Time Dependence of the Kepler Problem Solutions

So far we have only found the orbit solutions as functions $r(\theta)$. This of course describes much of the dynamics of the problem. But one does indeed frequently want the orbit as a function of time, so we obtain that result here.

Period of Elliptical Orbits

We can quickly obtain the period of elliptical orbits by using Kepler's second law, Equation 4.2. Kepler's second law tells us

$$\frac{dA}{dt} = \frac{1}{2}\frac{l_\theta}{\mu}$$

The area of the ellipse is $A = \pi\,a\,b$, so the period is the time required to sweep out the area of the ellipse,

$$\tau = \frac{A}{\frac{dA}{dt}} = \frac{\pi\,a\,b}{\frac{1}{2}\frac{l_\theta}{\mu}}$$

Let's write this in terms of the parameters of the orbit p and ϵ to obtain Kepler's third law:

$$\begin{aligned}\tau &= 2\,\pi\,\mu\,\frac{p^2}{(1-\epsilon^2)^{3/2}}\frac{1}{\mu\sqrt{G\,M\,p}} \\ &= 2\,\pi\sqrt{\frac{a^3}{G\,M}}\end{aligned}$$

The period depends only on a. Of course, a encodes information about the total energy E and the angular momentum l_θ. The implication of Kepler's third law for the solar system is that all orbits should lie on a single $\tau^2 \propto a^3$ curve because M, dominated by the sun, is almost the same for all planets.

quantity	symbol	formula(e)	sign	significance
angular momentum	l_θ	$\mu r^2\dot{\theta}$	≥ 0 $= 0$ gives trivial orbit	centrifugal potential (brings in effect of θ motion)
scale energy	E_{scale}	$\frac{1}{2}\frac{G\mu M}{p}$	> 0	scale energy $= \lvert E_{min}\rvert$ for attractive pot.
scale radius	p	$\frac{l_\theta^2}{G\mu^2 M}$	> 0 for attr. pot. < 0 for repul. pot.	sets scale of orbit
eccentricity	ϵ	$\sqrt{1+\frac{E}{E_{scale}}}$ $-\sqrt{1+\frac{E}{E_{scale}}}$	≥ 0 attr. pot. < -1 repul. pot.	sets shape of conic section, related to ratio of energy to scale energy
orbit center	x_c	$-\frac{\epsilon p}{1-\epsilon^2}$ $= -\epsilon a$	$= 0$ circular < 0 elliptical > 0 hyperbolic	
semimajor axis	a	$\frac{p}{1-\epsilon^2}$	> 0 circ./ellip. < 0 hyperbolic attr. > 0 hyperbolic repul.	distance from x_c to vertices along major axis
semiminor axis	b	$\frac{p}{\sqrt{\pm(1-\epsilon^2)}}$	> 0 attr. pot. < 0 repul. pot.	distance from x_c to vertices along minor axis (circ./ellip.) helps set asymptotic slope of trajectory (hyperbol.)
turning points	x_1 x_2	$\frac{p}{1+\epsilon}$ $-\frac{p}{1-\epsilon}$	> 0 < 0 circ./ellip. > 0 hyperbolic	turning points of motion relative to CM = focus 1 apsides for circ./ellip. orbits
For hyperbolic orbits, x_1 is the turning point for attractive potentials, x_2 the turning point for repulsive potentials.				

Table 4.1: Parameters for Keplerian orbits.

Full Time Dependence of Elliptical Orbits

Begin by defining a circle whose origin is the center of the elliptical orbit and whose radius is the semimajor axis length a. This circle circumscribes the elliptical orbit. The azimuthal angle describing points on the circle (measured around the center of the ellipse, not the origin) is denoted by ε and is called the **eccentric anomaly**. This is by way of comparison to the standard azimuthal angle θ at the origin, which is called the **true anomaly**. The circle is described by the curve

$$\frac{(x - x_c)^2}{a^2} + \frac{y^2}{a^2} = 1$$

Consider two points, one on the ellipse and one on the circle, that have the same x coordinate. See Figure 4.13 of Hand and Finch for a visual explanation. The values of the true anomaly and the eccentric anomaly for the two points can be related by the fact that they have the same x coordinate:

$$r(\theta)\cos\theta = x = x_c + a\,\cos\varepsilon = -a\,\epsilon + a\,\cos\varepsilon$$

where we have made use of the facts that distance from a focus to x_c is $a\epsilon$ and that one focus coincides with the origin. We can reduce this, making use of the polar form Equation 4.8 and the definition of a in terms of p and ϵ, Equation 4.12:

$$\begin{aligned} r\cos\theta &= -a\,(\epsilon - \cos\varepsilon) \\ \frac{1}{\epsilon}\,(p - r) &= -a\,(\epsilon - \cos\varepsilon) \\ r &= p + a\epsilon^2 - a\,\epsilon\,\cos\varepsilon \\ r &= a\,(1 - \epsilon\cos\varepsilon) \end{aligned} \tag{4.15}$$

Now, let's return to our integral form relating t to r, Equation 4.3, and change variables from r to ε:

$$\begin{aligned} t &= \int_{r(0)}^{r} dr' \left[\frac{2}{\mu}\,(E - U(r)) - \frac{l_\theta^2}{\mu^2\,r'^2}\right]^{-1/2} \\ &= \int_{r(0)}^{r} dr' \left[\frac{2}{\mu}\left(-\frac{G\,\mu\,M}{2\,a} + \frac{G\,\mu\,M}{r'}\right) - \frac{p\,G\,M}{r'^2}\right]^{-1/2} \\ &= \sqrt{2\,G\,M}\int_{r(0)}^{r} r'\,dr' \left[-\frac{r'^2}{2\,a} + r' - \frac{1}{2}\,p\right]^{-1/2} \\ &= \sqrt{2\,G\,M}\int_0^{\varepsilon} \frac{a\,(1 - \epsilon\,\cos\varepsilon')\,a\,\epsilon\sin\varepsilon' d\varepsilon'}{\left[-\frac{1}{2}\,a\,(1 - \epsilon\cos\varepsilon')^2 + a\,(1 - \epsilon\cos\varepsilon') - \frac{1}{2}\,a\,(1 - \epsilon^2)\right]^{1/2}} \\ &= \sqrt{2\,G\,M}\int_0^{\varepsilon} \frac{a\,(1 - \epsilon\,\cos\varepsilon')\,a\,\epsilon\sin\varepsilon' d\varepsilon'}{\left[\frac{a}{2}\,\epsilon^2\sin^2\varepsilon'\right]^{1/2}} \\ &= \sqrt{G\,M\,a^3}\int_0^{\varepsilon} (1 - \epsilon\cos\varepsilon')\,d\varepsilon' \\ t(\varepsilon) &= \sqrt{G\,M\,a^3}\,(\varepsilon - \epsilon\sin\varepsilon) \end{aligned} \tag{4.16}$$

This is **Kepler's equation**, a transcendental equation for $\varepsilon(t)$. From $\varepsilon(t)$, we can obtain $r(\varepsilon(t))$ from Equation 4.15 and $\theta(\varepsilon(t))$ from the polar form Equation 4.8. It's still not a closed-form solution, but better than the original integral equation.

For completeness, we note that the x and y coordinates can be written in terms of a and ε as

$$x = a\,(\cos\varepsilon - \epsilon) \qquad y = a\sqrt{1-\epsilon^2}\,\sin\varepsilon$$

The x equation is found from the original mapping between θ and ε, and y from $x^2+y^2=r^2$.

For realistic use of the above, one needs to know a, ϵ, and M. M is a parameter that is assumed to be known by other means (*e.g.*, in the solar system, M is dominated by the sun, which can be determined from orbits of other objects). It's hard to directly know a and ϵ unless one has traced out a fair part of the orbit. But one can get them indirectly by measuring r at two times t_1 and t_2; Equations 4.15 and 4.16 can be used to convert from $r(t_1)$ and $r(t_2)$ to a, ϵ, and $\varepsilon(t_1)$ and $\varepsilon(t_2)$.

Full Time Dependence of Hyperbolic Orbits

Here, instead of mapping from ellipse to a circle with $a\cos\varepsilon$ and $a\sin\varepsilon$, we map from a hyperbola using hyperbolic trigonometric functions $a\cosh\varepsilon$ and $a\sinh\varepsilon$. There is no obvious geometric interpretation of ε anymore, as it must satisfy $\varepsilon = 0$ for the point of closest approach and $\varepsilon \to \infty$ as $r \to \infty$. It is just a parameter. Our starting relation becomes

$$r(\theta)\cos\theta = x_c \mp |a|\,\cosh\varepsilon = |a\,\epsilon| \mp |a|\,\cosh\varepsilon = -a\,\epsilon + a\,\cosh\varepsilon$$

where the top sign is for an attractive potential. Remember that a and ϵ flip sign (a goes from negative to positive, ϵ from positive to negative) when the potential changes from attractive to repulsive. Following all the signs through, the important relations become

$$\begin{aligned}
r &= |a|\,(|\epsilon|\cosh\varepsilon \mp 1) = -a\,(\epsilon\cosh\varepsilon - 1) \\
t(\varepsilon) &= \sqrt{G\,M\,a^3}\,(|\epsilon|\sinh\varepsilon \mp \varepsilon) = \sqrt{G\,M\,a}\,a\,\,(-\epsilon\sinh\varepsilon + \varepsilon) \\
x &= |a\,\epsilon| \mp |a|\,\cosh\varepsilon = -a\,(\epsilon - \cosh\varepsilon) \\
y &= |a|\sqrt{\epsilon^2-1}\,\sinh\varepsilon = |b|\,\sinh\varepsilon
\end{aligned}$$

where the upper signs are for an attractive orbit and the lower signs for a repulsive one.

4.3 Scattering Cross Sections

Earlier, we studied the kinematics of elastic scattering of point particles (Section 1.3.3). In that section, we did not make any reference to the forces involved in scattering, we only considered what conclusions could be drawn from conservation laws. Thus, we were left with parameterized results, without any knowledge of how a given outcome (final state momentum vector or scattering angle) depended on the initial conditions.

Now that we have studied central force motion, obtaining qualitative results for arbitrary potentials and specific results for $\frac{1}{r}$ potentials, we have information about the dynamics of collisional interactions. We can use this to develop the concept of scattering cross section, which intimately uses the kinematics of collisions and the orbit information from central force motion.

The archetypal scattering problem we will consider is one involving a particle incident on a force center that scatters the incident particle via a conservative central force with potential $U(r)$. We have demonstrated that a two-particle system interacting via a conservative central force is equivalent to such a system when considered in its center-of-mass frame.

4.3.1 Setting up the Problem

Initial Conditions

As noted, we consider a particle incident on a force center subject to a scattering potential. It is assumed that the particle is asymptotically free, having enough energy to be in an unbound orbit. In order to have unbound orbits, the potential energy must go to zero at large r. The energy of the system is therefore

$$E = \frac{1}{2}\,\mu\, v_\infty^2$$

where v_∞ is the asymptotic particle speed as $r \to \infty$. We will use v_∞ to parameterize the initial energy of the system.

v_∞ only specifies an energy. We must also specify the geometry. The only free parameter left is l_θ, which we can see specifies the geometry as follows. Since the system is unbound, the trajectory must become straight lines at large radii, reflecting the incoming and outgoing velocity vectors. For example, for a $\frac{1}{r}$ potential, we know the ingoing and outgoing velocity vectors define the asymptotes of the hyperbolic orbit. We can calculate the distance between the scattering center and these straight lines where they come closest to the scattering center. This distance is defined to be the **impact parameter**, usually associated with the symbol b. This is displayed in Figure 4.3. The green line is the impact parameter. The impact parameter is related to l_θ. We know $l_\theta = \vec{r} \times \mu\,\dot{\vec{r}}$. At $r = \infty$, we know $\dot{\vec{r}}$ points in along the asymptote and $\vec{r}$ points to the origin, so the angle between $\vec{r}$ and $\dot{\vec{r}}$, which we call γ, is subtended by the impact parameter b. $\vec{r}$ is shown approximately in Figure 4.3 by the blue line. So,

$$l_\theta = r\,\mu\, v_\infty \sin\gamma = \mu\, v_\infty\, b$$

l_θ specifies the geometry through b, which fixes the position of the asymptote ($\vec{v}_\infty$) relative to the scattering center.

Scattering Angle

In Figure 4.3, the scattering angle, θ_*, is the angle between the incoming and outgoing velocity vectors. θ_* is determined by E and l_θ, or equivalently v_∞ and b, and the form of the potential function.

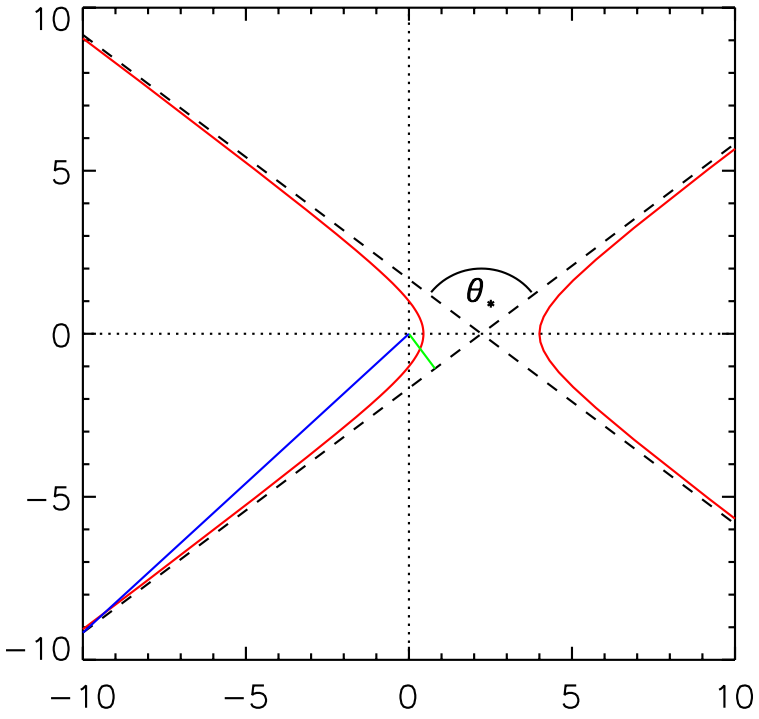


Figure 4.3: Scattering impact parameter illustration

4.3.2 The Generic Cross Section

Incident Beam

Now that we have made our definitions, let us bring in the concept of differential cross section. Suppose we have a beam of incoming particles, all with same velocity $\vec{v}_\infty$. Let the flux of particles F be the number of particles passing through a unit area in a unit time. If the beam has particle number density n, then the flux is

$$F = nv_\infty$$

Assume that the beam has a circular cross section and the axis of the beam points directly at the scattering center. The incoming particles will thus have a range of impact parameter values, ranging from $b = 0$ (along the beam axis) to $b = b_{max}$ (at the outer edge of the beam). The beam radius is b_{max} and its cross-sectional area is $A = \pi\, b_{max}^2$.

Differential Cross Section

The incident particles in the beam will be scattered into a range of angles depending on their input impact parameters (and the beam velocity). We define the **differential scattering cross section**, $\frac{d\sigma}{d\Omega}(\theta_*, \phi_*)$, via the probability of an incident particle being scattered into the solid angle $d\Omega$ in the direction (θ_*, ϕ_*) where θ_* is the polar angle measured from the beam axis and ϕ_* is the azimuthal angle around the beam axis. If $dN(\theta_*, \phi_*)$ is the number of particles per unit time scattered into the solid angle $d\Omega$ at (θ_*, ϕ_*), we define the differential

cross section via the relation

$$\frac{1}{A}\,\frac{d\sigma}{d\Omega}(\theta_*,\phi_*)\,d\Omega = \frac{dN(\theta_*,\phi_*)}{F\,A}$$

Let us explain the above. On the right side of first line, the denominator is the number of particles per unit time incident on the target from a beam of flux F and cross-sectional area A. Since the numerator is the number of particles per unit time that scatter into $d\Omega$ at (θ_*,ϕ_*), the right side is thus the fraction of particles that scatter into $d\Omega$ at (θ_*,ϕ_*); it is a probability. We include additional factors so that $\frac{d\sigma}{d\Omega}$ is defined only by the scattering force, not by parameters of the experiment. We have a $1/F$ on the right side but none on the left side because including the $1/F$ makes the ratio dN/F independent of F: if F goes up, dN goes up proportionally. However, we include a $1/A$ on the left side to cancel the $1/A$ on the right side because $dN(\theta_*,\phi_*)$ may not scale with A: if one adds cross-sectional area at a radius from which particles do not scatter into the particular solid angle $d\Omega$ at (θ_*,ϕ_*), $dN(\theta_*,\phi_*)$ will not increase when A increases. Hence the different treatment of F and A. Solving for the differential cross section gives

$$\frac{d\sigma}{d\Omega}(\theta_*,\phi_*) = \frac{1}{F}\,\frac{dN(\theta_*,\phi_*)}{d\Omega}$$

The beam area A has dropped out. $\frac{d\sigma}{d\Omega}$ has units of area per steradian; hence the name *cross section*. If we assume central force scattering, the problem is azimuthally symmetric about the beam axis and we may integrate over ϕ_* so that $d\Omega = 2\,\pi\,\sin\theta_* d\theta_*$. Furthermore, we know for central force scattering that there is a one-to-one correspondence between b and θ_* for a given v_∞. Therefore, the particles scattering into the interval $d\theta_*$ in polar angle come from some range db in impact parameter. Azimuthal symmetry of the problem lets us integrate over azimuthal angle in the beam also. We may relate db and $d\theta_*$ by requiring conservation of particle number:

$$F\,2\,\pi\,b\,db = -F\,\frac{d\sigma}{d\Omega}\,2\,\pi\,\sin\theta_* d\theta_*$$

A negative sign has been inserted under the assumption that the potential decreases in strength monotonically with radius: if you increase the impact parameter a little bit, the scattering angle should decrease, so a positive db implies a negative $d\theta_*$. Rewriting, we have

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_*}\left|\frac{db}{d\theta_*}\right|$$

That is, if we know the function $b(\theta_*, v_\infty)$, then we can determine the distribution of particles in scattering angle given a uniform incoming beam.

Total Cross Section

Once one has calculated $\frac{d\sigma}{d\Omega}$, it is formally a straightforward thing to calculate the **total cross section**:

$$\sigma = \int d\Omega\,\frac{d\sigma}{d\Omega} = 2\,\pi\int_0^\pi d\theta_*\sin\theta_*\frac{d\sigma}{d\Omega} = 2\,\pi\int_0^\infty db\,b$$

As one would expect, the total cross section is related to the probability that an incoming particle will be scattered to any angle. It can be viewed as a total "effective area" of the scattering center; the number of particles in the incoming beam that will be scattered is the same as if every particle within the central σ of the beam were scattered and all others left untouched.

Calculating $b(\theta_*)$

For a generic central potential, we can obtain a formula relating b and θ_* by returning to the integral relations that define the orbit. Recall Equation 4.5:

$$\theta(r) - \theta(0) = \pm l_\theta \int_{r(0)}^{r} \frac{dr'}{[r]^2} \left[2\,\mu\,(E - U(r)) - \frac{l_\theta^2}{r'^2} \right]^{-1/2}$$

θ is not θ_*, but it is related to it. Referring back to our scattering picture above, the scattering angle θ_* is the complement of the angle between the incoming and outgoing asymptotes. The angle between the asymptotes is related to the orbit angle as $r \to \infty$. For repulsive scattering, the angle between the asymptotes is just $|\theta_{out} - \theta_{in}|$, where these are the asymptotic orbit angles for the incoming and outgoing particles. For attractive scattering, since θ is measured from the $+x$-axis, the angle between the asymptotes is $2\,\pi - |\theta_{out} - \theta_{in}|$. So the scattering angle is

$$\theta_* = \mp \left(\pi - |\theta_{out} - \theta_{in}| \right)$$

where $-$ goes with the attractive potential. Next, $|\theta_{out} - \theta_{in}|$ is twice the angle between θ_{out} or θ_{in} and $\theta = 0$. $\theta = 0$ is obtained when $r = r_{min}$, the turning point. So, we have

$$\begin{aligned} \theta_* &= \mp \left(\pi - 2 \left| l_\theta \int_{r_{min}}^{\infty} \frac{dr'}{[r'(t)]^2} \left[2\,\mu\,(E - U(r)) - \frac{l_\theta^2}{r'^2} \right]^{-1/2} \right| \right) \\ &= \mp \left(\pi - 2 \left| \int_{r_{min}}^{\infty} \frac{b\,dr'}{r'^2} \left[1 - \frac{U(r)}{E} - \frac{b^2}{r'^2} \right]^{-1/2} \right| \right) \end{aligned}$$

We have used E instead of v_∞ to indicate the initial energy. Formally, now, we have $\theta_*(b)$, which we know is formally invertible because there is one-to-one relationship between b and θ_*. Thus, we can obtain the differential cross section for any scattering potential.

4.3.3 $\frac{1}{r}$ Potentials

For the $\frac{1}{r}$ potential, we can find the differential cross section explicitly because we have explicit relationships between θ and r.

Finding $b(\theta_*)$

We demonstrated earlier that the azimuthal angle θ of the orbit relative to the center of mass is limited to be $\cos\theta > -\frac{1}{\epsilon}$; this gives us θ_{out} and θ_{in}:

$$\theta_{out,in} = \pm \arccos\left(-\frac{1}{\epsilon} \right)$$

where $\theta_{out} = -\theta_{in}$ and the choice of sign for θ_{in} depends on initial conditions. So we have

$$\begin{aligned} |\theta_{out} - \theta_{in}| &= 2 \left| \arccos\left(-\frac{1}{\epsilon} \right) \right| \\ &= 2 \arccos\left(\mp \frac{1}{|\epsilon|} \right) \end{aligned}$$

where the $-$ is for an attractive potential. So we have

$$\theta_* = \mp\left[\pi - 2\arccos\left(\mp\frac{1}{|\epsilon|}\right)\right] = \pi - 2\arccos\frac{1}{|\epsilon|}\frac{1}{|\epsilon|} = \sin\frac{\theta_*}{2}$$

where the two different potentials have yielded the same result. With some work, we can write b in terms of ϵ. Start with Equation 4.9:

$$\begin{aligned} E &= \frac{l_\theta^2}{2\,\mu\,p^2}\left(\epsilon^2-1\right) \\ \frac{\mu\,p}{l_\theta} &= \sqrt{\frac{\mu}{2E}}\sqrt{\epsilon^2-1} \\ \frac{l_\theta}{G\,\mu\,M} &= \sqrt{\frac{\mu}{2E}}\sqrt{\epsilon^2-1} \\ \mu\,v_\infty\,b &= \frac{G\,\mu\,M}{\sqrt{2E/\mu}}\sqrt{\epsilon^2-1} \\ b &= \frac{G\,\mu\,M}{2E}\sqrt{\epsilon^2-1} \end{aligned}$$

So, we may now find $b(\theta_*)$:

$$b = \frac{G\,\mu\,M}{2E}\sqrt{\csc^2\frac{\theta_*}{2}-1} = \frac{G\,\mu\,M}{2E}\cot\frac{\theta_*}{2}$$

where we take the positive square root because $0 < \frac{\theta_*}{2} < \frac{\pi}{2}$.

Calculating the Differential Cross Section

We will need $\frac{db}{d\theta_*}$ to calculate the differential cross section, so, taking the derivative:

$$\left|\frac{db}{d\theta_*}\right| = \frac{1}{2}\frac{G\,\mu\,M}{2E}\csc^2\frac{\theta_*}{2} = \frac{b}{2}\csc\frac{\theta_*}{2}\sec\frac{\theta_*}{2}$$

Finally, using our formula for the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_*}\left|\frac{db}{d\theta_*}\right| = \frac{b^2}{4}\csc^2\frac{\theta_*}{2}\sec^2\frac{\theta_*}{2} = \left(\frac{G\,\mu\,M}{4E}\right)^2\frac{1}{\sin^4\frac{\theta_*}{2}}$$

Note that the result is independent of whether the scattering is attractive or repulsive. This result is the well known Rutherford scattering formula, first observed in the scattering of α particles (^{4}He nuclei) from gold nuclei in a foil. The observation that the scattering obeyed the simple expectation of a $\frac{1}{r}$ potential was an important piece of evidence that the atom consists of a charged nucleus surrounded mostly by empty space, as opposed to a "plum pudding" type model with electrons and protons intermixed uniformly in the atom.

The Total Cross Section

If one tries to calculate the total cross section from the Rutherford formula, one will end up with an infinite result. This is because, for a $\frac{1}{r}$ potential, the probability of scattering does not decrease sufficiently quickly with increasing b – the effective area of the scattering center is infinite. If the potential is made to converge more quickly (*e.g.*, by multiplying be an exponential decay), then a finite total cross section is obtained.

Chapter 5

Rotating Systems

We have not yet seriously studied dynamics of rotating systems. The subject breaks down into two topics: dynamics in rotating (and hence non-inertial) coordinate systems, and dynamics of rotating bodies in inertial coordinate systems. We must develop the theory of rotations first, which provides a language to use for discussing these two topics.

5.1 The Mathematical Description of Rotations

We develop the theory of rotations, progressing from infinitesimal rotations to finite rotations, in particular considering the group-theoretic aspects. Some of this material is found in Hand and Finch Chapters 7 and 8, but much is not.

We want to understand how rotation of the coordinate system affects the position vector $\vec{r}$. Such rotations will be of interest to us in two physical situations:

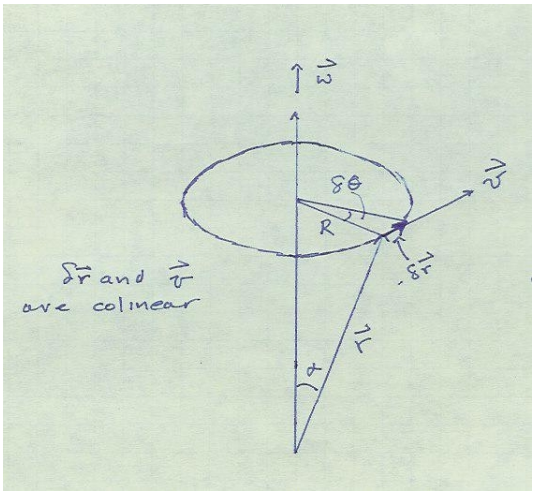
- Apparent motion in a rotating coordinate system of a point that is fixed relative to an inertial coordinate system.
- Rotation of a rigid body about some axis. Recall that a rigid body is a body for which the relative distance between any pair of points is fixed. Such a body may still rotate about any axis. A point P in the body follows rotational motion about the axis of motion relative to the non-rotating system.

We will consider the physical systems in the sections following this one. This section focuses on the mathematical language and general properties of rotational transformations.

5.1.1 Infinitesimal Rotations

Vector Cross-Product Version

Consider the following figure:



$\vec{r}$ points in an arbitrary direction in the non-rotating frame. The vector $\vec{r}$ is rotated by an angle $\delta\theta$ about the z axis. We are allowed to take the rotation axis to coincide with the z axis because $\vec{r}$ is arbitrary. If we define a vector $\delta\vec{\theta}$ that points along the axis of rotation (the z axis) and has magnitude $\delta\theta$, then the change $\delta\vec{r}$ in $\vec{r}$ is related to $\delta\vec{\theta}$ by

$$\delta\vec{r} \;=\; \delta\vec{\theta}\times\vec{r}$$

where $\times$ indicates a vector cross-product, and the rotated vector in the non-rotating system is

$$\vec{r}\,' \;=\; \vec{r}+\delta\vec{r}$$

The cross-product gives the correct direction for the displacement $\delta\vec{r}$ (perpendicular to the axis and $\vec{r}$) and the correct amplitude ($|\delta\vec{r}| = R\,\delta\theta = r\,\delta\theta\sin\alpha$). If we then divide by the time δt required to make this displacement, we have

$$\frac{\delta\vec{r}}{\delta t} = \frac{\delta\vec{\theta}}{\delta t} \times \vec{r} \Longrightarrow \dot{\vec{r}} = \vec{\omega} \times \vec{r}$$

where ω is the angular frequency of rotation about the axis and $\vec{\omega}$ points along the axis of rotation also.

Matrix Version

A more generic and therefore more useful way to look at a rotation is as a matrix operation on vectors. The infinitesimal rotation can be viewed as a matrix operation:

$$\vec{r}\,' = \vec{r} + \delta\theta\,\hat{z} \times \vec{r} = \begin{pmatrix} x - y\,\delta\theta \\ y + x\,\delta\theta \\ z \end{pmatrix} \equiv \mathbf{R}_{\delta\vec{\theta}}\,\vec{r}$$

with

$$\mathbf{R}_{\delta\vec{\theta}} = \begin{pmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1} + \delta\theta\,\mathbf{M}_z \qquad \mathbf{M}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where we have defined the infinitesimal rotation matrix $\mathbf{R}_{\delta\vec{\theta}}$ and the matrix $\mathbf{M}_z$. More generally, one can show that an infinitesimal rotation about an arbitrary axis can be written in matrix form using

$$\mathbf{R}_{\delta\vec{\theta}} = \mathbf{1} + \left(\delta\vec{\theta}\cdot\hat{x}\,\mathbf{M}_x + \delta\vec{\theta}\cdot\hat{y}\,\mathbf{M}_y + \delta\vec{\theta}\cdot\hat{z}\,\mathbf{M}_z\right) \equiv \mathbf{1} + \delta\vec{\theta}\cdot\vec{\mathbf{M}}$$

with

$$\mathbf{M}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{M}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathbf{M}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\vec{\mathbf{M}}$ is called the infinitesimal rotation **generator** because, obviously, it can be used to generate any infinitesimal rotation matrix $\mathbf{R}_{\delta\vec{\theta}}$ when combined with the rotation vector $\delta\vec{\theta}$. A simple way to write the generators is

$$(\mathbf{M}_i)_{jk} = -\epsilon_{ijk}$$

where ϵ_{ijk} is the completely antisymmetric Levi-Civita symbol (see Appendix A). It is useful to know that the above matrices satisfy the following relations:

$$\mathbf{M}_x^2 = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{M}_y^2 = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{M}_z^2 = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i \equiv [\mathbf{M}_i, \mathbf{M}_j] = \epsilon_{ijk}\,\mathbf{M}_k$$

Combining Infinitesimal Rotations

It is straightforward to see how to combine two infinitesimal rotations, even if they are not about the same axis. Clearly, our formula $\delta\vec{r} = \delta\vec{\theta} \times \vec{r}$ holds regardless of the orientation of $\delta\vec{\theta}$. So we obtain the result of two successive *infinitesimal* rotations $\delta\vec{\theta}_1$ and $\delta\vec{\theta}_2$ by

$$\begin{aligned} \vec{r}_2' &= \vec{r}_1' + \delta\vec{\theta}_2 \times \vec{r}_1' \\ &= \vec{r} + \delta\vec{\theta}_1 \times \vec{r} + \delta\vec{\theta}_2 \times (\vec{r} + \delta\vec{\theta}_1 \times \vec{r}) \\ &= \vec{r} + \left(\delta\vec{\theta}_1 + \delta\vec{\theta}_2\right) \times \vec{r} \end{aligned}$$

where in the last line we have dropped terms quadratic in the infinitesimal rotation angles. Thus, we see that the effect of two infinitesimal rotations $\delta\vec{\theta}_1$ and $\delta\vec{\theta}_2$ is found simply by obtaining the result of a rotation by the sum rotation vector $\delta\vec{\theta}_1 + \delta\vec{\theta}_2$. Obviously, if $\delta\vec{\theta}_1$ and $\delta\vec{\theta}_2$ are aligned, then the angles sum simply. But if the two rotation axes are not aligned, addition of the angle vectors describes the correct way to combine the rotations. In terms of matrices, we have

$$\begin{aligned} \vec{r}_2' &= \left(\mathbf{1} + \delta\vec{\theta}_2 \cdot \vec{\mathbf{M}}\right) \vec{r}_1' \\ &= \left(\mathbf{1} + \delta\vec{\theta}_2 \cdot \vec{\mathbf{M}}\right) \left(\mathbf{1} + \delta\vec{\theta}_1 \cdot \vec{\mathbf{M}}\right) \vec{r} \\ &= \left(\mathbf{1} + \left[\delta\vec{\theta}_1 + \delta\vec{\theta}_2\right] \cdot \vec{\mathbf{M}}\right) \vec{r} \end{aligned}$$

The implication of the addition rule for rotations is that angular velocities add in simple fashion also:

$$\vec{\omega}_{tot} = \vec{\omega}_1 + \vec{\omega}_2$$

NOTE: The simple addition rules for infinitesimal rotations and angular velocities do **not** in general hold for finite rotations, discussed in the next section.

5.1.2 Finite Rotations

There are two different ways of determining the appropriate form for finite rotations – integration of infinitesimal transformations and direct evaluation via direction cosines.

Integration of Infinitesimal Transformations

First, consider a finite rotation about a single axis, $\vec{\theta}$. That rotation can be built up as a product of infinitesimal rotations:

$$\mathbf{R}_{\vec{\theta}} = \lim_{N\to\infty} \left(\mathbf{1} + \frac{1}{N}\,\vec{\theta} \cdot \vec{\mathbf{M}}\right)^N$$

where $\delta\vec{\theta} = \frac{1}{N}\,\vec{\theta}$, which is infinitesimal in the limit $N \to \infty$. This can be rewritten (expand out the infinite product, or just realize that the above is one of the definitions of the exponential function):

$$\mathbf{R}_{\vec{\theta}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\vec{\theta} \cdot \vec{\mathbf{M}}\right)^n = \exp\left(\vec{\theta} \cdot \vec{\mathbf{M}}\right)$$

The second equality is not much use since an exponential with a matrix argument only has meaning as a power series expansion. If we now specialize to a rotation about the z axis, we get $\vec{\theta} \cdot \vec{\mathbf{M}} = \theta\, \mathbf{M}_z$. Using the relation above for $\mathbf{M}_z^2$, we can rewrite the sum as

$$\begin{aligned}
\mathbf{R} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \left[\frac{\theta^{2n}\,(-1)^n}{(2n)!} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\theta^{2n+1}\,(-1)^n}{(2n+1)!} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{M}_z \right] \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + (\cos\theta + \sin\theta\, \mathbf{M}_z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Of course, similar forms hold if the rotation is aligned with the x or y axis. For more general rotations, the parameterization in terms of the components of $\vec{\theta}$ is not particularly clear; we will develop a simpler parameterization, **Euler angles**, later.

Direction Cosines

Consider our rotation of the vector $\vec{r}$ to be a rotation of a frame F in which $\vec{r}$ is fixed relative to a nonrotating frame F'. In Hand and Finch, these are termed the "body" and "space" coordinate systems, respectively. The original vector has components (x, y, z) in the nonrotating frame. The transformation $\vec{r}\,' = \mathbf{R}\,\vec{r}$ gives the components of the vector in the nonrotating F' frame after the rotation; denote these components by (x', y', z'). In the F frame, the vector retains its original components (x, y, z) after rotation because the F frame rotates with the vector. Thus, the rotation matrix $\mathbf{R}$ provides a linear relationship between the components (x, y, z) and (x', y', z'). There are 9 obvious coefficients of this relationship, the 9 components of $\mathbf{R}$. One way to parameterize these 9 components is to consider how they relate to the angles between the F' coordinate axes $\hat{x}'$, $\hat{y}'$, and $\hat{z}'$ and the F coordinate axes $\hat{x}$, $\hat{y}$, and $\hat{z}$. It turns out that, in fact, $\mathbf{R}$ is the matrix of these dot products:

$$\mathbf{R} = \begin{pmatrix} \hat{x}' \cdot \hat{x} & \hat{x}' \cdot \hat{y} & \hat{x}' \cdot \hat{z} \\ \hat{y}' \cdot \hat{x} & \hat{y}' \cdot \hat{y} & \hat{y}' \cdot \hat{z} \\ \hat{z}' \cdot \hat{x} & \hat{z}' \cdot \hat{y} & \hat{z}' \cdot \hat{z} \end{pmatrix}$$

That this form works can be seen by applying it to the unit vectors of the F frame; we get back the unit vectors of the F frame written in terms of the unit vectors of the F' frame. Since rotation is a linear operation,

$$\mathbf{R}\left(\alpha\,\vec{a} + \beta\,\vec{b}\right) = \alpha\,\mathbf{R}\,\vec{a} + \beta\,\mathbf{R}\,\vec{b}$$

the same matrix is valid for rotating an arbitrary vector $\vec{r}$. In the case of a simple rotation about the z axis, the above direction cosine form is obviously consistent with the matrix operator form given above. The two forms must be consistent in general since the linearity of the relationship between $\vec{r}$ and $\vec{r}\,'$ allows there to be only one such rotation matrix.

Orthogonality of Finite Rotation Matrices

Coordinate rotation matrices are very much like the rotation matrices we obtained for transforming between generalized and normal coordinates in the coupled oscillation problem (Section 3.2). In particular, rotation matrices must be **orthogonal** matrices ($\mathbf{R}^T = \mathbf{R}^{-1}$) because they must be **norm-preserving**. It is intuitively obvious that rotation of a vector about an axis must preserve its length, but it can be seen to be explicitly true by considering an infinitesimal rotation:

$$\left|\vec{r}\,'\right|^2 = (\vec{r} + \delta\vec{r}) \cdot (\vec{r} + \delta\vec{r}) = |\vec{r}|^2 + 2\,\delta\vec{r} \cdot \vec{r} + |\delta\vec{r}|^2 = |\vec{r}|^2$$

where the second term vanishes because $\delta\vec{r}$ is normal to $\vec{r}$ and the third term has been dropped because it is of second order in $\delta\vec{r}$. If infinitesimal rotations preserve length, then so do finite rotations. If we require this condition to be true, we obtain

$$\vec{r}^{\,T}\vec{r} = \vec{r}\,'^{\,T}\vec{r}\,' = (\mathbf{R}\,\vec{r})^T\,(\mathbf{R}\,\vec{r}) = \vec{r}^{\,T}\,\mathbf{R}^T\,\mathbf{R}\,\vec{r} \quad \Longrightarrow \quad \mathbf{R}^T\mathbf{R} = \mathbf{1} \quad \Longrightarrow \quad \mathbf{R}^T = \mathbf{R}^{-1}$$

(We have used the ^{T} notation for displaying dot products of vectors.) Hence, we obtain the orthogonality condition.

Orthogonality implies that the columns of $\mathbf{R}$, treated as vectors, are in fact orthonormalized:

$$\delta_{ij} = \left(\mathbf{R}^T\mathbf{R}\right)_{ij} = \sum_k R_{ki}R_{kj}$$

(We refer to the components of $\mathbf{R}$ by R_{ab}, dropping the boldface notation because each component is just a number.) There are 6 conditions implied by the above, leaving only 3 independent components of $\mathbf{R}$. As we will see later, these 3 degrees of freedom can be parameterized in terms of the **Euler angles**.

One can easily see that the "transposed" relation also holds. Norm preservation also implies

$$\vec{r}\,'^{\,T}\vec{r}\,' = \vec{r}^{\,T}\vec{r} = \left(\mathbf{R}^T\vec{r}\,'\right)^T\left(\mathbf{R}^T\vec{r}\,'\right) = \vec{r}\,'^{\,T}\mathbf{R}\,\mathbf{R}^T\vec{r}\,' \quad \Longrightarrow \quad \mathbf{R}\,\mathbf{R}^T = \mathbf{1}$$

That is, $\mathbf{R}^T$ is the "right inverse" of $\mathbf{R}$ also.[1] Written out in components, we thus have

$$\delta_{ij} = \left(\mathbf{R}\mathbf{R}^T\right)_{ij} = \sum_k R_{ik}R_{jk}$$

The rows of $\mathbf{R}$ also are othonormal.

5.1.3 Interpretation of Rotations

We at this point should comment on the two different possible physical interpretations of the rotation matrices $\mathbf{R}$. We made a similar distinction when considering coordinate transformations in Section 2.1.10.

Active transformations

In an active transformation, we think of the transformation as actively rotating the particle whose position is given by the vector $\vec{r}(t)$ relative to the coordinate axes. The rotations of the form $\vec{\omega} \times \vec{r}$ that we began with are really of that form. The coordinate system that the rotation is relative to is inertial, the coordinate system in which the rotating vector is fixed is noninertial.

[1] We could have used the fact that, for square matrices, left inverses and right inverses are always the same. But it's nice to see it directly.

Passive transformation

We think of a passive transformation as simply a relabeling of points in space according to a new coordinate system. In this picture, the coordinate system in which the particle is at rest is inertial and the rotating system is not.

Why there can be confusion

The difficulty arises because the transformations relating the two systems are mathematically identical in the two cases, but the physical interpretation is very different. We use the mathematical equivalence to relate one type of transformation to another to allow us to write down the rules. But, the definition of which system is inertial differs between the cases, so we must be very careful.

5.1.4 Scalars, Vectors, and Tensors

We have so far discussed rotations only in terms of their effect on vectors, objects that we have an intuitive feel for. But the link between rotations and vectors is much deeper, leading us to the generic concept of **tensors**.

Vectors and Vector Coordinate Representations

As hinted at in the discussion of the interpretation of rotations, there are multiple concepts described by the same mathematics. As indicated above, we tend to think of a rotation as actively rotating a vector from one position in physical space to another. The rotating coordinate system is noninertial and thus physics is modified there.

But, regardless of which coordinate system is inertial, we are frequently required to transform vectors from one coordinate system to another. This is the passive sense of a rotation transformation. The vector is not undergoing any dynamics, it is simply being looked at in a different frame. The vector, as a physics entity, does not change by looking at it in a different coordinate system. But its **coordinate representation** – the set of numbers that give the components of the vector along the coordinate axes – does change. Rotation matrices thus not only provide the means to dynamically rotate a vector about some axis, but they also provide a method to obtain the coordinate representation in one frame from that in another frame.

It is important to recognize the difference between a vector and its coordinate representation: a vector is a physical entity, independent of any coordinate system, while its coordinate representations let us "write down" the vector in a particular coordinate system. To make this clear, we will use the standard vector notation $\vec{r}$ to denote the "coordinate-independent" vector and underlined notation $\underline{\vec{r}}$ and $\underline{\vec{r}}'$ to denote its coordinate representations in the F and F' coordinate systems.

Formal Definitions of Scalars, Vectors, and Tensors

We have so far relied on some sort of unstated, intuitive definition of what a vector is. We may, on the other hand, make use of the properties of the coordinate representations of vectors to **define** what is meant by the term "vector." A vector $\vec{v}$ is defined to be an object with coordinate representations in different frames that are related by our orthogonal

rotation matrices:

$$\begin{aligned} \underline{\vec{v}}' &= \mathbf{R}\,\underline{\vec{v}} \\ \left(\underline{\vec{v}}'\right)_i &= \sum_j (\mathbf{R})_{ij}\,(\underline{\vec{v}})_j \end{aligned}$$

where in the second line we have written out the relation between coordinate representations in the two frames component-by-component. One might worry that the above definition of a vector is circular because rotation matrices are to some extent defined in terms of vectors. This worry can be eliminated by taking the direction cosine definition of rotation matrices – that definition rests only on the idea of coordinate systems and direction cosines.

We will use as a convenient shorthand the following:

$$v_i' = R_{ij}\,v_j$$

Let us explain the shorthand carefully. First, repeated indices are defined to imply summation (the Einstein summation convention), so $\sum$ symbols are not given explicitly. Second, as noted earlier, the quantity R_{ij} refers to the ij component of the matrix $\mathbf{R}$. Finally, the quantity v_j is the jth component of the coordinate representation $\underline{\vec{v}}$, $v_j = (\underline{\vec{v}})_j$. Similarly, $v_i' = (\underline{\vec{v}}')_i$. That is, the vector object is referred to with an arrow as $\vec{a}$ while the coordinate representations are underlined with an arrow, $\underline{\vec{a}}$ and $\underline{\vec{a}}'$, and the components of the representations are referred to using subscripts but no underlines or arrows, a_i and a_i'.[2]

We have an intuitive definition of a **scalar** as an object whose coordinate representation is independent of frame. Explicitly, the transformation property of a scalar s with coordinate representations $\underline{s}$ and $\underline{s}'$ is

$$\underline{s}' = \underline{s}$$

for any pair of frames. Since the representation of s is independent of frame, there is no distinction between s, $\underline{s}$, and $\underline{s}'$.

We can see, for example, that the norm of the position vector is a scalar by relating its values in two different frames:

$$\underline{\vec{r}}' \cdot \underline{\vec{r}}' = r_i'\,r_i' = R_{ij}\,r_j\,R_{ik}\,r_k = \delta_{jk}\,r_j\,r_k = r_j\,r_j = \underline{\vec{r}} \cdot \underline{\vec{r}}$$

where we have used $\underline{\vec{r}} = \mathbf{R}^T\,\underline{\vec{r}}'$ and the orthonormality property $R_{ij}\,R_{ik} = \delta_{jk}$. In general, the dot product $\vec{a} \cdot \vec{b}$ of two vectors is a scalar by the same argument.

The generalization of scalars and vectors is **tensors**, or, more specifically, **rank n tensors**. A rank n tensor $\mathcal{T}$ is an object that has coordinate representation $\underline{\mathcal{T}}$ with N^n components $T_{i_1 \cdots i_n}$ (where N is the dimensionality of the physical space, $N = 3$ for what we have so far considered) with transformation properties

$$T'_{i_1 \cdots i_n} = R_{i_1 j_1} \cdots R_{i_n j_n} T_{j_1 \cdots j_n}$$

A vector is a rank 1 tensor and a scalar is a rank 0 tensor. A rank 2 tensor has coordinate representations that look like square $N \times N$ matrices; what distinguishes a rank 2 tensor

[2] We note that there is no need for underlines for the coordinate representation a_i because a subscript without an arrow implies consideration of a coordinate representation.

from a simple matrix is the relation between the coordinate representations in different frames. It is important to remember the distinction!

An alternate, but equivalent, definition of a rank n tensor is an object whose product with n vectors is a scalar for any n vectors. That is, if we claim $\mathcal{T}$ is a rank n tensor with representation $T_{i_1\cdots i_n}$ in a particular frame, then, for n arbitrary vectors $\{\vec{a}_i\}$ with components $\{a_{i,j_i}\}$ in that frame, if we calculate in each frame the quantity

$$\underline{s} = T_{i_1\cdots i_n}\, a_{1,i_1}\cdots a_{n,i_n}$$

it is required that s be the same in every frame regardless of the choice of the $\{\vec{a}_i\}$. We can see the equivalence of the two definitions by equating the component representations of s calculated in different frames:

$$\begin{aligned} \underline{s}' &= \underline{s} \\ T'_{i_1\cdots i_n}\, a'_{1,i_1}\cdots a'_{n,i_n} &= T_{p_1\cdots p_n}\, a_{1,p_1}\cdots a_{n,p_n} \\ T'_{i_1\cdots i_n}\, R_{i_1 j_1}\, a_{1,j_1}\cdots R_{i_n j_n}\, a_{n,j_n} &= T_{p_1\cdots p_n}\, a_{1,p_1}\cdots a_{n,p_n} \end{aligned}$$

Since the above must hold for all possible choices of the $\{\vec{a}_i\}$, we may conclude

$$\begin{aligned} T'_{i_1\cdots i_n}\, R_{i_1 j_1}\cdots R_{i_n j_n} &= T_{j_1\cdots j_n} \\ R_{k_1 j_1}\cdots R_{k_n j_n}\, T'_{i_1\cdots i_n}\, R_{i_1 j_1}\cdots R_{i_n j_n} &= R_{k_1 j_1}\cdots R_{k_n j_n}\, T_{j_1\cdots j_n} \\ T'_{i_1\cdots i_n}\, \delta_{k_1 i_1}\cdots\delta_{k_n i_n} &= R_{k_1 j_1}\cdots R_{k_n j_n}\, T_{j_1\cdots j_n} \\ T'_{k_1\cdots k_n} &= R_{k_1 j_1}\cdots R_{k_n j_n}\, T_{j_1\cdots j_n} \end{aligned}$$

where we have used the "transposed" orthonormality condition $R_{kj}\, R_{ij} = \delta_{ki}$. Thus, we recover our other definition of a rank n tensor.

Rank 2 tensors are special because their coordinate representations $\underline{\mathcal{T}}$ look like simple $N\times N$ matrices. In particular, the transformation of a rank 2 tensor has a simple matrix form:

$$\begin{aligned} T'_{ij} &= R_{ik}\, R_{jl}\, T_{kl} = R_{ik}\, T_{kl}\, R_{lj} \\ \underline{\mathcal{T}}' &= \mathbf{R}\,\underline{\mathcal{T}}\,\mathbf{R}^T = \mathbf{R}\,\underline{\mathcal{T}}\,\mathbf{R}^{-1} \end{aligned}$$

where $\underline{\mathcal{T}}$ and $\underline{\mathcal{T}}'$ are $N\times N$ matrices. The last expression is the **similarity transformation** of the $N\times N$ matrix $\underline{\mathcal{T}}$ by the orthogonal matrix $\mathbf{R}$.

Examples of Tensors

- One obvious rank 2 tensor is the outer product of two vectors:

  $$T_{ij} = a_i\, b_j \quad \text{or} \quad \mathcal{T} = \vec{a}\,\vec{b}^T$$

  Since each vector transforms as a rank 1 tensor, it is obvious that the above product transforms as a rank 2 tensor.

- More generally, if we take a rank m tensor with coordinate representation components $U_{i_1\cdots i_m}$ and a rank n tensor with coordinate representation components $V_{j_1\cdots j_n}$ and **contract** over – *i.e.*, match up indices and sum, the generalization of a dot product – any p pairs of indices, then the resulting set of quantities is a rank $m+n-2p$ tensor. Proving it is clearly a tedious exercise in index arithmetic relying on the rotation matrix orthogonality relation $R_{ki}\, R_{kj} = \delta_{ij}$ and its transpose relation $R_{ik}\, R_{jk} = \delta_{ij}$. Taking $p = 0$ as a special case gives us the simple outer product of the two tensors, which gives the previous example when both tensors are rank 1.

- The identity matrix is a rank 2 tensor and, in fact, it is **isotropic**, meaning that its coordinate representation is the same in all frames. Let's just try transforming it to see this:

$$1'_{ij} = R_{ik}\, R_{jl}\, 1_{kl} = R_{ik}\, R_{jl}\, \delta_{kl} = R_{ik}\, R_{jk} = \delta_{ij}$$

(We used the "transposed" orthonormality condition $R_{ik}\, R_{jk} = \delta_{ij}$.) So, we see that the identity matrix has representation δ_{ij} in any frame and that the representations in different frames are related by the appropriate transformation relations.

- We can demonstrate that the ϵ_{ijk} Levi-Civita symbol is an isotropic rank 3 tensor. Let's explicitly calculate the effect of the transformation rule on it:

$$\epsilon'_{ijk} = R_{il}\, R_{jm}\, R_{kn}\, \epsilon_{lmn}$$

We may evaluate the above by recognizing that the "transposed" orthonormality condition on $\mathbf{R}$ implies that the rows of $\mathbf{R}$ look like N mutually orthonormal vectors in N-dimensional space. (Here we use the term vector more loosely – we have no need to prove that these rows behave like vectors in rotated frames, we only need the fact that their component representations in a given frame looks like that of N orthonormal vectors.) Denote these "vectors" by $\vec{\mathbf{R}}^{\,r}_i$, where $\left(\vec{\mathbf{R}}^{\,r}_i\right)_j = R_{ij}$. (The ^{r} superscript indicates we are treating the rows, rather than the columns, of $\mathbf{R}$ as vectors.) With this notation, the above product looks like $\vec{\mathbf{R}}^{\,r}_i \cdot \left(\vec{\mathbf{R}}^{\,r}_j \times \vec{\mathbf{R}}^{\,r}_k\right)$. In $N = 3$ dimensions, the expression will only be nonvanishing when the triplet ijk is a cyclic or anticyclic combination; and the expression will have magnitude 1 and take the sign of the permutation (cyclic or anticyclic). These are exactly the properties of ϵ_{ijk}, so we have

$$\epsilon'_{ijk} = \epsilon_{ijk}$$

So the Levi-Civita symbol is an isotropic rank 3 tensor for $N = 3$ (and for arbitrary N, though we will not prove it here). Note that this implies some properties of $\vec{\mathbf{M}}$:

1. When treated as a single rank 3 tensor $\mathcal{M}$ with coordinate representation components $M_{ijk} = \left(\vec{\mathbf{M}}_i\right)_{jk} = -\epsilon_{ijk}$, $\mathcal{M}$ is clearly an isotropic rank 3 tensor. For this particularly interesting case, we will take the symbol $\vec{\mathbf{M}}$ to stand for the rank 3 tensor $\mathcal{M}$. Since $\vec{\mathbf{M}}$ is isotropic, there is no distinction between $\underline{\vec{\mathbf{M}}}$ and $\vec{\mathbf{M}}$.
2. Given a vector $\vec{a}$, the quantity $\vec{a} \cdot \vec{\mathbf{M}}$ has in frames F and F' coordinate representations $\underline{\vec{a}} \cdot \underline{\vec{\mathbf{M}}} = \underline{\vec{a}} \cdot \vec{\mathbf{M}}$ and $\underline{\left(\vec{a} \cdot \vec{\mathbf{M}}\right)'} = \underline{\vec{a}}' \cdot \underline{\vec{\mathbf{M}}}' = \underline{\vec{a}}' \cdot \vec{\mathbf{M}}$, where the last step in each case is possible because $\vec{\mathbf{M}}$ is isotropic. Thus, only the coordinate representation of the vector $\vec{a}$ need be changed to write $\vec{a} \cdot \vec{\mathbf{M}}$ in different frames.

- With the above, we may show that the operator $\vec{a} \times$ is a rank 2 tensor. We are referring to the operator, not just the vector $\vec{a}$. The operation $\vec{a} \times \vec{b}$ is written as

$$\left(\vec{a} \times \vec{b}\right)_i = \epsilon_{ijk}\, a_j\, b_k = (\epsilon_{ijk}\, a_j)\, b_k = \left(\vec{a} \cdot \vec{\mathbf{M}}\right)_{ik} b_k$$

which shows that $\vec{a} \times$ looks like the operation of a matrix on a vector. Since we know $\vec{a}$ is a vector and $\vec{\mathbf{M}}$ is a rank 3 tensor, the contraction over one index must yield a rank 2 tensor as discussed above. But, since we did not explicitly prove the general

relation, let's prove this specific case explicitly using the transformation properties of $\vec{a}$ and $\vec{\mathbf{M}}$:

$$\begin{aligned}\underline{\left(\vec{a}\cdot\vec{\mathbf{M}}\right)'}_{ij} = -a'_k\epsilon'_{kij} &= -R_{kl}\,a_l\,R_{km}\,R_{in}\,R_{jp}\,\epsilon_{mnp}\\ &= -\delta_{lm}\,a_l\,R_{in}\,R_{jp}\,\epsilon_{mnp}\\ &= -R_{in}\,R_{jp}\,a_m\,\epsilon_{mnp}\\ &= R_{in}\,R_{jp}\underline{\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{M}}\right)}_{np}\end{aligned}$$

So, indeed, $\vec{a}\cdot\vec{\mathbf{M}}$, and thus $\vec{a}\times$, transforms like a rank 2 tensor. This last fact will be useful in the next section on dynamics in rotating systems.

5.1.5 Comments on Lie Algebras and Lie Groups

We make some qualitative comments on Lie algebras and groups. Such concepts are not incredibly useful here, but serve to allow analogies to be made between the physical rotations we have talked about here and transformations in more complex systems. We progress through some definitions of mathematical objects that lead up to Lie algebras and Lie groups.

Groups

A **group** is defined to be a set of elements with a binary operation rule that specifies how combinations of pairs of elements yield other members of the group. The set must be **closed** under the operation – binary combinations of members of the set may only yield other members of the set – to be called a group. In addition, the binary operation must be associative, $a\,(b\,c) = (a\,b)\,c$, there must be an identity element 1 such that $a\,1 = a$, and each element must have an inverse such that $a^{-1}\,a = a\,a^{-1} = 1$. Note that the group operation need not be commutative.

Fields

A **field** is a group that has two kinds of operations, addition and multiplication. It is a group under each of the operations separately, and in addition satisfies distributivity: $a\,(b+c) = a\,b + a\,c$.

Group Algebras

A **group algebra** is a combination of a field F (with addition $+$ and multiplication $\cdot$) and a group G (with multiplication $*$), consisting of all *finite* linear combinations of elements of G with coefficients from F, $a\,g + b\,h$, where a and b belong to F and g and h belong to G. The group operations on F and G continue to work:

$$\begin{aligned}a\,g + b\,g &= (a+b)\,g\\ a\cdot\sum_i a_i\,g_i &= \sum_i (a\cdot a_i)\,g_i\\ \left(\sum_i a_i\,g_i\right) * \left(\sum_j b_j\,h_j\right) &= \sum_{i,j}(a_i\cdot b_j)\,(g_i * h_j)\end{aligned}$$

Lie Algebras

A **Lie algebra** is a group algebra with the additional conditions that the group elements $\{\tau_i\}$ belonging to G satisfy the commutation relations

$$[\tau_i, \tau_j] \equiv \tau_i \, \tau_j - \tau_j \, \tau_i = c_{ij}^k \tau_k$$

The $\{c_{ij}^k\}$ are called the **structure constants** of the Lie algebra. They satisfy $c_{ij}^k = -c_{ji}^k$. (Actually, the real definition of a Lie algebra is slightly more generic, allowing for definition of a commutator without G being part of a group algebra.) A Lie algebra may have a finite or infinite number of group members, though in general we will only consider finite ones. The matrices $\mathbf{M}_x$, $\mathbf{M}_y$, and $\mathbf{M}_z$ defined earlied form a Lie algebra with the real numbers as the field F.

Lie Groups

A **Lie group** is the exponential of the Lie algebra, consisting of all possible elements

$$a_m = \exp\left(\sum_k \theta_m^k \tau_k\right)$$

where the θ_m^k are members of the field F and the τ_k are members of the group G. The exponential is defined in terms of its infinite power series expansion, which is well-defined for a group. Note that the members of the Lie group are not members of the Lie algebra because, by dint of the power series expansion, they are *infinite* linear combinations of members of the Lie algebra. The Lie group is entirely separate from the Lie algebra. The Lie group is, as its name indicates, a group. Moreover, thanks to the method of definition, the group is differentiable with respect to the members of the field.

Representations

Our rotation matrices are a particular **representation** of the rotation group in three dimensions, $O(3)$, which is a Lie group. The $\mathbf{M}$ matrices are a representation of the Lie algebra that generate the Lie group. The word representation is used because the matrices are not really necessary; it is only the group and field rules that are fundamental. The representation in terms of matrices provides one way that those rules can be realized, but not the only way.

Significance

These definitions serve only to illustrate that there is a fundamental structure underlying rotation matrices, that they form a Lie group can be generated from a small set of elements forming a Lie algebra. Lie algebras and groups are found everywhere in physics because they are the means by which continuous transformations are performed. As we saw in Section 2.1.10, continuous coordinate transformations under which a Lagrangian is invariant are of great significance because they give rise to conserved canonical momenta. The continuous parameters of such symmetry transformations are the field members θ_m^k, so the conserved momenta are given by the group members τ_k. When rotations are symmetry transformations, we thus naturally see that angular momentum $\vec{l} = \vec{p}^T \vec{\mathbf{M}} \vec{r}$ is conserved. We will see another Lie group, the Lorentz group, in connection with special relativity.

5.2 Dynamics in Rotating Coordinate Systems

We use the formalism developed in the last section to describe dynamics in rotating, and hence non-inertial, coordinate systems. We follow Hand and Finch Chapter 7 conceptually, but employ the more formal mathematics developed in the previous section.

As a general point, all rotating coordinate systems are non-inertial since Newton's first and second laws are not trivially obeyed in them (as we shall see). Therefore, we neglect to reinforce the word "rotating" with the word "non-inertial" in the text. We will also in general only consider two coordinate systems – one that is inertial and one that is rotating relative to the inertial system but otherwise fixed. That is, we will not consider a rotating system that is also moving relative to an inertial system. One can consider such systems, but we must first develop the basics of describing rotating systems. Given these restrictions, we will simply refer to reference frames as "rotating" and "non-rotating", with the implication that the non-rotating system is inertial and the rotating system is rotating but not otherwise moving with respect to the inertial system. We will also use the term "body" for the rotating system and "space" for the non-rotating system, which make sense when referring to rotation of a rigid body.

We will figure out how to do the physics in the noninertial frame by transforming to an inertial frame, doing the physics in the inertial frame we understand, and transforming back to the noninertial frame.

The notation in Hand and Finch (and in most other texts) on this subject is extremely confusing. We are very explicit about coordinate systems here to be clear.

5.2.1 Newton's Second Law in Rotating Coordinate Systems

We obtain a version of Newton's second law in rotating coordinate systems, deriving fictitious forces that must be included when solving for the dynamics in a rotating system.

Notation

We will use the notation introduced in Section 5.1.4 to distinguish vectors and tensors and their coordinate representations. Briefly, that notation is

object	notation
vector without reference to a coordinate system	$\vec{r}$
coordinate representation of a vector	$\underline{\vec{r}}$
components of coordinate representation of a vector	r_i
scalar without reference to a coordinate system	s
coordinate representation of a scalar	$\underline{s}$
tensor	$\mathcal{T}$
coordinate representation of a tensor	$\underline{\mathcal{T}}$
components of coordinate representation of a tensor	$T_{i_1 \ldots i_n}$
rotation matrix	$\mathbf{R}$
components of rotation matrix	R_{ij}

Since our rotating and non-rotating coordinate systems will usually be denoted by F and F', the vector $\vec{r}$ will have its coordinate representations in these systems written as $\underline{\vec{r}}$ and $\underline{\vec{r}}'$ and the components of these representations written as r_i and r'_i.

For $\vec{\omega}$ and $\dot{\vec{\omega}}$, the angular velocity and acceleration of the rotating system relative to the non-rotating system, it may be unclear what is meant by a representation in the rotating frame

because, strictly speaking, $\vec{\omega}$ and $\dot{\vec{\omega}}$ would vanish if *measured* in the rotating frame. However, $\vec{\omega}$ and $\dot{\vec{\omega}}$ are well-defined vectors in the fixed frame with coordinate representations there, so we can obtain their coordinate representations in the rotating frame using a rotation coordinate transformation. Explicitly, $\vec{\omega}$ and $\dot{\vec{\omega}}$ refer to the vectors in a coordinate-system-independent sense, $\underline{\vec{\omega}}'$ and $\underline{\dot{\vec{\omega}}}'$ refer to their representations in the non-rotating system (where they are measured), with components ω_i' and $\dot{\omega}_i'$, while $\underline{\vec{\omega}}$ and $\underline{\dot{\vec{\omega}}}$, with components ω_i and $\dot{\omega}_i$, refer *not to measurements of these vectors in the rotating system*, but rather to the quantities *obtained by applying the appropriate coordinate system transformation* to the fixed-frame representations $\underline{\vec{\omega}}'$ and $\underline{\dot{\vec{\omega}}}'$. The symbols $\underline{\vec{\omega}}$ and $\underline{\dot{\vec{\omega}}}$ are thus rather nonphysical and nonintuitive. To avoid confusion, we will never use the symbols $\underline{\vec{\omega}}$ and $\underline{\dot{\vec{\omega}}}$ but rather leave explicit the necessary transformation operators.

Position and Velocity in Rotating and Non-Rotating Frames

We consider a rotating frame F and a non-rotating frame F'. We first need to transform the coordinates and velocities from F to F' so we can do the physics in F'. Suppose some particle describes a path with coordinate representation $\underline{\vec{r}}(t)$ relative to the rotating (body) frame. That path has coordinate representation as $\underline{\vec{r}}'(t)$ in the non-rotating (space) frame. At any given time, the coordinate representations in the two frames are related by $\mathbf{R}(t)$, a rotation matrix with possible time dependence of the rotation angle. Thus, we have

$$\underline{\vec{r}}'(t) \;=\; \mathbf{R}(t)\,\underline{\vec{r}}(t)$$

Note the sense of $\mathbf{R}(t)$*: it converts from the rotating system to the non-rotating system, not the other way!* Now, let us take a time derivative so we can determine the relation between the velocities in the two frames:

$$\begin{aligned}\frac{d}{dt}\,\underline{\vec{r}}'(t) &= \left(\frac{d}{dt}\,\mathbf{R}(t)\right)\underline{\vec{r}}(t) + \mathbf{R}(t)\frac{d}{dt}\,\underline{\vec{r}}(t)\\ &= \dot{\mathbf{R}}(t)\,\underline{\vec{r}}(t) + \mathbf{R}(t)\frac{d}{dt}\,\underline{\vec{r}}(t)\end{aligned}$$

The velocity relative to the non-rotating system has two components – one due to rotation of the rotating system, the other due to motion relative to the rotating system. But, what is meant by $\mathbf{R}(t)\,\frac{d}{dt}\,\underline{\vec{r}}(t)$ and why is it not just $\frac{d}{dt}\,\underline{\vec{r}}'(t)$?[3]

This confusion is similar to the issue noted above with regard to $\vec{\omega}$ and $\dot{\vec{\omega}}$. $\mathbf{R}\,\frac{d}{dt}\,\underline{\vec{r}}$ is simply the set of numbers obtained by applying the coordinate transformation to the elements of $\frac{d}{dt}\,\underline{\vec{r}}$. An example may clarify this. Consider a particle moving in a radial line on a rotating table, at constant speed relative to the table. Its $\frac{d}{dt}\,\underline{\vec{r}}(t)$ vector is constant in the rotating frame, $\frac{d}{dt}\,\underline{\vec{r}}(t) = c\,\hat{x}$, and its position vector $\underline{\vec{r}}(t)$ is evolving linearly, $\underline{\vec{r}}(t) = c\,t\,\hat{x}$. Then its velocity in the non-rotating frame will have two pieces. The first is $\dot{\mathbf{R}}(t)\,\underline{\vec{r}}(t)$ piece, which corresponds to instantaneous circular motion in the non-rotating frame due to the rotation of the rotating frame and is easily seen to be $c\,t\,\omega\,\hat{\phi}'$ where $\hat{\phi}'$ is the unit vector for the polar coordinate in the plane of motion in the non-rotating frame. The second piece is the $\mathbf{R}\,\frac{d}{dt}\,\underline{\vec{r}}(t)$ piece due to motion relative to the rotating frame. If at $t = 0$, $\hat{x}$ coincides with the $\hat{x}'$ axis of the non-rotating frame, then $\mathbf{R}\,\frac{d}{dt}\,\underline{\vec{r}}(t)$ is $c\,\hat{\rho}'$ where $\hat{\rho}'$ is the radial unit vector in the non-rotating system. That is, the contribution of $\mathbf{R}\,\frac{d}{dt}\,\underline{\vec{r}}(t)$ is $c\,\hat{\rho}'$ while the total velocity relative to the non-rotating frame is $\frac{d}{dt}\,\underline{\vec{r}}'(t) = c\,\hat{\rho}' + c\,t\,\omega\,\hat{\phi}'$.

[3] We have carefully avoided using $\underline{\vec{v}}$ and $\underline{\vec{v}}'$ symbols here for reasons that will become apparent.

Thus, in the strict sense of the definition of a vector as a set of quantities that obey a specific set of transformation rules, the set of numbers $\frac{d}{dt}\,\underline{\vec{r}}(t)$ and the set of numbers $\frac{d}{dt}\,\underline{\vec{r}}'(t)$ are not different coordinate representations of the same vector, but really they are representations of *different vectors*! They are different vectors because the time derivatives are referenced to different coordinate systems. *When discussing rotating systems, time derivatives must in general be specified relative to a particular frame.* Hand and Finch and other texts use the notation $|_{body}$ and $|_{space}$ to make the distinction. We will write write $\underline{\vec{v}}_{body}$ for the coordinate representation of $\frac{d}{dt}\,\underline{\vec{r}}$ in the rotating frame and $\underline{\vec{v}}'_{space}$ for the coordinate representation of $\frac{d}{dt}\,\underline{\vec{r}}'$ in the non-rotating frame.

It would be consistent for $\underline{\vec{v}}_{body}$ and $\underline{\vec{v}}'_{body} = \mathbf{R}\,\underline{\vec{v}}_{body}$ to refer to the two coordinate representations of the vector given by measuring the velocity of the particle relative to the rotating frame, and for $\underline{\vec{v}}'_{space}$ and $\underline{\vec{v}}_{space} = \mathbf{R}^T\underline{\vec{v}}'_{space}$ to refer to the coordinate representations of the vector given by measuring the velocity of the particle relative to the non-rotating frame. Hand and Finch actively make use of symbols analogous to $\underline{\vec{v}}'_{body}$ and $\underline{\vec{v}}_{space}$ (their equivalents are $\mathbf{v}|'_{\text{body}}$ and $\mathbf{v}|_{\text{space}}$). We feel this is confusing because the concepts implied by these symbols are nonphysical and nonintuitive: one cannot measure the components of the coordinate representation $\underline{\vec{v}}'_{body}$ because the vector $\vec{v}_{body}$ is fundamentally a rotating-frame quantity. One obtains $\underline{\vec{v}}'_{body}$ only through a mathematical transformation of the measured components $\underline{\vec{v}}_{body}$. And similarly for the relation between $\underline{\vec{v}}'_{space}$ and $\underline{\vec{v}}_{space}$. We will keep the rotation matrices explicit and never use $\underline{\vec{v}}'_{body}$ and $\underline{\vec{v}}_{space}$.

Given the above, we rewrite our original equation as

$$\underline{\vec{v}}'_{space}(t) = \dot{\mathbf{R}}(t)\,[\mathbf{R}(t)]^T\,\underline{\vec{r}}'(t) + \mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) \tag{5.1}$$

Finally, we note that the use of the $|_{body}$ and $|_{space}$ symbols in conjunction with derivatives is somewhat subtle. For example, we do not write $\underline{\vec{v}}'_{space} = \frac{d}{dt}\,\underline{\vec{r}}'\big|_{space}$. $\underline{\vec{r}}'$ is the non-rotating frame coordinate representation of $\vec{r}$, so writing $\frac{d}{dt}\,\underline{\vec{r}}'\big|_{body}$ would make no sense at all. The derivative $\frac{d}{dt}\,\underline{\vec{r}}'$ *must* be a derivative with respect to the non-rotating frame. On the other hand, if we use the coordinate-free notation $\frac{d}{dt}\,\vec{r}$, then we must indeed specify $|_{body}$ or $|_{space}$ because no coordinate representation is implied. So, we will see the $|_{body}$ and $|_{space}$ designators in some places and not in others.

Rewriting Using Instantaneous Angular Velocity

If $\underline{\vec{v}}_{body}$ vanishes and the motion is simple rotational, we expect $\underline{\vec{v}}'_{space}(t) = \underline{\vec{\omega}}' \times \underline{\vec{r}}'(t)$. We shall see that we can rewrite the above using this form for more generic circumstances. There are two ways to see this:

- **Indirect method** (à la Hand and Finch)

  First, we show that the matrix $\dot{\mathbf{R}}(t)\,[\mathbf{R}(t)]^T$ is antisymmetric. $\mathbf{R}$ is orthogonal, so $\mathbf{R}\,\mathbf{R}^T = \mathbf{I}$. If we take the time derivative, we find

$$\dot{\mathbf{R}}\,\mathbf{R}^T + \mathbf{R}\,\dot{\mathbf{R}}^T = 0$$
$$\dot{\mathbf{R}}\,\mathbf{R}^T = -\left(\dot{\mathbf{R}}\,\mathbf{R}^T\right)^T$$

  The last line indicates the matrix $\dot{\mathbf{R}}\,\mathbf{R}^T$ is antisymmetric.

Next, antisymmetric matrices only have three independent components (the diagonal vanishes and the lower-off-diagonal elements are related to the upper off-diagonal elements), so can in fact be rewritten in terms of a vector:

$$A_{ij} = \sum_k \epsilon_{ijk}\, a_k$$

(ϵ_{ijk} is the completely antisymmetric Levi-Civita symbol, see Appendix A.) If we now rewrite the action of $\mathbf{A}$ on a vector $\vec{b}$, we obtain

$$\left[\mathbf{A}\,\vec{b}\right]_i = \sum_j A_{ij}\, b_j = \sum_{j,k} \epsilon_{ijk} a_k\, b_j = \left[\vec{b}\times\vec{a}\right]_i = -\left[\vec{a}\times\vec{b}\right]_i$$

That is, the action of $\dot{\mathbf{R}}(t)\,[\mathbf{R}(t)]^T$ is identical to that of vector cross product. We know that when $\underline{\vec{v}}_{body}$ vanishes, the full expression, Equation 5.2, must reduce to $\underline{\vec{\omega}}' \times \underline{\vec{r}}'$, so we make the identification of $\dot{\mathbf{R}}(t)\,[\mathbf{R}(t)]^T$ with the instantaneous angular velocity cross-product operator, $\underline{\vec{\omega}}'(t)\times$.

- **Direct method** (using generators)
  Let's just calculate $\dot{\mathbf{R}}$ based on the matrix form. We have to be a bit careful to avoid formalistic hitches. For example, if we just blindly start calculating from the matrix exponential form, we obtain:

$$\begin{aligned}
\dot{\mathbf{R}} &= \frac{d}{dt}\left[\sum_{n=0}^{\infty}\frac{\left(\vec{\theta}(t)\cdot\vec{\mathbf{M}}\right)^n}{n!}\right] \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n-1}\left(\vec{\theta}(t)\cdot\vec{\mathbf{M}}\right)^k\left[\frac{d}{dt}\left(\vec{\theta}(t)\cdot\vec{\mathbf{M}}\right)\right]\left(\vec{\theta}(t)\cdot\vec{\mathbf{M}}\right)^{n-k-1} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n-1}\left(\vec{\theta}(t)\cdot\vec{\mathbf{M}}\right)^k\left(\vec{\omega}(t)\cdot\vec{\mathbf{M}}\right)\left(\vec{\theta}(t)\cdot\vec{\mathbf{M}}\right)^{n-k-1}
\end{aligned}$$

  Obviously, what we would like to do is commute $\vec{\omega}(t)\cdot\vec{\mathbf{M}}$ with the other factors so that we would get in the end $\dot{\mathbf{R}} = \vec{\omega}\cdot\vec{\mathbf{M}}\,\mathbf{R}$. The calculation needed is:

$$\begin{aligned}
\left(\vec{\theta}\cdot\vec{\mathbf{M}}\right)\left(\vec{\omega}\cdot\vec{\mathbf{M}}\right) &= \sum_{i,j}\theta_i\,\omega_j\,\mathbf{M}_i\,\mathbf{M}_j = \sum_{i,j}\theta_i\,\omega_j\,(\mathbf{M}_j\,\mathbf{M}_i + \epsilon_{ijk}\,\mathbf{M}_k) \\
&= \left(\vec{\omega}\cdot\vec{\mathbf{M}}\right)\left(\vec{\theta}\cdot\vec{\mathbf{M}}\right) + \vec{\mathbf{M}}\times\left(\vec{\theta}\times\vec{\omega}\right)
\end{aligned}$$

  While the last term is sure to vanish in simple cases (*e.g.*, simple circular motion of a single particle), it is not clear that it vanishes in all cases. To obtain $\dot{\mathbf{R}}$, we can instead use the definition of a derivative and the form for an infinitesimal rotation:

$$\begin{aligned}
\dot{\mathbf{R}}(t) &= \lim_{dt\to 0}\frac{1}{dt}\left[\mathbf{R}(t+dt) - \mathbf{R}(t)\right] \\
&= \lim_{dt\to 0}\frac{1}{dt}\left[\left(\mathbf{I} + \vec{\omega}(t)\,dt\cdot\vec{\mathbf{M}}\right) - \mathbf{I}\right]\mathbf{R}(t) \\
&= \lim_{dt\to 0}\vec{\omega}(t)\cdot\vec{\mathbf{M}}\,\mathbf{R}(t) = \vec{\omega}(t)\cdot\vec{\mathbf{M}}\,\mathbf{R}(t)
\end{aligned}$$

(Incidentally, by comparison to our previous commutation attempt, this result demonstrates that $\vec{\theta} \times \vec{\omega}$ does in general vanish.) We have not been careful yet about what frame's coordinate representation for $\vec{\omega}$ is used above. Based on our expectation that the result should reduce to $\underline{\vec{\omega}}' \times \underline{\vec{r}}'(t)$ when $\underline{\vec{v}}_{body} = 0$, the appropriate representation for $\vec{\omega}$ is $\underline{\vec{\omega}}'$, the non-rotating-frame representation. With that specification and the above mathematical result, we see trivially that

$$\dot{\mathbf{R}}\,\mathbf{R}^T = \underline{\vec{\omega}}' \cdot \vec{\mathbf{M}}\,\mathbf{R}\,\mathbf{R}^T = \underline{\vec{\omega}}' \cdot \vec{\mathbf{M}}$$

Finally, we make use of the form for $\vec{\mathbf{M}}$ given earlier, $(\mathbf{M}_i)_{jk} = -\epsilon_{ijk}$:

$$\left(\underline{\vec{\omega}}' \cdot \vec{\mathbf{M}}\right)_{ij} \;=\; \sum_k \omega'_k \left(\mathbf{M}_k\right)_{ij} = -\sum_k \omega'_k\,\epsilon_{kij} = \sum_k \epsilon_{ikj}\,\omega'_k$$

So, finally, we have

$$\left(\dot{\mathbf{R}}\,\mathbf{R}^T \underline{\vec{r}}'\right)_i \;=\; \left(\underline{\vec{\omega}}' \cdot \vec{\mathbf{M}}\,\underline{\vec{r}}'\right)_i = \sum_j \left(\underline{\vec{\omega}}' \cdot \vec{\mathbf{M}}\right)_{ij} r'_j = \sum_{jk} \epsilon_{ikj}\,\omega'_k\,r'_j = \left(\underline{\vec{\omega}}' \times \underline{\vec{r}}'\right)_i$$

Either way you derive it, we have the end result

$$\underline{\vec{v}}'_{space}(t) = \underline{\vec{\omega}}'(t) \times \underline{\vec{r}}'(t) + \mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) \tag{5.2}$$

If there is ever ambiguity about how to calculate $\underline{\vec{\omega}}'(t)$, then one simply has to obtain $\mathbf{R}(t)$ and calculate $\dot{\mathbf{R}}(t)\,[\mathbf{R}(t)]^T$.

Acceleration and Fictitious Forces

Let's obtain the acceleration from the above equation. We take the time derivative in the non-rotating frame:[4]

$$\begin{aligned}
\frac{d}{dt}\,\underline{\vec{v}}'_{space}(t) &= \dot{\underline{\vec{\omega}}}'(t) \times \underline{\vec{r}}'(t) + \underline{\vec{\omega}}'(t) \times \frac{d}{dt}\,\underline{\vec{r}}'(t) + \dot{\mathbf{R}}(t)\,\underline{\vec{v}}_{body}(t) + \mathbf{R}(t)\,\frac{d}{dt}\,\underline{\vec{v}}_{body}(t) \\
&= \dot{\underline{\vec{\omega}}}'(t) \times \underline{\vec{r}}'(t) + \underline{\vec{\omega}}'(t) \times \left[\underline{\vec{\omega}}'(t) \times \underline{\vec{r}}'(t) + \mathbf{R}(t)\,\underline{\vec{v}}_{body}(t)\right] \\
&\quad + \dot{\mathbf{R}}(t)\;[\mathbf{R}(t)]^T\,\mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) + \mathbf{R}(t)\,\underline{\vec{a}}_{body}(t) \\
&= \underline{\vec{\omega}}'(t) \times \left[\underline{\vec{\omega}}'(t) \times \underline{\vec{r}}'(t)\right] + 2\,\underline{\vec{\omega}}'(t) \times \mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) + \dot{\underline{\vec{\omega}}}'(t) \times \underline{\vec{r}}'(t) \\
&\quad + \mathbf{R}(t)\,\underline{\vec{a}}_{body}(t)
\end{aligned}$$

$\underline{\vec{a}}_{body} = \frac{d}{dt}\,\underline{\vec{v}}_{body}(t)$ is the coordinate representation in the rotating frame of the acceleration relative to the rotating frame. The left side of above equation gives the true force because it is calculated in the non-rotating frame. The term $\underline{\vec{a}}_{body}(t)$ gives the apparent force in the rotating frame. Thus, we can obtain a relation between the rotating-frame apparent force $\vec{F}_{app}$ with coordinate representation $\underline{\vec{F}}_{app} = m\,\underline{\vec{a}}_{body}$ and the non-rotating frame true force $\vec{F}_{true}$ with coordinate representation $\underline{\vec{F}}'_{true} = m\,\frac{d}{dt}\,\underline{\vec{v}}'_{space}$:

$$\begin{aligned}
&\underline{\vec{F}}_{app} = \\
&\mathbf{R}^T(t)\left[\underline{\vec{F}}'_{true} - m\,\underline{\vec{\omega}}'(t) \times \left[\underline{\vec{\omega}}'(t) \times \underline{\vec{r}}'(t)\right] - 2\,m\,\underline{\vec{\omega}}'(t) \times \mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) - m\,\dot{\underline{\vec{\omega}}}'(t) \times \underline{\vec{r}}'(t)\right]
\end{aligned} \tag{5.3}$$

[4]Note that, as discussed earlier, we do not need to specify $|_{body}$ or $|_{space}$ for the derivatives in the following because the derivatives are acting on coordinate representations, not coordinate-free vectors. We shall see later where we will need to make these specifications.

$\vec{F}_{app}$ and $\vec{F}_{true}$ are different vectors, not just different coordinate representations of the same force vector. Unlike with velocities and accelerations, the rotating-frame representation $\underline{\vec{F}}_{true}$ of the non-rotating frame true force vector $\vec{F}_{true}$ makes physical sense since no derivatives are involved. We shall see, in fact, that our results below will be most useful when the coordinate representation of $\vec{F}_{true}$ is most easily specified in the rotating frame. On the other hand, it is difficult to find a reason to use $\underline{\vec{F}}'_{app}$ even though it is as physically reasonable as $\underline{\vec{F}}_{true}$.

The above form is still not that useful because it relates the apparent force in the rotating frame to quantities $\underline{\vec{r}}'(t)$ and $\mathbf{R}(t)\,\underline{\vec{v}}_{body}(t)$ in the non-rotating frame. But, we can obtain a version that makes use of only rotating frame quantities. Let's rewrite the above, replacing the $\underline{\vec{\omega}}'\times$ operator with the quantity from which it was derived, $\dot{\mathbf{R}}\mathbf{R}^T$. The various terms are (drop the explicit time dependence for convenience):

$$\begin{aligned}
\mathbf{R}^T\underline{\vec{\omega}}'\times\left[\underline{\vec{\omega}}'\times\underline{\vec{r}}'\right] &= \mathbf{R}^T\left(\dot{\mathbf{R}}\mathbf{R}^T\right)\left(\dot{\mathbf{R}}\mathbf{R}^T\right)\underline{\vec{r}}' = \left(\mathbf{R}^T\dot{\mathbf{R}}\right)\left(\mathbf{R}^T\dot{\mathbf{R}}\right)\mathbf{R}^T\underline{\vec{r}}' \\
&= \left(\mathbf{R}^T\dot{\mathbf{R}}\mathbf{R}^T\mathbf{R}\right)\left(\mathbf{R}^T\dot{\mathbf{R}}\mathbf{R}^T\mathbf{R}\right)\underline{\vec{r}} \\
&= \left(\mathbf{R}^T\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}}\,\mathbf{R}\right)\left(\mathbf{R}^T\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}}\,\mathbf{R}\right)\underline{\vec{r}} \\
\mathbf{R}^T\underline{\vec{\omega}}'\times\mathbf{R}\,\underline{\vec{v}}_{body} &= \mathbf{R}^T\left(\dot{\mathbf{R}}\mathbf{R}^T\right)\mathbf{R}\,\underline{\vec{v}}_{body} \\
&= \left(\mathbf{R}^T\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}}\,\mathbf{R}\right)\underline{\vec{v}}_{body} \\
\mathbf{R}^T\dot{\underline{\vec{\omega}}}'\times\underline{\vec{r}}' &= \mathbf{R}^T\left[\frac{d}{dt}\left(\dot{\mathbf{R}}\mathbf{R}^T\right)\right]\underline{\vec{r}}' = \mathbf{R}^T\left[\frac{d}{dt}\,\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}}\right]\mathbf{R}\,\underline{\vec{r}} \\
&= \left(\mathbf{R}^T\dot{\underline{\vec{\omega}}}'\cdot\vec{\mathbf{M}}\,\mathbf{R}\right)\underline{\vec{r}}
\end{aligned}$$

Further simplification of the above requires us to understand what $\mathbf{R}^T\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}}\,\mathbf{R}$ is. For those not wanting to rely on the formal definition of tensors, the result can be obtained by intuitive means.[5] The more formal demonstration makes use of two facts shown in Section 5.1.4: a) $\vec{\mathbf{M}}$ is an isotropic rank 3 tensor, and b) for $\vec{\omega}$ a vector, the contraction $\vec{\omega}\cdot\vec{\mathbf{M}}$ is therefore a rank 2 tensor. Thus, if $\mathbf{R}$ converts coordinate representations in the frame F to representations in the frame F', then the two representations of $\vec{\omega}\cdot\vec{\mathbf{M}}$ are related by

$$\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}} = \mathbf{R}\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\cdot\vec{\mathbf{M}}\right)\mathbf{R}^T \Longleftrightarrow \mathbf{R}^T\left(\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}}\right)\mathbf{R} = \left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\cdot\vec{\mathbf{M}}$$

[5] Since $\vec{\omega}\cdot\vec{\mathbf{M}}$ looks like a matrix, the quantity $\vec{a}^T\left(\vec{\omega}\cdot\vec{\mathbf{M}}\right)\vec{b}$ looks like a number. We in general would like simple numbers to be independent of frame rotation. We know that the coordinate representations of $\vec{a}$ are related by $\underline{\vec{a}}' = \mathbf{R}\,\underline{\vec{a}}$ and similarly for $\vec{b}$. Requiring invariance of $\vec{a}^T\left(\vec{\omega}\cdot\vec{\mathbf{M}}\right)\vec{b}$ implies

$$\begin{aligned}
\underline{\vec{a}}'^T\underline{\left(\vec{\omega}\cdot\vec{\mathbf{M}}\right)'}\,\underline{\vec{b}}' &= \underline{\vec{a}}^T\underline{\left(\vec{\omega}\cdot\vec{\mathbf{M}}\right)}\,\underline{\vec{b}} \\
\underline{\vec{a}}'^T\left(\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}}\right)\underline{\vec{b}}' &= \left(\mathbf{R}^T\underline{\vec{a}}'\right)^T\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\cdot\vec{\mathbf{M}}\right)\left(\mathbf{R}^T\underline{\vec{b}}'\right) \\
&= \underline{\vec{a}}'^T\mathbf{R}\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\cdot\vec{\mathbf{M}}\right)\mathbf{R}^T\underline{\vec{b}}'
\end{aligned}$$

which can only hold for arbitrary $\vec{a}$ and $\vec{b}$ if $\underline{\left(\vec{\omega}\cdot\vec{\mathbf{M}}\right)'} = \mathbf{R}\underline{\left(\vec{\omega}\cdot\vec{\mathbf{M}}\right)}\mathbf{R}^T$, or equivalently $\left(\underline{\vec{\omega}}'\cdot\vec{\mathbf{M}}\right) = \mathbf{R}\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\cdot\vec{\mathbf{M}}\right)\mathbf{R}^T$. As in the text, we use $\mathbf{R}^T\underline{\vec{\omega}}'$ instead of $\underline{\vec{\omega}}$ because the latter is nonphysical.

We do not use the symbol $\underline{\vec{\omega}}$ for $\mathbf{R}^T\underline{\vec{\omega}}'$ but rather leave the rotation matrices explicit for reasons discussed before: the rotating-frame coordinate representation $\underline{\vec{\omega}}$ of the non-rotating frame angular velocity $\vec{\omega}$ with non-rotating frame representation $\underline{\vec{\omega}}'$ is a nonphysical quantity; $\underline{\vec{\omega}}$ cannot be measured directly, while $\underline{\vec{\omega}}'$ can. The same argument holds for $\dot{\vec{\omega}}$, so

$$\mathbf{R}^T\left(\underline{\dot{\vec{\omega}}}'\cdot\vec{\mathbf{M}}\right)\mathbf{R} = \left[\mathbf{R}^T\underline{\dot{\vec{\omega}}}'\right]\cdot\vec{\mathbf{M}}$$

With the above transformation rules and our previous results, we find

$$\begin{aligned}
\mathbf{R}^T\underline{\vec{\omega}}'\times\left[\underline{\vec{\omega}}'\times\underline{\vec{r}}'\right] &= \left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\cdot\vec{\mathbf{M}}\right)\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\cdot\vec{\mathbf{M}}\right)\underline{\vec{r}} = \left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\times\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\times\underline{\vec{r}}\right)\\
\mathbf{R}^T\underline{\vec{\omega}}'\times\mathbf{R}\underline{\vec{v}}_{body} &= \left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\cdot\vec{\mathbf{M}}\right)\underline{\vec{v}}_{body} = \left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\times\underline{\vec{v}}_{body}\\
\mathbf{R}^T\underline{\dot{\vec{\omega}}}'\times\underline{\vec{r}}' &= \left[\mathbf{R}^T\underline{\dot{\vec{\omega}}}'\right]\cdot\vec{\mathbf{M}}\,\underline{\vec{r}} = \left[\mathbf{R}^T\underline{\dot{\vec{\omega}}}'\right]\times\underline{\vec{r}}
\end{aligned}$$

Finally, then, our result is

$$\begin{aligned}
\underline{\vec{F}}_{app} &= \underline{\vec{F}}_{true}\\
&\quad - m\left[\mathbf{R}^T\underline{\vec{\omega}}'\right](t)\times\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right](t)\times\underline{\vec{r}}(t)\right) - 2m\left[\mathbf{R}^T\underline{\vec{\omega}}'\right](t)\times\underline{\vec{v}}_{body}(t)\\
&\quad - m\left[\mathbf{R}^T\underline{\dot{\vec{\omega}}}'\right](t)\times\underline{\vec{r}}(t)
\end{aligned} \tag{5.4}$$

Now, on the right side, we have all the additional terms written in terms of coordinates and velocities in the rotating frame, as we wanted. For the angular velocity and acceleration, we must first obtain their coordinate representations in the non-rotating frame and then calculate their representations in the rotating frame by applying $\mathbf{R}^T$. For the true force, we may either calculate its non-rotating frame representation and transform to the rotating frame or we may calculate its rotating frame representation directly depending on which is more convenient.

The additional terms are called "fictitious forces" because they add like forces to the true force term. They are named as follows:

- **Centrifugal force**: The first term, $m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})$, is the well-known centrifugal force, pushing objects outward even when there is no apparent force acting, independent of the speed of the object in the rotating frame. The centrifugal force is essentially an implication of Newton's first law – a particle wants to continue on a straight line in an inertial frame, so it moves outward in a rotating frame under the influence of no force.
- **Coriolis force**: The second term, $2\,m\,\vec{\omega}\times\vec{v}_{body}(t)$, is the Coriolis force, which is an apparent force exerted on objects due to their velocity. The coriolis force again is an expression of Newton's first law, including the effect of motion in the noninertial frame – moving objects do not follow straight-line paths in a rotating system, so a force must be invoked to explain that.
- **Euler force**: This term is not widely discussed. It is an additional correction due to angular acceleration of the rotating frame. The easiest way to get an intuitive feel for it is to consider a particle sitting fixed to a frictionless turntable whose rotation is accelerating. If the turntable were moving at fixed angular velocity, there would need to be a force applied to cancel the fictitious centrifugal force. These forces are both radial in the rotating frame. If the turnable is accelerating, there needs to be an additional force in the tangential ($\hat{\phi}$) direction to keep the particle from slipping as the turntable is accelerated; this force would counteract the Euler force.

Correspondence to Textbooks

The following is somewhat redundant with our discussion of the distinction between the rotating and non-rotating system forms of particle velocity earlier. It is an important and subtle distinction, so it is worth reiterating.

Many texts are somewhat sloppy or misleading in their notation on this topic. For example, Hand and Finch give the general formula (Equation 7.31)

$$\left.\frac{d\vec{e}}{dt}\right|_{space} = \vec{\omega} \times \vec{e} + \left.\frac{d\vec{e}}{dt}\right|_{body}$$

where $_{space}$ indicates the time derivative should apply to motion relative to the non-rotating frame and $_{body}$ indicates the time derivative should apply to motion relative to the rotating frame. They neglect to say anything about the frame in which the coordinate representations of the vectors should be evaluated. Recall from our earlier discussion of $\vec{v}_{body}$ and $\vec{v}_{space}$ that, to write down a coordinate representation of the time derivative of a vector, one has to specify two things: a) relative to which frame the derivative is calculated, which in many texts is specified by $|_{body}$ or $|_{space}$;[6] and b) in which frame the coordinate representation is calculated, which is usually specified using primed or unprimed coordinates matching the frame. For example, to calculate the velocity of a particle, one first has to say whether one wants to know the velocity relative to the rotating frame or the non-rotating frame. These are two *different* vectors. Then one has to specify which representation one wants for the vector, the rotating frame or non-rotating frame representation. Confusion arises because the quantities $\underline{\vec{v}}'_{body}$ ($= \mathbf{R}\,\underline{\vec{v}}_{body}$, written in Hand and Finch as $\mathbf{v}|'_{body}$) and $\underline{\vec{v}}_{space}$ ($= \mathbf{R}^T\underline{\vec{v}}'_{space}$, written in Hand and Finch as $\mathbf{v}|_{space}$) are physically unmeasurable and therefore nonintuitive.

So, one can make sense of the equation by remembering that $|_{body}$ and $|_{space}$ refer to different vectors, while primes or lack of primes refer to coordinate representations. Also, $\vec{\omega}$ is *always* referenced to the space frame because it vanishes in the body frame, which is why no $|_{body}$ or $|_{space}$ modifier is needed. Primes or lack thereof do apply to $\vec{\omega}$, though, to obtain the appropriate coordinate representation (which must match that chosen for the vector whose time derivatives are being considered).

This discussion hopefully explains two statements in Hand and Finch. First, after Equation 7.31,

> Vectors $\vec{e}$ and $\vec{\omega}$ can be expressed in either the body or space coordinates when doing the calculation.

It would be more precise to say

> One can do the calculation using either the body or space-frame coordinate representations of the vectors $\vec{e}$ and $\vec{\omega}$. On the left side, we consider the change in $\vec{e}$ relative to the space frame and on the right side the change in $\vec{e}$ relative to the body frame. $\vec{\omega}$ is always measured relative to the space frame, though we must write it in the coordinate representation that matches the one chosen for $\vec{e}$.

[6] Note that these modifiers are necessary because Hand and Finch's equation contains derivatives of coordinate-free vectors, not of coordinate representations.

Second, Equations 7.32 and 7.34 of Hand and Finch are coordinate-free versions of our Equation 5.4:

$$\vec{F}_{app} = \vec{F}_{true} - m\,\vec{\omega}(t) \times [\vec{\omega}(t) \times \vec{r}(t)] - 2\,m\,\vec{\omega}(t) \times \vec{v}_{rel}(t) - m\,\dot{\vec{\omega}}(t) \times \vec{r}(t) \tag{5.5}$$

Hand and Finch leave unstated that all vectors must be evaluated using representations in the same frame, either in the non-rotating or rotating frame. Our Equation 5.4 is this equation with all terms evaluated in the rotating frame representation (and which necessitates $\mathbf{R}^T$ factors in front of $\underline{\vec{\omega}}'$ and $\underline{\dot{\vec{\omega}}}'$). We could obtain a version with all terms evaluated in the non-rotating frame representation by applying $\mathbf{R}$ to both sides.

Thornton writes a confusing equation (Equation 10.12) similar to the aforementioned Equation 7.31 of Hand and Finch:

$$\left(\frac{d}{dt}\,\vec{Q}\right)_{fixed} = \left(\frac{d}{dt}\,\vec{Q}\right)_{rotating} + \vec{\omega} \times \vec{Q}$$

Again, it is not emphasized that: a) the derivatives appearing on the left and right side are really different vectors because the derivatives are taken with reference to different frames (though the $|_{fixed}$ and $|_{rotating}$ notation implies this); and b) one must choose a coordinate system in which to evaluate the representations. Thornton makes the even more confusing statement

> We note, for example, that the angular acceleration $\dot{\vec{\omega}}$ is the same in both the fixed and rotating systems:
>
> $$\left(\frac{d}{dt}\,\vec{\omega}\right)_{fixed} = \left(\frac{d}{dt}\,\vec{\omega}\right)_{rotating} + \vec{\omega} \times \vec{\omega}$$
>
> because $\vec{\omega} \times \vec{\omega}$ vanishes and $\dot{\vec{\omega}}$ designates the common value in the two systems.

What does it mean to calculate $\dot{\vec{\omega}}$ in the rotating system? That's obviously nonsense; what is meant is that $\dot{\vec{\omega}}$ is measured relative to the fixed system, but one can calculate the representation of the vector in the rotating system and that representation is identical to its representation in the fixed system.

Goldstein does a better job of explaining the distinctions, but is still somewhat indirect about it. The text around Equations 4.82 and 4.86 is

> The time *rate of change* of the vector $\vec{G}$ as seen by two observers is then . . . :
>
> $$\left(\frac{d}{dt}\,\vec{G}\right)_{space} = \left(\frac{d}{dt}\,\vec{G}\right)_{body} + \vec{\omega} \times \vec{G}$$
>
> . . . Here the subscripts $_{space}$ and $_{body}$ indicate the time derivatives observed in the space and body (rotating) system of axes, respectively. The resultant vector equation can then of course be resolved along any desired set of axes, fixed or moving. But again note that the time rate of change is only relative to the specified coordinate system, components may be taken along another set of coordinate axes only *after* the differentiation has been carried out.

The phrase "observed in" is used where we have used "relative to" or "referenced to" and the phrase "resolved along along any desired set of axes" where we have used "coordinate representation."

Clearly, the formulae that are frequently quoted are correct, but one must know how to interpret the notation. Our notation here, while more cumbersome, is hopefully clearer. We *never* write nonphysical quantities like $\frac{d}{dt}\,\underline{\vec{Q}}\Big|_{space}$ or $\frac{d}{dt}\,\underline{\vec{Q}'}\Big|_{body}$ but rather we always leave the rotation matrices explicit, writing $\mathbf{R}^T\,\frac{d}{dt}\,\underline{\vec{Q}'}\Big|_{space}$ and $\mathbf{R}\,\frac{d}{dt}\,\underline{\vec{Q}}\Big|_{body}$.

5.2.2 Applications

Deflection of a Plumb Line and a Falling Particle

Section 7.8 of Hand and Finch does two standard examples: calculation of the deflection of a plumb line from the true gravity vector due to centrifugal force, and the deviation of the path of a falling particle from both the true gravity vector (due to both centrifugal and Coriolis forces) and from the plumb line (due to the Coriolis force that does not affect the plumb line). The deflected plumb line gives the direction of the effective gravitational acceleration vector $\vec{g}_{eff}$. Experimentally, if one performs the falling particle experiment, but uses a plumb line as the reference, the plumb line will have already been affected by centrifugal force and thus the falling particle will appear to experience only the Coriolis force deflection.

Foucault's Pendulum

A classic example of dynamics in rotating frames is Foucault's pendulum. Foucault's pendulum is a spherical pendulum (pendulum free to move in θ and ϕ, not just θ) whose plane of oscillation precesses relative to the rotating frame. We do this example. Hand and Finch use Lagrangian techniques, so we do it by a complementary Newtonian method.

Assume the pendulum is mounted at latitude λ. Our rotating coordinate system is one fixed to the rotating earth at the location of the pendulum, with x pointing east, y pointing north, and z normal to the surface (pointing outward along a radial line from the center of the earth). The polar coordinate for the pendulum will have value zero when the pendulum points along the radial line and the azimuthal coordinate will be measured from east through north. We will make the small angle approximation and also assume ω is much smaller than the pendulum natural frequency. The pendulum position is therefore

$$\begin{aligned}\underline{\vec{r}}(t) &= \hat{x}\,x + \hat{y}\,y + \hat{z}\,l\,(1 - \cos\theta)\\ &\approx \hat{x}\,x + \hat{y}\,y\end{aligned}$$

The angular velocity vector in the rotating system is $\mathbf{R}^T\underline{\vec{\omega}}' = \omega\,(\hat{y}\cos\lambda + \hat{z}\sin\lambda)$; one could obtain this by calculating $\underline{\vec{\omega}}'$ in a non-rotating frame and then applying $\mathbf{R}^T$, but it's pretty easy to just write down the result. See Figure 7.10 of Hand and Finch if the form is not obvious.

The only true forces acting on the system are gravity and the rope tension. These forces have a complicated representation in the non-rotating frame, but have the simple representation in the rotating frame $\underline{\vec{F}}_{true} = -m\,g\,\hat{z} - T\hat{r}$ where $\hat{r}$ is the unit vector for the bob position.

There will be fictitious centrifugal and Coriolis forces acting:

$$\begin{aligned}
\underline{\vec{F}}_{cen} &= -m\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\times\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\times\underline{\vec{r}}\right)\\
&= -m\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\times\left[\omega\left(\hat{y}\,\cos\lambda+\hat{z}\,\sin\lambda\right)\times\left(\hat{x}\,x+\hat{y}\,y\right)\right]\\
&= -m\,\omega^2\,\left(\hat{y}\,\cos\lambda+\hat{z}\,\sin\lambda\right)\times\left[-x\,\hat{z}\,\cos\lambda+x\,\hat{y}\,\sin\lambda-y\,\hat{x}\sin\lambda\right]\\
&= -m\,\omega^2\,\left[-x\,\hat{x}+y\,\hat{z}\,\cos\lambda\sin\lambda-y\,\hat{y}\sin^2\lambda\right]\\
\underline{\vec{F}}_{cor} &= -2\,m\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]\times\frac{d}{dt}\,\underline{\vec{r}}(t)\\
&= -2\,m\left[\omega\left(\hat{y}\,\cos\lambda+\hat{z}\,\sin\lambda\right)\right]\times\left[\left(\dot{x}\,\hat{x}+\dot{y}\,\hat{y}\right)\right]\\
&\approx 2\,m\,\omega\left(\dot{x}\hat{z}\cos\lambda-\dot{x}\,\hat{y}\sin\lambda+\dot{y}\,\hat{x}\sin\lambda\right)
\end{aligned}$$

The centrifugal term produces small changes to the restoring force. The Coriolis term produces forces transverse to the first-order motion (the $\hat{x}$ and $\hat{y}$ terms) and a small change in the tension in the pendulum. The centrifugal term is of order $\omega^2\,d$ where d is the typical displacement in the xy plane. The Coriolis term is of order $\omega\,d\,\omega_0$ where $\omega_0\gg\omega$ is the natural frequency of the pendulum. So we can drop the centrifigual term entirely. We will also drop the $\hat{z}$ Coriolis term because it causes no precession. So, the forces we will consider are:

$$\begin{aligned}
\underline{\vec{F}}_{app} &= -m\,g\,\hat{z}-T\,\hat{r}+2\,m\,\omega\,\sin\lambda\,\left(\dot{y}\,\hat{x}-\dot{x}\,\hat{y}\right)\\
&= -m\,g\,\hat{z}+T\left(-\frac{x}{l}\,\hat{x}-\frac{y}{l}\hat{y}+\hat{z}\right)+2\,m\,\omega\,\sin\lambda\,\left(\dot{y}\,\hat{x}-\dot{x}\,\hat{y}\right)
\end{aligned}$$

So we obtain three equations of motion:

$$\begin{aligned}
m\,\ddot{x} &= -T\,\frac{x}{l}+2\,m\,\dot{y}\,\omega\,\sin\lambda\\
m\,\ddot{y} &= -T\,\frac{y}{l}-2\,m\,\dot{x}\,\omega\,\sin\lambda\\
m\,\ddot{z} &= -mg+T
\end{aligned}$$

We can of course use the last equation to eliminate T, so we end up with two differential equations

$$\begin{aligned}
\ddot{x}-2\,\dot{y}\,\omega\,\sin\lambda+\frac{g}{l}\,x &= 0\\
\ddot{y}+2\,\dot{x}\,\omega\,\sin\lambda+\frac{g}{l}\,y &= 0
\end{aligned}$$

Combine the two first equations via $q=x+i\,y$ to obtain

$$\ddot{q}+2\,i\,\dot{q}\,\omega\,\sin\lambda+\frac{g}{l}\,q = 0$$

The equation is a damped simple harmonic oscillator with $Q=(2\,i\,\omega\,\sin\lambda)^{-1}$. Imaginary Q actually implies oscillation, with solution

$$\begin{aligned}
q(t) &= A\,\exp\left(-\frac{t}{2Q}\right)\exp(\pm i\,\omega_c\,t)\\
&= A\,\exp(-i\,\omega\,\sin\lambda)\exp(\pm i\,\omega_c\,t)
\end{aligned}$$

with

$$\omega_c=\sqrt{\frac{g}{l}}\sqrt{1+\omega^2\sin^2\lambda}$$

Since we assumed ω is small compared to the natural frequency of oscillation, we can approximate $\omega_c = \sqrt{\frac{g}{l}}$, the natural frequency of the pendulum. $\omega \sin\lambda$ is the angular frequency for amplitude to be traded between the x and y components, so it is the precession angular frequency. The precession period is

$$T = \frac{2\pi}{\omega \sin\lambda}$$

A bit of intuitive explanation: the pendulum's angular momentum relative to a non-rotating frame is actually fixed because there are no torques aside from gravity acting on it. An apparent torque appears because the angular momentum appears to change in the rotating frame. It is a fictitious torque, and the precession is fictitious; it is simply a kinematic result of the rotating reference frame.

5.2.3 Lagrangian and Hamiltonian Dynamics in Rotating Coordinate Systems

It may be surprising that we can do Lagrangian and Hamiltonian dynamics in rotating coordinate systems. It should not be, as rotating coordinates are just an example of generalized coordinates. The trick, as always, is to write down the energy functions in an inertial system where they are well understood and then use the transformation from the inertial coordinates to generalized coordinates to rewrite the energies in terms of coordinates in the rotating coordinate system. We do the Foucault's pendulum example. This material is discussed in Hand and Finch Section 5.8 and in the Foucault's pendulum example in Section 7.10.

Obtaining and Using the Lagrangian

The kinetic energy in the non-rotating frame is obviously

$$T = \frac{1}{2}\, m\, |\underline{\vec{v}}'_{space}|^2$$

As we know, $\underline{\vec{v}}'_{space}$ consists of an $\vec{\omega} \times \vec{r}$ term to account for the motion of the rotating system and a $\underline{\vec{v}}_{body}$ term to account for motion relative to the rotating system. Recall from the Newtonian discussion of Foucault's pendulum:

$$\left[\mathbf{R}^T \underline{\vec{\omega}}'\right] \times \underline{\vec{r}} \;=\; \omega\left[-y\,\hat{x}\sin\lambda + x\,\hat{y}\,\sin\lambda - x\,\hat{z}\,\cos\lambda\right]$$

(this form is specific to the problem at hand, where we have motion only in x and y in the rotating frame – in general, $\left[\mathbf{R}^T \underline{\vec{\omega}}'\right] \times \underline{\vec{r}}$ may have more terms). We also have

$$\underline{\vec{v}}_{body} = \dot{x}\,\hat{x} + \dot{y}\,\hat{y}$$

Thus we have

$$\begin{aligned} T &= \frac{1}{2}\, m \left(\left|\underline{\vec{v}}_{body}\right|^2 + 2\underline{\vec{v}}_{body} \cdot \left[\mathbf{R}^T \underline{\vec{\omega}}'\right] \times \underline{\vec{r}} + \left|\left[\mathbf{R}^T \underline{\vec{\omega}}'\right] \times \underline{\vec{r}}\right|^2 \right) \\ &= \frac{1}{2}\, m \left(\left(\dot{x}^2 + \dot{y}^2\right) + 2\,\omega\,\sin\lambda\,\left(-y\,\dot{x} + x\,\dot{y}\right) + \omega^2\,\sin^2\lambda\left(x^2 + y^2\right) + \omega^2\,\cos^2\lambda\, x^2 \right) \\ &\approx \frac{1}{2}\, m \left(\left(\dot{x}^2 + \dot{y}^2\right) + 2\,\omega\,\sin\lambda\,\left(-y\,\dot{x} + x\,\dot{y}\right) \right) \end{aligned}$$

where we have dropped the ω^2 terms because $\omega^2 \ll \omega_0^2 = \frac{g}{l}$ and we will have a $\omega_0^2\left(x^2 + y^2\right)$ contribution from the potential energy. The potential energy is easy ($m\,g\,l\,(1 - \cos\theta) \approx \frac{1}{2}\, m\, g\, l\, \theta^2$):

$$U = \frac{1}{2}\, m\, g\, l\, \frac{x^2 + y^2}{l^2}$$

So the Lagrangian is

$$L \approx \frac{1}{2}\, m \left(\dot{x}^2 + \dot{y}^2\right) + m\,\omega\, \sin\lambda\, \left(-y\,\dot{x} + x\,\dot{y}\right) - \frac{1}{2}\, m\, g\, l\, \frac{x^2 + y^2}{l^2}$$

The canonical momenta are

$$p_x = m\,\dot{x} - y\,\omega\, \sin\lambda \qquad\qquad p_y = m\,\dot{y} + x\,\omega\, \sin\lambda$$

The Euler-Lagrange equations are

$$\begin{aligned}
\frac{d}{dt}\left(m\,\dot{x} - y\,\omega\, \sin\lambda\right) &= \dot{y}\,\omega\, \sin\lambda - m\,g\,\frac{x}{l} \qquad &\Longrightarrow \qquad \ddot{x} + \frac{g}{l}\,x &= 2\,\dot{y}\,\omega\, \sin\lambda \\
\frac{d}{dt}\left(m\,\dot{y} + x\,\omega\, \sin\lambda\right) &= -\dot{x}\,\omega\, \sin\lambda - m\,g\,\frac{y}{l} \qquad &\Longrightarrow \qquad \ddot{y} + \frac{g}{l}\,y &= -2\,\dot{x}\,\omega \sin\lambda
\end{aligned}$$

which are the same coupled equations we found via Newtonian methods.

The Hamiltonian and Hamilton's Equations

With the canonical momenta, we may calculate the Hamiltonian

$$\begin{aligned}
H &= p_x\,\dot{x} + p_y\,\dot{y} - L \\
&\approx \frac{p_x^2 + p_y^2}{2\,m} - \omega\, \sin\lambda\, (x\,p_y - y\,p_x) + \frac{1}{2}\, m\, g\, l\, \frac{x^2 + y^2}{l^2}
\end{aligned}$$

Hamilton's equations will be

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} + y\,\omega\, \sin\lambda \qquad &\dot{p}_x &= -\frac{\partial H}{\partial x} = p_y\,\omega\, \sin\lambda - m\,g\,\frac{x}{l} \\
\dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{m} - x\,\omega\, \sin\lambda \qquad &\dot{p}_y &= -\frac{\partial H}{\partial y} = -p_x\,\omega\, \sin\lambda - m\,g\,\frac{y}{l}
\end{aligned}$$

Note that

$$H = H_{\omega=0} - l_z\,\omega\, \sin\lambda$$

Generalization

We made one approximation in the above – that the ω^2 term in the kinetic energy could be ignored – and we used a potential energy specific to the particular problem. Obviously, we could have done the analysis for any potential energy function – the canonical momenta would be unchanged. We could have also left the ω^2 terms in. Since they contain only x and y, they would actually look like potential energy terms, contributing an "effective potential energy," much like the centrifugal potential term in the central-force problem.

5.3 Rotational Dynamics of Rigid Bodies

5.3.1 Basic Formalism

First, we discuss the concepts of energy and angular momentum for a rotating rigid body and use these to determine the basic equations of rotational motion.

Review of Physical Quantities for a System of Particles

In Section 1.3, we also derived expressions for the linear (Equations 1.16 and 1.18) and angular momentum (Equations 1.22 and 1.23) and kinetic (Equations 1.24 and 1.25) and potential energies (Equations 1.28 and 1.29) of a system of particles. Our definitions and results were:

total mass	$M = \sum_a m_a$
center of mass	$\vec{R} = \frac{1}{M}\sum_a m_a \vec{r}_a$
total linear momentum	$\vec{P} = \sum_a m_a \dot{\vec{r}}_a = M\,\dot{\vec{R}}$
Newton's second law, translational motion	$\frac{d\vec{P}}{dt} = \sum_a \vec{F}_a^{(e)} = \vec{F}^{(e)}$
center-of-mass coordinates	$\vec{s}_a = \vec{r}_a - \vec{R}$
total angular momentum	$\vec{L} = \vec{R}\times\vec{P} + \sum_a \vec{s}_a \times \vec{q}_a = \vec{R}\times M\,\dot{\vec{R}} + \sum_a \vec{s}_a \times m_a\,\dot{\vec{s}}_a$
Newton's second law, rotational motion	$\dot{\vec{L}} = \sum_a \vec{r}_a \times \vec{F}_a^{(e)}$
total kinetic energy	$T = \frac{1}{2}M\,\dot{R}^2 + \sum_a \frac{1}{2}m_a\,\dot{s}_a^2 = \frac{\dot{P}^2}{2\,M} + \sum_a \frac{1}{2}m_a\,\dot{s}_a^2$
total external potential energy	$U^{(e)} = \sum_a U_a(\vec{r}_a)$
total internal potential energy	$U^{(i)} = \sum_{a,b,\,b<a} \tilde{U}_{ab}(\vec{r}_{ab})$

where $\vec{F}_a^{(e)}$ is the external force acting on the ath particle, $\vec{F}^{(e)}$ is the total external force, $U_a(\vec{r}_a)$ is the potential energy of the ath particle due to external conservative forces and $U_{ab}(\vec{r}_{ab})$ is the internal potential energy due to a strong-form central force between the ath and bth particles. Nonconservative external forces and non-central forces between particles of the system are not considered here.

Kinematics of a Rigid Body

Recall in Section 1.3 that we defined a **rigid body** to be a collection of particles with positions $\{\vec{r}_a\}$ such that the distance between particles $r_{ab} = |\vec{r}_a - \vec{r}_b|$ are constant, though their orientations may change. Assuming a rigid body permits us to conclude that the center-of-mass frame coordinates satisfy

$$\dot{\vec{s}}_a = \vec{\omega}\times\vec{s}_a$$

with a common $\vec{\omega}$ for all a. This can be seen as follows.

- If r_{ab} are all fixed, then the s_{ab} are also all fixed. This implies that $\dot{\vec{s}}_a \cdot \vec{s}_a = 0$. This is to some extent intuitively obvious – in a rigid body, all motion is only rotational. But it can be proven explicitly as follows. The center of mass is fixed, so we have

$$\begin{aligned} m_a \vec{s}_a + \sum_{b \neq a} m_b \, \vec{s}_b &= 0 \\ m_a \vec{s}_a &= -\sum_{b \neq a} m_b \, \vec{s}_b \\ \left(m_a + \sum_{b \neq a} m_b \right) \vec{s}_a &= \sum_{b \neq a} m_b \, (\vec{s}_a - \vec{s}_b) \\ \vec{s}_a &= M^{-1} \sum_{b \neq a} m_b \, \vec{s}_{ab} \end{aligned}$$

  Therefore, dotting the above equation into its time derivative, we have

$$\begin{aligned} \dot{\vec{s}}_a \cdot \vec{s}_a &= M^{-2} \sum_{b \neq a \, c \neq a} m_b \, m_c \, \dot{\vec{s}}_{ab} \cdot \vec{s}_{ac} \\ &= M^{-2} \sum_{b, c \neq a, c \leq b} m_b \, m_c \left(\dot{\vec{s}}_{ab} \cdot \vec{s}_{ac} + \dot{\vec{s}}_{ac} \cdot \vec{s}_{ab} \right) \\ &= M^{-2} \sum_{b, c \neq a, c \leq b} m_b \, m_c \frac{d}{dt} \left(\vec{s}_{ab} \cdot \vec{s}_{ac} \right) \end{aligned}$$

  For $c = b$, the quantity in parentheses is s_{ab}^2, which we know to be constant by our assumption of a rigid body. We can prove that the same term vanishes for $b \neq c$ also. Consider the triplet of particles a, b, and c with $b, c \neq a$ and $c < b$. These three particles form a triangle such that $\vec{s}_{bc} = \vec{s}_{ac} - \vec{s}_{ab}$, so

$$s_{bc}^2 = s_{ac}^2 + s_{ab}^2 - 2 \, \vec{s}_{ac} \cdot \vec{s}_{ab}$$

  All three vector magnitudes must be constant because we have assumed a rigid body. Therefore, the last term is constant. But that term is exactly the quantity whose time derivative is being taken. Hence, the right side of the above equation vanishes and $\dot{\vec{s}}_a \cdot \vec{s}_a = 0$.

- Because $\dot{\vec{s}}_a \cdot \vec{s}_a = 0$, there must be a vector $\vec{\omega}$ such that $\dot{\vec{s}}_a = \vec{\omega} \times \vec{s}_a$, with $\vec{\omega}$ the same for all a. This can be seen by considering pairs of particles a, b. Suppose that each particle had its own $\vec{\omega}$, denoted by $\vec{\omega}_a$ and $\vec{\omega}_b$. Then we have

$$\begin{aligned} 0 &= \dot{\vec{s}}_{ab} \cdot \vec{s}_{ab} = (\vec{\omega}_a \times \vec{s}_a - \vec{\omega}_b \times \vec{s}_b) \cdot (\vec{s}_a - \vec{s}_b) = -(\vec{\omega}_a \times \vec{s}_a) \cdot \vec{s}_b - (\vec{\omega}_b \times \vec{s}_b) \cdot \vec{s}_a \\ &= -\omega_{a,j} \, \epsilon_{ijk} \, s_{a,k} \, s_{b,i} - \omega_{b,j} \, \epsilon_{ijk} \, s_{b,k} \, s_{a,i} = (\omega_{a,j} - \omega_{b,j}) \, \epsilon_{ijk} s_{a,i} \, s_{b,k} \\ &= -(\vec{\omega}_a - \vec{\omega}_b) \cdot (\vec{s}_a \times \vec{s}_b) \end{aligned}$$

  Since $\vec{s}_a \times \vec{s}_b$ is arbitrary, $\vec{\omega}_a = \vec{\omega}_b$ must hold. Since it holds for any pairs, associativity ensures there is a single $\vec{\omega}$ for all a.

This is obviously a vast simplification of the dynamics – relative to the center of mass, the only dynamics is rotational motion about some angular velocity axis $\vec{\omega}$.

Energy and Angular Momentum of a Rigid Body

With the above simplification that the motion of a rigid body relative to its center of mass is purely rotational, we may obtain specific forms for the energies and angular momentum (neglecting the terms due to translational motion of the center of mass):

- **Kinetic Energy**
  We have $\dot{\vec{s}}_a = \vec{\omega} \times \vec{s}_a$, so the kinetic energy of rotation is
  $$T_{rot} = \sum_a \frac{1}{2} m_a \left(\vec{\omega} \times \vec{s}_a\right)^2 = \sum_a \frac{1}{2} m_a \left(\omega^2 s_a^2 - \left(\vec{\omega} \cdot \vec{s}_a\right)^2\right)$$
  where we have used a vector identity from Appendix A.3, which is equivalent to $\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$.
- **Potential Energy**
  Since we are dealing with a rigid body with $\vec{r}_{ab}$ fixed, the internal potential energy $U^{(i)}$ is a constant and can be ignored. The external potential energy does not simplify any further until the form of the single-particle external potential energy is specified.
- **Angular Momentum**
  Again, using $\dot{\vec{s}}_a = \vec{\omega} \times \vec{s}_a$ allows simplification:
  $$\vec{L} = \sum_a \vec{s}_a \times m_a\, \dot{\vec{s}}_a = \sum_a m_a\, \vec{s}_a \times \left(\vec{\omega} \times \vec{s}_a\right) = \sum_a m_a \left(s_a^2\, \vec{\omega} - \left(\vec{\omega} \cdot \vec{s}_a\right) \vec{s}_a\right)$$

Moment of Inertia Tensor

The above relations can be unified if we define the rank 2 **moment of inertia tensor** $\mathcal{I}$ with coordinate representation

$$\begin{aligned} I_{ij} &= \sum_a m_a \left(s_a^2 \delta_{ij} - s_{a,i}\, s_{a,j}\right) \qquad (5.6) \\ \mathcal{I} &= \sum_a m_a \left(\vec{s}_a \cdot \vec{s}_a\, \mathbf{1} - \vec{s}_a\, \vec{s}_a^{\,T}\right) \end{aligned}$$

where $\mathbf{1}$ is the rank 2 identity tensor. $\mathcal{I}$ is a rank 2 tensor because $\mathbf{1}$ is an isotropic rank 2 tensor and $\vec{s}_a$ is a vector. The coordinate representation of $\mathcal{I}$ in any given frame is simply given by writing the above formula using the coordinate representation of $\vec{s}_a$, given by $\underline{\vec{s}}_a$ or $s_{a,i}$ following our standard notation (see Section 5.1.4).

Using the above definition, the rotational kinetic energy and angular momentum can be written as

$$\begin{aligned} T_{rot} &= \frac{1}{2}\, \vec{\omega}^{\,T} \mathcal{I}\, \vec{\omega} = \frac{1}{2}\, \omega_i\, I_{ij}\, \omega_j \qquad (5.7) \\ \vec{L} &= \mathcal{I}\, \vec{\omega} \iff L_i = I_{ij}\, \omega_j \qquad (5.8) \end{aligned}$$

Clearly, these definitions ensure that T_{rot} is a scalar and $\vec{L}$ is a vector under rotational transformations, as one would expect.

We note that the moment of inertia tensor for a continuous rigid body can be written as the continuous limit of the above expression:

$$I_{ij} = \int d^3 r \rho(\vec{s}) \left(s^2 \delta_{ij} - s_i\, s_j\right)$$

For example

$$\begin{aligned} I_{xx} &= \int dx\, dy\, dx \rho(\vec{r}) \left(y^2 + z^2\right) \\ I_{xy} &= \int dx\, dy\, dx \rho(\vec{r})\, x\, y \end{aligned}$$

Principal Axes

The physical interpretation of $\mathcal{I}$ is most easily seen by obtaining the **principal axes** of the system. The coordinate representation of $\mathcal{I}$ is a symmetric matrix in any frame, so it may be diagonalized. Let us denote by F' the frame in which we initially calculate the coordinate representation of $\mathcal{I}$, denoted by $\underline{\mathcal{I}}'$. Let us use F and $\underline{\mathcal{I}}$ to refer to the frame in which $\mathcal{I}$ is automatically diagonal and its coordinate representation in that frame. The rotation matrix that diagonalizes $\underline{\mathcal{I}}'$ is the matrix that rotates from F to F'. Let the eigenvalues be I_1, I_2, and I_3 (the **principal moments** and let the eigenvectors be denoted by $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ (the **principal axes**).[7] Let $\underline{\vec{e}}_i'$ and $e'_{i,j}$ refer to the coordinate representations and components thereof of the eigenvectors in the F' frame. Then the rotation matrix $\mathbf{R}$ that transforms from F to F' is

$$(\mathbf{R})_{ij} = R_{ij} = e'_{j,i}$$

That is, $\mathbf{R}$ has as its columns the coordinate representations of the eigenvectors in the F' frame. To be absolutely clear, the coordinate representations in the two frames will be:

- Arbitrary non-principal-axis frame F':

$$\begin{aligned} I'_{ij} &= \begin{pmatrix} I'_{11} & I'_{12} & I'_{13} \\ I'_{21} & I'_{22} & I'_{23} \\ I'_{31} & I'_{32} & I'_{33} \end{pmatrix} = R_{ik}\, R_{jl}\, I_{kl} = \left(\mathbf{R}\underline{\mathcal{I}}\,\mathbf{R}^T\right)_{ij} \\ e'_{i,j} &= R_{ji} = R_{jk}\delta_{ki} = R_{jk}e_{i,k} = \left(\mathbf{R}\,\underline{\vec{e}}_i\right)_j \end{aligned}$$

- Principal-axis frame F:

$$\begin{aligned} I_{ij} &= \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = R^T_{ik}\, R^T_{jl}\, I'_{kl} = R_{ki}\, R_{lj}\, I'_{kl} = \left(\mathbf{R}^T\,\underline{\mathcal{I}}'\,\mathbf{R}\right)_{ij} \\ e_{i,j} &= \delta_{ji} = R_{kj}R_{ki} = R_{kj}e'_{i,k} = \left(\mathbf{R}^T\underline{\vec{e}}_i\right)_j \end{aligned}$$

Note: our convention for the direction of the rotation is the opposite of that in Hand and Finch: here, $\mathbf{R}$ rotates from the principal axis frame to the initial frame, while for Hand and Finch $\mathbf{R}$ rotates from the original frame to the principal axis frame. We do this for consistency with our discussion of rotating and nonrotating frames in Section 5.2 – in general, the principal axis frame F will be rotating and noninertial while there will be some inertial nonrotating reference frame F' in which the representation $\underline{\mathcal{I}}'$ is not diagonal.

Physical Significance of the Moment of Inertia Tensor

What is the moment of inertia tensor? Formally, it is a matrix that tells us how the angular velocity converts to kinetic energy, or, more importantly, the relation between

[7] The eigenvalues are positive and the eigenvectors real because the kinetic energy is nonnegative. The arguments to prove these facts are analogous to those made in Section 3.2 regarding normal modes.

angular velocity and angular momentum. Angular velocity and angular momentum may not necessarily point in the same direction if $\mathcal{I}$ has off-diagonal elements. They will only align if $\vec{\omega}$ points along a principal axis of the body.

But let's try to understand $\mathcal{I}$ at a more physical level. The inertia tensor is an analogy to mass for linear momentum. Recall that mass is essentially the conversion factor between linear velocity and linear momentum. Velocity is an entirely "kinematic" quantity, having nothing to do with the particular body under study. Momentum is a "dynamical" quantity in that it fully determines what happens in a particular physical situation. Mass is the intrinsic property of an object that converts from the kinematic velocity to the dynamical momentum.

In linear dynamics, we need only one number for the mass because of the isotropy of the physical situation consisting of empty space and a point particle: the conversion between velocity and momentum does not depend on the direction of the velocity because there is nothing yet that breaks spherical symmetry.

The moment of inertia tensor is the angular analogue of mass – given the kinematic quantity angular velocity, it converts to the dynamical quantity angular momentum. However, a generic rigid body breaks the isotropy of space because it is in general not spherically symmetric. Thus, depending on which axis of the body is given angular velocity, the conversion to angular momentum is different. This is not surprising. For example, given a long rod of length L and radius R, with $R \ll L$, if you give it angular speed ω about its long axis, it will have much less angular momentum or kinetic energy than if you give it the same angular speed about an axis perpendicular to its long axis. The rod breaks the spherical symmetry of space.

So it is now not surprising that we need something more complicated than a single scalar to represent angular inertia. If we think about it, we see that more information is needed than can be stored in a vector, too. If a rigid body has some arbitrary orientation relative to some frame F', then specifying the angular inertia of the body in F' requires six numbers: we require three numbers to specify the orientation of the body's principal axes (two numbers specify the direction of the 3-axis, one additional number specifies the az angle of the 1-axis about the 3-axis. The 2-axis's direction is then forced by requiring orthogonality to the 1- and 3-axes and a right-handed coordinate system) and we require three additional numbers to specify the inertia about those three axes (the three principal moments). Six numbers are needed, requiring something at least as complex as a second-rank tensor. We see from the way the inertia tensor was defined that it is indeed the minimally complex object that can provide the necessary six degrees of freedom, a second-rank tensor. When symmetries are present in the rigid body, the number of degrees of freedom decreases. In particular, if the body is spherical (or has the same inertia tensor as a sphere), then the spherical symmetry of space is restored. This is reflected in the fact that the three principal moments of a sphere are identical and therefore you can pick any three mutually orthogonal directions as the principal axes of a sphere. Only one number is truly needed to describe the inertia of the sphere, which is the common principal moment.

The inertia tensor can be better understood by considering the linear analogy – suppose the magnitude of the linear momentum arising from making a mass move at a certain speed depended on the direction in which the mass was moving. In general, we would again need six numbers to fully describe this "mass tensor" – the orientation of the principal axes (axes that yield linear momentum aligned with linear velocity) and the conversion factors

from speed to momentum for these three principal axes. This in fact happens for electrons and holes in some crystals because the crystal breaks the spherical symmetry of space. The "effective mass" may be large for motion in one direction and much less for another direction.

Examples

It is useful to calculate the inertia tensor of some example systems.

- A pendulum of length l with two bobs, one of mass m_1 at the end of the rigid support and one of mass m_2 halfway down. Let the system be at rest so the masses lie along the z axis. Assume the rod and bobs have zero physical extent.

  The x and y coordinates of both masses vanish. The z coordinates are l and $\frac{l}{2}$. Thus, the moment of inertia tensor in this frame is

$$I_{ij} = \begin{pmatrix} m_1\, l^2 + \frac{1}{4}\, m_2\, l^2 & 0 & 0 \\ 0 & m_1\, l^2 + \frac{1}{4}\, m_2\, l^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  This coordinate system is the principal axis system. Angular velocities about the x and y axes will result in angular momentum – these are normal pendulum motion. Angular velocity about the z axis yields no angular momentum because the rod simply rotates about its axes, leaving the bobs fixed in space.

- A sphere of radius a.

  Let's calculate first the I_{33} component. It is

$$\begin{aligned} I_{33} &= \rho \int r^2\, dr\, \sin\theta d\theta\, d\phi \left(x^2 + y^2\right) \\ &= \rho \int_0^a r^2\, dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\, r^2\, \sin^2\theta \\ &= 2\,\pi\,\rho \int_0^a r^4\, dr \int_0^\pi \left(1 - \cos^2\theta\right) \sin\theta d\theta \\ &= 2\,\pi\,\rho\, \frac{a^5}{5} \left(2 - \frac{2}{3}\right) = \frac{8\,\pi}{15}\,\rho\, a^5 = \frac{2}{5}\, M\, a^2 \end{aligned}$$

  By symmetry, I_{11} and I_{22} are the same. The off-diagonal elements vanish because the integrand is odd in any coordinate and the object is symmetric in any coordinate. So

$$I_{ij} = \frac{2}{5}\, M\, a^2 \delta_{ij}$$

  Because $\underline{\mathcal{I}}$ is proportional to the identity tensor, the coordinate representation of $\mathcal{I}$ is the same in any frame.

- A circular hoop of radius a.

  Let the hoop's rotational symmetry axis be the z axis. I_{33} will then obviously be $M\, a^2$. The calculation for the other two axes is a bit trickier; let's do I_{11}:

$$\begin{aligned} I_{11} &= \frac{M}{2\pi a} \int_0^{2\pi} a\, d\phi \left(y^2 + z^2\right) = \frac{M}{2\pi} \int_0^{2\pi} d\phi a^2\, \sin^2\phi = \frac{M\, a^2}{2\pi} \frac{1}{2} \int_0^{2\pi} (1 - \cos 2\phi) \\ &= \frac{1}{2}\, M\, a^2 \end{aligned}$$

I_{22} will be the same by symmetry. The off-diagonal components vanish as for the sphere, because the object is symmetric in each coordinate axis. Thus

$$I_{ij} = \begin{pmatrix} \frac{1}{2} M a^2 & 0 & 0 \\ 0 & \frac{1}{2} M a^2 & 0 \\ 0 & 0 & M a^2 \end{pmatrix}$$

- A **symmetric top** is an object for which two of the principal moments are equal, $I_1 = I_2 \neq I_3$ for example. A **prolate ellipsoid** is a symmetric top with $I_3 < I_1 = I_2$, indicating it is thinner about the 3 axis than about the other two, like a football. An **oblate ellipsoid** is the converse, with $I_3 > I_1 = I_2$ and is fatter about the 3 axis than the other two, like a squashed sphere. Symmetric tops need not be ellipsoidal, but their behavior is always the same as one of these ellipsoids because the principal moments are the same. Finally, a **rotator** has one vanishing moment of inertia and thereby only rotates in a plane. This is typical of molecules.

Displaced Axis Theorem

The **displaced axis theorem** provides us with a method to calculate the moment of inertia tensor relative to an origin that is not the center of mass. Why would we want to do this when we already have a description of the kinematics that nicely separates the motion of the center of mass and rigid body motion about the center of mass? In some cases, if the center of mass is moving noninertially, it is easier to obtain equations of motion for rotation about the displaced axis than about the center of mass. For example, a cylinder, sphere, or hoop rolling on an inclined plane. The center of mass accelerates, so we will have trouble applying Newton's laws in the accelerating frame. Another case is when a system consists of multiple rigidly connected objects, each of which has an easy-to-calculate moment of inertia tensor about its own center of mass, which we can then convert to a moment of inertia about the center of mass of the ensemble system.

The theorem states that, if the moment of inertia tensor about the center of mass has coordinate representation $\underline{\mathcal{I}}'$ in some particular choice of frame F', then, in a frame F'' that is translated by a constant vector $\vec{a}$ from F', with coordinate representation a'_i in F', the moment of inertia tensor has coordinate representation

$$I''_{ij} = I'_{ij} + M \left(a'^2 \delta_{ij} - a'_i \, a'_j \right)$$

The theorem can be seen to be true by realizing that motion about the new origin consists of the original rotational motion plus translational motion of a pointlike object of mass M at position $-\underline{\vec{a}}'$ relative to the F'' origin. It is trivial to calculate the moment of inertia of a point mass; it is the added term. Adding this new term linearly to the original term is justified by the fact that the angular momentum breaks down linearly into translational and rotational pieces, and that the original coordinate representation $\underline{\mathcal{I}}'$ provides the rotational piece and the new term provides the translational piece.

A dumbbell example is provided in Hand and Finch Section 8.2. An example of the first type, rolling motion, is provided by considering a rolling sphere. Let the x and y axes be in the plane in which the sphere is rolling. The instantaneous axis of rotation is the point of contact of the sphere with the plane, perpendicular to the direction of motion of the sphere. The axis is thus displaced by $-a\,\hat{z}$ from the sphere's center of mass. The moment of inertia

tensor is therefore

$$I''_{ij} = I'_{ij} + M\left(a^2\delta_{ij} - a^2\delta_{i3}\,\delta_{j3}\right) = \begin{pmatrix} \frac{7}{5}\,M\,a^2 & 0 & 0 \\ 0 & \frac{7}{5}\,M\,a^2 & 0 \\ 0 & 0 & \frac{2}{5}\,M\,a^2 \end{pmatrix}$$

Since the x and y axes are the axes about which the rolling motion can occur, the x and y principal moments increase. Rotation about the z axis is unchanged since it is the direction perpendicular to the plane in which rolling motion occurs.

Euler Angles

As we have explained before, the orthonormality conditions on rotation matrices imply that three parameters determine uniquely a rotation matrix. One standard parameterization in use is Euler angles.

The **Euler angles** are three angles that describe how to obtain the rotated coordinate system from the original one. They are illustrated in the Figure 5.1. Briefly, the three angles consist of a rotation ϕ about the z axis, a rotation θ about the new x-axis, and a rotation ψ about the new z axis.

As usual, we have two coordinate systems, body (F) and space (F'). We will obtain F from F' by the above three rotations. By our usual convention, the matrix $\mathbf{R}$ describing the full rotation will convert coordinate representations from the body system to the space system

$$\underline{\vec{r}}' = \mathbf{R}\,\underline{\vec{r}}$$

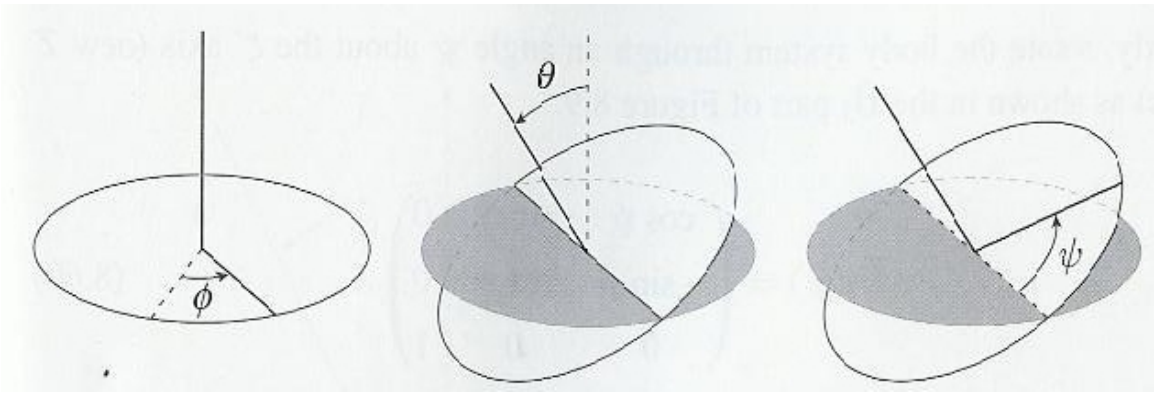
Our three steps are explicitly:

1. ϕ
   Define a system F_1 that is rotated by an angle ϕ about $\hat{z}'$ from F'. The rotation matrix that converts vectors from F_1 to F' is

   $$\mathbf{R}_1(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
   $$\underline{\vec{r}}' = \mathbf{R}_1(\phi)\,\underline{\vec{r}}_1$$

   where $\underline{\vec{r}}_1$ is the coordinate representation of $\vec{r}$ in F_1. As a check, a point that is along the x_1 axis (coordinate representation $\underline{\vec{r}}_1 = (a, 0, 0)$) will by the above relations have coordinate representation $\underline{\vec{r}}' = (a\,\cos\phi, a\,\sin\phi, 0)$, as expected. Note that the sign of ϕ is perhaps opposite of what one might expect: a rotation of F_1 by $-\phi$ about $\hat{z} = \hat{z}_1$ gives F'. This unfortunate nonintuitive sign arises because our convention says that $\mathbf{R}$ rotates from the rotating frame to the rotating frame, but ϕ is conventially defined in the opposite direction.
2. θ
   Define a system F_2 that is rotated by an angle θ about $\hat{x}_1$ *relative to* F_1. Note that $\hat{x}_1$ is, by our notation, the unit vector along the x axis of the F_1 system. The rotation matrix for this rotation alone is

   $$\mathbf{R}_2(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$
   $$\underline{\vec{r}}_1 = \mathbf{R}_2(\theta)\,\underline{\vec{r}}_2$$



$\mathbf{R}_1^T(\phi)$ $\mathbf{R}_2^T(\theta)$ $\mathbf{R}_3^T(\psi)$

Figure 5.1: Hand and Finch Figure 8.9, definition of the Euler angles, shown sequentially.

The form can be checked by considering a point along the y_2 axis with F_2 coordinate representation $\vec{r}_2 = (0, a, 0)$. By the above rotation matrix, its coordinate representation in the F_1 frame is $\vec{r}_1 = (0, a\cos\theta, a\sin\theta)$ as one would expect. We have the same nonintuitive sign as for $\mathbf{R}_1$.

3. ψ

   Define a system F that is rotated by an angle ψ about $\hat{z}_2$, where $\hat{z}_2$ is the z axis of the F_2 system. The rotation matrix for this rotation has a similar form to $\mathbf{R}_1$:

$$\begin{aligned} \mathbf{R}_3(\psi) &= \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \vec{r}_2 &= \mathbf{R}_3(\psi)\,\vec{r} \end{aligned}$$

   The check is the same as the check of $\mathbf{R}_1$. We have the same nonintuitive sign as for $\mathbf{R}_1$.

The overall rotation matrix for going from F to F' is just the product of the above three matrices, in the appropriate order:

$$\begin{aligned} \mathbf{R}(\phi,\theta,\psi) &= \mathbf{R}_1(\phi)\,\mathbf{R}_2(\theta)\,\mathbf{R}_3(\psi) \\ &= \begin{pmatrix} c_\psi c_\phi - c_\theta s_\phi s_\psi & -s_\psi c_\phi - c_\theta s_\phi c_\psi & s_\theta s_\phi \\ c_\psi s_\phi + c_\theta c_\phi s_\psi & -s_\phi s_\psi + c_\theta c_\phi c_\psi & -s_\theta c_\phi \\ s_\theta s_\psi & s_\theta c_\psi & c_\theta \end{pmatrix} \\ \vec{r}\,' &= \mathbf{R}(\phi,\theta,\psi)\,\vec{r} \end{aligned} \tag{5.9}$$

where $c_\psi = \cos\psi$, $s_\psi = \sin\psi$, etc. is a convenient shorthand. Do not throw up your hands in frustration at this matrix – it actually has a very sensible form when you look at it. First, the z components are related by a simple θ rotation because there is only one rotation that does not hold the z axis fixed. Second, the xz and yz terms are relatively simple because they arise from only two of the rotations. The remaining terms are more complicated, but do follow a pattern. The Euler angles are summarized in Figure 5.2.

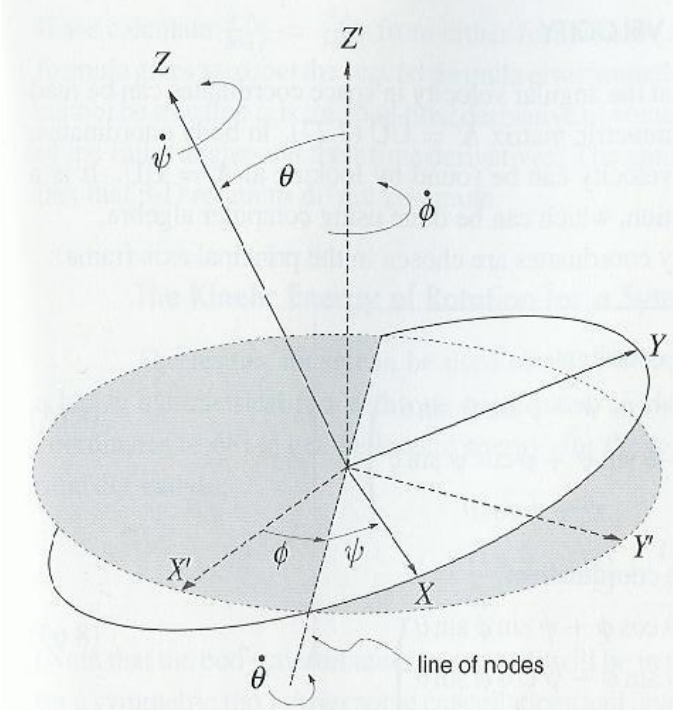


Figure 5.2: Hand and Finch Figure 8.10, definition of the Euler angles, shown together.

Relation of Euler Form to Single-Axis Rotation

A nice interpretation of $\mathbf{R}$ is found by realizing that, in spite of its complicated form, its physical meaning is that of a rotation about a single axis; that is, $\mathbf{R}(\phi, \theta, \psi)$ is related by a similarity transformation to a simple rotation about the z-axis:

$$\mathbf{R}(\phi, \theta, \psi) = \mathbf{C} \begin{pmatrix} \cos\Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{C}^T$$

We to some extent expect this result – we proved earlier that, for a rigid body, all motion about the center of mass consists of rotation about a common angular velocity vector $\vec{\omega}$ – but this result is more generic, being true for arbitrary rotation matrices in $N = 3$ dimensions. A somewhat vague proof is given in Hand and Finch; we attempt to give a more rigorous one here. It requires a bit of math, though. Working step-by-step:

1. **Orthogonal matrices have unit determinant**
   The defining relation of orthogonal matrices is

   $$\mathbf{R}\,\mathbf{R}^T = \mathbf{1}$$

   Taking the determinant of both sides and using the facts $|\mathbf{A}\,\mathbf{B}| = |\mathbf{A}||\mathbf{B}|$ and $|\mathbf{A}^T| = |\mathbf{A}|$, we obtain

   $$|\mathbf{R}| = |\mathbf{R}^T| = \pm 1$$

Now, all rotation matrices must have only the + sign. This can be seen quickly as follows. From our discussion of infinitesimal rotations in terms of the generator matrices $\vec{\mathbf{M}}$, one can see explicitly that an infinitesimal rotation about any single coordinate axis has positive determinant (to linear order in the angle $\delta\theta$ – the corrections are $\mathcal{O}(\delta\theta)^2$), and thus determinant +1. Since infinitesimal rotations commute, one can construct an infinitesimal rotation about any direction from the product of three such rotations; such an infinitesimal rotation must therefore also have determinant +1. It holds that any finite rotation can be built up as an infinite product of infinitesimal rotations, and also that the determinant of a product is the product of the determinants. Since each infinitesimal rotation has determinant +1 to linear order, the product must also have determinant +1 to linear order. Hence, any rotation matrix must have determinant +1.

2. **Orthogonal matrices satisfy $|\mathbf{A} - \mathbf{1}| = 0$**
   Next, consider the mathematical identity for orthogonal matrices

$$(\mathbf{R} - \mathbf{1})\,\mathbf{R}^T = \mathbf{1} - \mathbf{R}^T$$

   Taking the determinant of both sides and using the aforementioned properties of determinants, we obtain

$$|\mathbf{R} - \mathbf{1}| = |\mathbf{1} - \mathbf{R}^T|$$

   Now, it holds that the determinant of any matrix is the same as that of its transpose (because one can calculate the determinant by taking minors over a row or a column), so we also have

$$|\mathbf{1} - \mathbf{R}^T| = |\mathbf{1} - \mathbf{R}|$$

   Therefore,

$$|\mathbf{R} - \mathbf{1}| = |\mathbf{1} - \mathbf{R}|$$

   That is, the determinant of $\mathbf{R} - \mathbf{1}$ is equal to the determinant of its negative. But, for an arbitrary matrix $\mathbf{A}$, it holds

$$|-\mathbf{A}| = (-1)^N |\mathbf{A}|$$

   where N is the dimensionality of the space, $N = 3$. Thus, $|\mathbf{R} - \mathbf{1}|$ is equal to the negative of itself, which is only possible if it vanishes. Thus, we know

$$|\mathbf{R} - \mathbf{1}| = 0$$

3. **Orthgonal matrices in $N = 3$ dimensions have exactly one eigenvector with eigenvalue = 1**
   Finally, let us look for eigenvalues and eigenvectors of $\mathbf{R}$. As you well know, if $\vec{a}$ is an eigenvector of $\mathbf{R}$ with eigenvalue λ, then

$$\mathbf{R}\,\vec{a} = \lambda\,\vec{a} \Longleftrightarrow (\mathbf{R} - \lambda\mathbf{1})\,\vec{a} = 0$$

   Nontrivial solutions $\vec{a}$ are found only if there exist solutions λ to the secular equation

$$|\mathbf{R} - \lambda\mathbf{1}| = 0$$

Our proof that $|\mathbf{R} - \mathbf{1}| = 0$ shows that there is at least one solution, $\lambda = 1$. Since $\mathbf{R}$ and the eigenvalue are real, the eigenvector must be real also. Thus, we are assured that there is some vector $\vec{a}$ such that $\mathbf{R}\,\vec{a} = \vec{a}$.

More generally, in $N = 3$, since the determinant of $\mathbf{R}$ has to be 1 and the determinant is the product of all the eigenvalues, one infers that any remaining eigenvalues come in complex conjugate pairs. Therefore, unless $\mathbf{R} = \mathbf{1}$, there is *exactly one* eigenvalue $+1$ and one corresponding eigenvector.

4. **Interpretation of result: orthogonal matrices in $N = 3$ dimensions always look like a rotation about an axis**

   The statement that there is an eigenvector with unit eigenvalue is equivalent to the statement that $\mathbf{R}$ is a rotation about $\vec{a}$ because $\mathbf{R}$ leaves any vector that is a multiple of $\vec{a}$ unchanged. The uniqueness of the eigenvector implies that the rotation axis is unique.

   Note how the proof depended on the fact that the number of dimensions N is odd. Also, the proof that there is a unique axis of rotation relied on $N = 3$. As a check, we note that there is no eigenvector for rotations in $N = 2$ space – all vectors in the 2-dimensional plane change under any rotation.

5. **Rewriting the Euler-angle matrix in terms of the single-rotation matrix**

   The statement that a rotation matrix **always** looks like a single rotation about an axis is interesting, but not yet computationally useful. The statement does not result in a much simplified rotation matrix unless $\vec{a}$ is already aligned with one of the axes of the coordinate system in which we have specified $\mathbf{R}$. The generic simplification is that $\mathbf{R}$ is always only a similarity transform away from a single-rotation matrix.

   We are assured that there is at least one rotation matrix $\mathbf{C}$ that rotates to F' from a system in which $\vec{a}$ is the z axis; it is the matrix $\mathbf{C}$ that solves the linear set of equations

$$\mathbf{C} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \vec{a}$$

   Note that $\mathbf{C}$ is not completely specified since the above provides only two independent equations for the three free parameters of an orthogonal matrix. (The third equation is related to the other two through the orthonormality condition on $\mathbf{C}$.) Regardless, there exists at least one $\mathbf{C}$.

   Our Euler-angle rotation matrix provides the relationship

$$\underline{\vec{r}}\,' = \mathbf{R}(\phi, \theta, \psi)\, \underline{\vec{r}}$$

   We usually think of this equation as relating the coordinate representations in two different frames of a single vector. But we have discussed many times how this is equivalent to saying that $\mathbf{R}$ actually rotates a vector so that its coordinate representation *in a single frame* changes from $\underline{\vec{r}}$ to $\underline{\vec{r}}\,'$. Viewed that way, $\underline{\vec{r}}$ and $\underline{\vec{r}}\,'$ are different vectors in the same frame rather than the same vector in different frames.[8] Using this alternative viewpoint, we may then rewrite $\underline{\vec{r}}\,'$ and $\underline{\vec{r}}$ using $\mathbf{C}$:

$$\mathbf{C}\, \underline{\vec{r}}\,'_{\vec{a}} = \mathbf{R}(\phi, \theta\, \psi)\, \mathbf{C}\, \underline{\vec{r}}_{\vec{a}}$$

[8] Note that we will momentarily subvert our notation here; normally, $\underline{\vec{r}}$ and $\underline{\vec{r}}\,'$ denote representations of the same vector $\vec{r}$ in different frames F and F'. Here, they denote representations in the same frame of different vectors $\vec{r}$ and $\vec{r}\,'$.

where the $_{\vec{a}}$ subscript indicates the coordinate representation of the vector in the frame in which the rotation axis $\vec{a}$ is the z axis. Applying $\mathbf{C}^T$ on the left, we have

$$\underline{\vec{r}}'_{\vec{a}} = \mathbf{C}^T \mathbf{R}(\phi, \theta\, \psi)\, \mathbf{C}\, \underline{\vec{r}}_{\vec{a}}$$

But we know by the definition of the $\vec{a}$ frame that the representations $\underline{\vec{r}}_{\vec{a}}$ and $\underline{\vec{r}}'_{\vec{a}}$ in this frame are related by the simple rotation form

$$\begin{aligned} \underline{\vec{r}}'_{\vec{a}} &= \mathbf{R}_{\vec{a}}\, \underline{\vec{r}}_{\vec{a}} \\ \mathbf{R}_{\vec{a}} &= \begin{pmatrix} \cos\Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore, we have

$$\mathbf{R}(\phi, \theta, \psi) = \mathbf{C}\, \mathbf{R}_{\vec{a}}\, \mathbf{C}^T$$

Now that we have the relation between $\mathbf{R}(\phi, \theta, \psi)$ and a single rotation through an angle Φ, we want to know how the angles themselves are related. This is found by making use of the fact that the trace of a product of matrices is cyclic[9], which allows us to see

$$\mathrm{Tr}[\mathbf{R}_{\vec{a}}] = \mathrm{Tr}[\mathbf{C}\, \mathbf{R}(\phi, \theta, \psi)\, \mathbf{C}^T] = \mathrm{Tr}[\mathbf{C}^T \mathbf{C}\, \mathbf{R}(\phi, \theta, \psi)] = \mathrm{Tr}[\mathbf{R}(\phi, \theta, \psi)]$$

which is, explicitly

$$\begin{aligned} 1 + 2\, \cos\Phi &= (1 + \cos\theta) \cos(\phi + \psi) + \cos\theta \\ \cos\left(\frac{\Phi}{2}\right) &= \cos\left(\frac{\phi + \psi}{2}\right) \cos\left(\frac{\theta}{2}\right) \end{aligned}$$

Thus, we are able to obtain the actual rotation angle from the Euler angles.

Euler Angles and Angular Velocity

How are the rate of change of Euler angles related to angular velocity? We will of course need to know this to make much use of Euler angles.

Recall our earlier result (Section 5.2) that, if $\mathbf{R}$ is the rotation matrix that transforms from the body frame F to the space frame F', then the angular velocity cross product operator $\vec{\omega}\times = \vec{\omega} \cdot \vec{\mathbf{M}}$ has coordinate representations in the two frames

$$\begin{aligned} \underline{\left(\vec{\omega} \cdot \vec{\mathbf{M}}\right)'} &= \left(\underline{\vec{\omega}}' \cdot \vec{\mathbf{M}}\right) = \dot{\mathbf{R}}\, \mathbf{R}^T \\ \underline{\left(\vec{\omega} \cdot \vec{\mathbf{M}}\right)} &= \mathbf{R}^T \underline{\left(\vec{\omega} \cdot \vec{\mathbf{M}}\right)'} \mathbf{R} = \mathbf{R}^T \left(\dot{\mathbf{R}}\, \mathbf{R}^T\right) \mathbf{R} = \mathbf{R}^T\, \dot{\mathbf{R}} \end{aligned}$$

In general, $\left(\vec{a} \cdot \vec{\mathbf{M}}\right)_{jk} = -a_i\, \epsilon_{ijk}$, so one can recover ω_i or ω'_i from the above by just reading off components or using $a_i = -\frac{1}{2}\, \epsilon_{ijk} \left(\vec{a} \cdot \vec{\mathbf{M}}\right)_{jk}$ or, equivalently, $\vec{a} = \frac{1}{2}\, \vec{\mathbf{M}} \left(\vec{a} \cdot \vec{\mathbf{M}}\right)$.[10] Obviously, the calculation is somewhat painful. The result is

$$\underline{\vec{\omega}}' = \begin{pmatrix} \dot{\theta}\, c_\phi + \dot{\psi}\, s_\phi\, s_\theta \\ \dot{\theta}\, s_\phi - \dot{\psi}\, c_\phi\, s_\theta \\ \dot{\phi} + \dot{\psi}\, c_\theta \end{pmatrix} \qquad \mathbf{R}^T \underline{\vec{\omega}}' = \begin{pmatrix} \dot{\theta}\, c_\psi + \dot{\phi}\, s_\psi\, s_\theta \\ -\dot{\theta}\, s_\psi + \dot{\phi}\, c_\psi\, s_\theta \\ \dot{\psi} + \dot{\phi}\, c_\theta \end{pmatrix} \tag{5.10}$$

[9] This is easy to see: the trace of a matrix $\mathbf{A}$ is $\mathrm{Tr}[\mathbf{A}] = A_{ii}$, so the trace of a product of three matrices $\mathbf{A}\,\mathbf{B}\,\mathbf{C}$ is $\mathrm{Tr}[\mathbf{A}\,\mathbf{B}\,\mathbf{C}] = A_{ij}\, B_{jk}\, C_{ki}$, which clearly can be rewritten as $C_{ki}\, A_{ij}\, B_{jk} = \mathrm{Tr}[\mathbf{C}\,\mathbf{A}\,\mathbf{B}]$ or $B_{jk}\, C_{ki}\, A_{ij} = \mathrm{Tr}[\mathbf{B}\,\mathbf{C}\,\mathbf{A}]$.

[10] This is seen by recognizing $\epsilon_{ijk}\, \epsilon_{ljk} = 2\, \delta_{il}$ and therefore $-\frac{1}{2}\, \epsilon_{ijk} \left(\vec{a} \cdot \vec{\mathbf{M}}\right)_{jk} = \frac{1}{2}\, \epsilon_{ijk}\, \epsilon_{ljk}\, a_l = \delta_{il}\, a_l = a_i$.

(Note that we, as before, write $\mathbf{R}^T\underline{\vec{\omega}}'$ rather than the misleading symbol $\underline{\vec{\omega}}$; see below for further discussion.) That the above makes sense can be seen by looking at a diagram showing the axes about which ϕ, θ, and ψ are defined and projecting these axes onto the F' or F axes. The projections are reasonably obvious. Such a diagram, taken from Hand and Finch, is provided in Figure 5.2.

As noted above, one will often see $\underline{\vec{\omega}}$ written instead of $\mathbf{R}^T\underline{\vec{\omega}}'$. As explained in Section 5.2, we avoid use of the symbol $\underline{\vec{\omega}}$ because it suggests an angular velocity measured in the rotating system, which should vanish. Using the notation $\mathbf{R}^T\underline{\vec{\omega}}'$ emphasizes that the object under discussion is the rotating-frame representation of the angular velocity measured in the nonrotating frame, whose representation in the nonrotating frame is $\underline{\vec{\omega}}'$.

Integrability and Euler Angles

Is it possible to obtain from $\vec{\omega}(t)$ the full time evolution of the system? That is, is $\vec{\omega}(t)$ an integrable function that can give us $\phi(t)$, $\theta(t)$, and $\psi(t)$? The answer is no, and we can see this by assuming it is possible and obtaining a contradiction.

Suppose $\underline{\vec{\omega}}'$ were integrable. Then it would be the total time derivative of some function $\underline{\vec{\Lambda}}'(\phi,\theta,\psi)$. Generically, we would then be able to write using the chain rule

$$\underline{\vec{\omega}}' = \frac{d}{dt}\,\underline{\vec{\Lambda}}' = \frac{\partial\underline{\vec{\Lambda}}'}{\partial\phi}\,\dot{\phi} + \frac{\partial\underline{\vec{\Lambda}}'}{\partial\theta}\,\dot{\theta} + \frac{\partial\underline{\vec{\Lambda}}'}{\partial\psi}\,\dot{\psi}$$

This expression must equal our previous expression for $\underline{\vec{\omega}}'$, so, matching up coefficients of the independent time derivatives in each angle, we obtain

$$\frac{\partial\underline{\vec{\Lambda}}'}{\partial\phi} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \frac{\partial\underline{\vec{\Lambda}}'}{\partial\theta} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \qquad \frac{\partial\underline{\vec{\Lambda}}'}{\partial\psi} = \begin{pmatrix} \sin\phi\,\sin\theta \\ -\cos\phi\,\sin\theta \\ \cos\theta \end{pmatrix}$$

Now, if we require that mixed partial derivatives commute in the second partial derivatives, we will obtain contradictions. For example,

$$\frac{\partial\,\Lambda'_x}{\partial\,\psi\,\partial\,\phi} = 0 \qquad \frac{\partial\,\Lambda'_x}{\partial\,\phi\,\partial\,\psi} = \cos\phi\,\sin\theta$$

So, $\underline{\vec{\omega}}'$ (and hence $\vec{\omega}$) is not integrable.

This lack of integrability arises, in some sense, because of the noncommutativity of 3-dimensional rotations. We can see this by counterexample. As above, suppose $\vec{\omega}$ is integrable and is the total time derivative of a function $\vec{\Lambda}$. $\vec{\Lambda}$ is a vector representing the orientation of the rigid body (as a function of time). Now, we know angular velocities add as vectors (we proved this in Section 5.1), and we know integration is a linear operation, so orientation vectors $\vec{\Lambda}$ must add linearly as vectors. Such addition is a commutative operation, indicating that two rotations obtained by integrating two angular velocity vectors would commute. But we know this is not true: rotations about different axes do not commute.

Another, related way of looking at it is that the orientation of a rigid body is, by dint of it being given by the moment of inertia tensor, a rank 2 tensor, not a vector. Any rigid body has three principal axes. To specify how the rigid body is oriented in a given frame, we must specify the orientation of not one, but two, of these three body axes. More information than is available in a vector is required.

Rotational Kinetic Energy for a Symmetric Top in Terms of Euler Angles

For a symmetric top in its principal axis (body) frame, we have that the moment of inertia tensor is diagonal and the first two principal components are equal, giving for the kinetic energy

$$T = \frac{1}{2} I_1 \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2$$

(We succumb to convention and use ω_i for the component $\left[\mathbf{R}^T \underline{\vec{\omega}}'\right]_i$ for the sake of legibility.) The body frame angular velocity was given in the previous section. Because we get to add ω_1^2 and ω_2^2, the expression greatly simplifies (though use of $\cos^2 + \sin^2 = 1$ identities), giving

$$T = \frac{1}{2} I_1 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 \tag{5.11}$$

Since T is a scalar, the above form is valid for the kinetic energy in any frame whose origin is the center of mass. If the center of mass is displaced from the origin, an additional translational term needs to be added.

One explanatory note: One tends to get confused about how one can specify a kinetic energy of rotation in the body frame since the body does not rotate in the body frame. This is the recurring nonintuitive fact that, when we specify $\vec{\omega}$ in the body frame representation, we are simply transforming the $\vec{\omega}$ representation in the inertial frame F' to the body frame F. We are *not* measuring the angular velocity with respect to F; that would of course vanish. Similarly, just because we calculate T from body-frame representations $\underline{\mathcal{I}}$ and $\mathbf{R}^T\underline{\vec{\omega}}'$ doesn't mean that T is a kinetic energy due to motion relative to the body frame. In fact, since the coordinate-free vector $\vec{\omega}$ is the angular velocity of the rigid body relative to the inertial frame F', T must be the kinetic energy as measured in that inertial frame. This makes sense, as T is a quantity that is inherently only measurable in the space frame. We are free to calculate it in the body frame representation because we know to transform its constituent tensors to the body frame representation.

5.3.2 Torque-Free Motion

Now that we have the formalism in hand to calculate the energy and angular momentum of a rigid body in a rotating frame, we may apply it to a number of standard problems. We begin with torque-free motion so that we may better understand the relationship between angular velocity and angular momentum.

The Euler angles are not necessary for torque-free motion because the angular momentum vector is constant in the space frame. That is, in symmetric top problems, there are three vectors we need to consider: the angular momentum vector, the angular velocity vector, and the symmetry axis of the top. If all three are free to move, then the Euler angles provide a convenient parameterization. But if one is fixed, then one of the Euler angles can be fixed to be zero, in which case the Euler angle matrix will simplify greatly. So there's no point in starting with the complicated full matrix. We will find the Euler angle matrix useful in the problem of a symmetric top subject to a torque.

We will continue to use $\mathbf{R}^T\underline{\vec{\omega}}'$ instead of $\underline{\vec{\omega}}$, but we will use use ω_i for the component $\left[\mathbf{R}^T\underline{\vec{\omega}}'\right]_i$ for the sake of legibility. We run into the same issue with other angular velocity vectors (spin angular velocity $\vec{\Omega}$, precession angular velocity $\vec{\omega}_P$) and the angular momentum $\vec{L}$, and we deal with them notationally in the same way.

Generic Torque-Free Motion: Euler's Equations

Our goal in this section is to obtain enough information to give the time evolution of the orientation of the rigid body in the space frame. We shall do this by first obtaining the time evolution of the body-frame representation of the angular velocity vector, $\mathbf{R}^T\underline{\vec{\omega}}'(t)$, and then obtaining the rigid body orientation in the space frame from it.

Our starting point for discussing torque-free motion is the requirement that the total angular momentum be conserved in the nonrotating frame F' (the nonrotating frame because it is the inertial one):

$$\frac{d}{dt}\underline{\vec{L}}' = 0$$

We may obtain the rotating frame version $\frac{d}{dt}\left[\mathbf{R}^T\underline{\vec{L}}'\right]$ using our general results, Equations 5.1 and 5.2.

$$\frac{d}{dt}\underline{\vec{L}}' = \underline{\vec{\omega}}' \times \underline{\vec{L}}' + \mathbf{R}\,\frac{d}{dt}\left[\mathbf{R}^T\underline{\vec{L}}'\right]$$

Recall that $\frac{d}{dt}\underline{\vec{L}}'$ refers to a time derivative with respect to the space frame because $\underline{\vec{L}}'$ is the representation in the space frame, and similarly $\frac{d}{dt}\left[\mathbf{R}^T\underline{\vec{L}}'\right]$ refers to a time derivative with respect to the body frame because $\mathbf{R}^T\underline{\vec{L}}'$ is the representation in the body frame.[11] As we did in Section 5.2, we may apply $\mathbf{R}^T$ from the left, which we know has the generic effect of converting F' quantities to F quantities.[12] This gives

$$\begin{aligned} 0 &= \left[\mathbf{R}^T\underline{\vec{\omega}}'\right] \times \left[\mathbf{R}^T\underline{\vec{L}}'\right] + \frac{d}{dt}\left[\mathbf{R}^T\underline{\vec{L}}'\right] \\ \frac{d}{dt}\left[\mathbf{R}^T\underline{\vec{L}}'\right] &= -\left[\mathbf{R}^T\underline{\vec{\omega}}'\right] \times \left[\mathbf{R}^T\underline{\vec{L}}'\right] \end{aligned} \tag{5.12}$$

where we have made use of conservation of angular momentum in the nonrotating frame. One obvious interpretation of the above is that there is a fictitious torque in the body frame that is analogous to the Coriolis force. to better understand the nature of this torque, let us first reduce $\mathbf{R}^T\underline{\vec{L}}'$:

$$\mathbf{R}^T\underline{\vec{L}}' = \mathbf{R}^T\left[\underline{\mathcal{I}}'\underline{\vec{\omega}}'\right] = \mathbf{R}^T\underline{\mathcal{I}}'\mathbf{R}\,\mathbf{R}^T\underline{\vec{\omega}}' = \underline{\mathcal{I}}\,\mathbf{R}^T\,\underline{\vec{\omega}}'$$

where we have transformed the space-frame representation $\underline{\mathcal{I}}'$ to the body-frame representation $\underline{\mathcal{I}}$ in the usual way. The body frame is the principal axis frame, so $\underline{\mathcal{I}}$ is diagonal, giving

$$\mathbf{R}^T\underline{\vec{L}}' = \begin{pmatrix} I_1\,\omega_1 \\ I_2\,\omega_2 \\ I_3\,\omega_3 \end{pmatrix}$$

[11] Recalling some general points made in Section 5.2: If we want to transform $\frac{d}{dt}\underline{\vec{L}}'$ to the body frame, we would write $\mathbf{R}^T\frac{d}{dt}\underline{\vec{L}}'$, and similarly $\mathbf{R}\,\frac{d}{dt}\left[\mathbf{R}^T\underline{\vec{L}}'\right]$ transforms $\frac{d}{dt}\left[\mathbf{R}^T\underline{\vec{L}}'\right]$ to the space frame. We never put $|_{space}$ and $|_{body}$ labels on derivatives of coordinate representations because the derivative of a coordinate representation *can only be* the derivative with respect to that particular frame.

[12] Quickly: $\mathbf{R}^T\left[\underline{\vec{\omega}}' \times \underline{\vec{L}}'\right] = \mathbf{R}^T\left(\underline{\vec{\omega}}' \cdot \vec{\mathbf{M}}\right)\underline{\vec{L}}' = \mathbf{R}^T\mathbf{R}\left(\left[\mathbf{R}^T\underline{\vec{\omega}}'\right] \cdot \vec{\mathbf{M}}\right)\mathbf{R}^T\underline{\vec{L}}' = \left[\mathbf{R}^T\underline{\vec{\omega}}'\right] \times \left[\mathbf{R}^T\underline{\vec{L}}'\right]$.

where we write ω_i for the components of the coordinate representation $\mathbf{R}^T\underline{\vec{\omega}}'$ for the sake of brevity. Therefore,

$$\begin{aligned} I_1 \frac{d}{dt}\omega_1 &= \omega_3 L_2 - \omega_2 L_3 = \omega_2 \omega_3 (I_2 - I_3) \\ I_2 \frac{d}{dt}\omega_2 &= \omega_1 L_3 - \omega_3 L_1 = \omega_1 \omega_3 (I_3 - I_1) \\ I_3 \frac{d}{dt}\omega_3 &= \omega_2 L_1 - \omega_1 L_2 = \omega_1 \omega_2 (I_1 - I_2) \end{aligned} \tag{5.13}$$

These are known as **Euler's equations**. They tell us how the components of the body-frame representation $\mathbf{R}^T\underline{\vec{\omega}}'$ evolve due to the fictitious torque present in the body frame. As one expects, the fictitious torque is determined completely by the angular velocity of the rotating (body) frame and the principal moments of the rigid body.

We may draw two generic conclusions from the torque-free Euler's equations:

- The angular velocity component along a given axis is constant if the other two principal moments are equal. This is equivalent to the object having rotational symmetry about the given axis, such as a symmetric top.
- Using Equation 5.13, we may also see that the overall angular velocity is constant in both the space and body frames if it is aligned with any principal axis. If, for example, ω_1 and ω_2 vanish at $t = 0$, then all the time derivatives vanish at $t = 0$, and so the angular velocity components maintain their initial values with $\omega_1 = \omega_2 = 0$ and ω_3 nonzero and constant. The body-frame angular velocity is thus constant and aligned with a principal axis. If the angular velocity is along a principal axis in the body frame, then the body frame is in fact rotating about that principal axis and so the representations $\underline{\vec{\omega}}'$ and $\mathbf{R}^T\underline{\vec{\omega}}'$ are equal. Since we have just seen that the latter is constant, then so is the former.[13]

With the full time evolution of $\mathbf{R}^T\underline{\vec{\omega}}'$ in hand, we can now obtain the orientation of the body as a function of time. First, it holds that

$$\mathbf{R}^T\underline{\vec{L}}' = \underline{\mathcal{I}}\,\mathbf{R}^T\underline{\vec{\omega}}'$$

If we apply $\mathbf{R}$ to the above, we obtain

$$\mathbf{R}\left(\underline{\mathcal{I}}\,\mathbf{R}^T\underline{\vec{\omega}}'\right) = \underline{\vec{L}}'$$

Now, recall that Euler's equations have given us $\mathbf{R}^T\underline{\vec{\omega}}'$, that $\underline{\mathcal{I}}$ is known because it is the body-frame representation and is thus given by the principal moments, that $\underline{\vec{L}}'$ is a constant set by the initial conditions, and also that rotation matrices only have three degrees of freedom. Thus, we have a set of three linear equations for the three degrees of freedom of $\mathbf{R}$ at any instant of time t. We can thus obtain $\mathbf{R}(t)$, which completely defines the orientation of the body relative to the space frame as a function of time. Because a rigid body is fully

[13] One might have the following worry. The principal axes are in general rotating in the space frame. So what does it mean for the angular velocity, which we specify in the space frame, to be aligned with a principal axis? The resolution of this quandary is simply that, if the angular velocity is aligned with a principal axis, then the body frame rotates about that axis relative to the space frame, and thus that principal axis is in fact fixed in the space frame, too. The components of $\underline{\vec{\omega}}'$ and $\mathbf{R}^T\underline{\vec{\omega}}'$ along that axis are identical and the other components vanish, so $\vec{\omega}$, $\vec{L}$, and that principal axis are collinear in all frames.

specified by its three principal moments and the orientation of its principal axes, it holds that $\mathbf{R}(t)$, along with the principal moments, fully determines the rigid body in the space frame at any time.

Torque-Free Motion of a Symmetric Top – Intuitive Explanation

In the following, we will calculate the motion of a torque-free symmetric top whose angular momentum and spin vectors are misaligned. Before we dive into details, let's think about what the solution should look like.

The setup is as follows. A symmetric top has $I_1 = I_2 \neq I_3$. The initial conditions include rotation with angular speed Ω about the 3 axis of the top, with the value of Ω to be determined by the other initial conditions. These other conditions we will specify are the magnitude of the angular momentum vector L and the total kinetic energy T.

Though we will use L and T to specify the initial conditions, we can see that specifying L and T is equivalent to specifying the inital total angular speed and the tilt angle between the initial angular velocity vector $\vec{\omega}_0$ and the top's 3-axis. *A priori*, one might worry that one has to specify the full orientation of the top (3 numbers) and the initial angular velocity (3 numbers). But we can see that the number of true degrees of freedom is much fewer. Since there is no external torque, the initial absolute orientation of the body and $\vec{\omega}_0$ relative to the lab inertial frame is irrelevant; only the relative orientation of the two will matter. So that eliminates the three degrees of freedom associated with the body orientation. Next, because we assume a symmetric top, it does not matter what the azimuthal angle of $\vec{\omega}_0$ is in the body frame; only the length ω_0 and the polar angle γ_0 between the 3-axis and $\vec{\omega}_0$ are of consequence. For specificity, we will assume $\vec{\omega}_0$ is in the plane formed by the body 2 and 3 axes at $t = 0$. Using the fact that $\underline{\mathcal{I}}$ is diagonal in the body frame, we may write $\vec{\underline{L}}$ and T in terms of ω_0 and γ_0:

$$\underline{\vec{\omega}}_0 = \omega_0 \begin{pmatrix} 0 \\ \sin\gamma_0 \\ \cos\gamma_0 \end{pmatrix} \qquad \underline{\mathcal{I}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$$\vec{\underline{L}} = \underline{\mathcal{I}}\,\underline{\vec{\omega}}_0 = \omega_0 \begin{pmatrix} 0 \\ I_1 \sin\gamma_0 \\ I_3 \cos\gamma_0 \end{pmatrix} \qquad T = \frac{1}{2}\,\underline{\vec{\omega}}_0^T \underline{\mathcal{I}}\,\underline{\vec{\omega}}_0 = \frac{1}{2}\,\omega_0^2 \left(I_1 \sin^2\gamma_0 + I_3 \cos^2\gamma_0\right)$$

We thus see that specifying ω_0 and γ_0 is equivalent to specifying L and T.

We note that the above expressions suggest characteristic values of the angular momentum and kinetic energy:

$$L_* = I_3\,\omega_0 \qquad T_* = \frac{1}{2}\,I_3\,\omega_0^2 = \frac{1}{2}\,\frac{L_*^2}{I_3}$$

The initial angular momentum and kinetic energy can be written in terms of these characteristic values:

$$L = L_* \sqrt{1 + \left(\frac{I_1^2}{I_3^2} - 1\right)\sin^2\gamma_0} \qquad T = T_* \left(1 + \left(\frac{I_1}{I_3} - 1\right)\sin^2\gamma_0\right)$$

That is, L and T scale with the characteristic values L_* and T_* that are set only by I_3 and ω_0, and then they vary away from these characteristic values due to I_1 and γ_0. In

particular, L_* and T_* are maximum values if $I_1 < I_3$ (oblate) and are minimum values if $I_1 > I_3$ (prolate). Once we have obtained the full solution, we will revisit this point and make plots of the various parameters.

We can see from the above that T deviates from $\frac{L^2}{2\,I_3} = \frac{1}{2}\,I_3\,\omega_0^2$ when $\gamma_0 \neq 0$; that is, when there is an angle between the 3-axis and the angular velocity. Another way of saying this is that $\vec{\omega}_0$, $\vec{L}$, and T are trivially related only when $\vec{\omega}_0$ points along a body axis. That body axis must be the 3-axis because we have assumed there is some angular velocity along the 3-axis in the initial conditions. If $\vec{\omega}_0$ is misaligned with the body axis, then the relationship between $\vec{\omega}_0$, $\vec{L}$, and T admits the freedom parameterized by the angle γ_0.

Since the spin angular velocity can only point along the 3-axis, misalignment of $\vec{\omega}_0$ and the 3-axis implies an additional component of angular velocity nonaligned with the spin. We will denote this additional term by $\vec{\omega}_P$, foreshadowing its physical significance as the precession frequency of the top.

In what direction does the extra angular velocity component $\vec{\omega}_P$ point, and how does it affect the dynamics? Since there is no torque and the kinetic energy is constant, we expect that, whatever happens, all angular velocities will stay constant in length, but possibly may change direction over time. (One could invoke complex cancellations to keep the net angular momentum and kinetic energy constant, but that's really baroque.) If $\vec{\omega}_P$ had a component $\vec{\omega}_{P,\perp}$ perpendicular to $\vec{L}$, then the top's 3 axis would precess both around $\vec{L}$ (due to the $\vec{\omega}_{P,||}$ piece) as well as around an axis perpendicular to $\vec{L}$. Because $T = \frac{1}{2}\left(\vec{\Omega}^T\vec{L} + \vec{\omega}_P^T\vec{L}\right)$, a change in the direction of $\vec{\Omega}$ that is not simple precession about $\vec{L}$ makes the first term time-dependent. The second term only obtains a contribution $\vec{\omega}_{P,||}$, so one would need $\vec{\omega}_{P,||}$ to have a time-dependent length. Thus, we would have a picture in which the top's 3 axis is both precessing about $\vec{L}$ and an axis perpendicular to $\vec{L}$, with the speed of precession about $\vec{L}$ varying in time. One again would require complicated cancellations to keep $\vec{L}$ and T constant while all these other quantities are varying. The simpler alternative is to claim that $\vec{\omega}_{P,\perp}$ must vanish. This allows $\vec{\Omega}^T\vec{L}$ to be constant, which then permits $\vec{\omega}_{P,||}$ to be constant. We get simple precession of the 3 axis about $\vec{L}$ while the top is spinning about the 3 axis with angular speed Ω. It requires some more careful work to prove that this indeed is the solution (*i.e.*, to exclude the baroque possibilities), but it is important to build up the intuition to lead you to the likely physical behavior.

Torque-Free Motion of a Symmetric Top in the Body Frame

Before finding the space-frame solution, let's also explicitly solve the problem using Euler's equations to calculate the motion in the body frame. Our mental picture is of a symmetric top, spinning about its 3 axis, with an angular momentum vector that is misaligned with the spin axis because the kinetic energy and angular momentum are mismatched. A symmetric top has $I_1 = I_2$, so the three Euler equations take on a very simple form:

$$\begin{aligned}
\frac{d}{dt}\,\omega_1 &= \omega_2\,\omega_3\left(1 - \frac{I_3}{I_1}\right) \\
\frac{d}{dt}\,\omega_2 &= \omega_1\,\omega_3\left(\frac{I_3}{I_1} - 1\right) \\
\frac{d}{dt}\,\omega_3 &= 0
\end{aligned}$$

From these coupled first order equations, we may obtain two uncoupled second order equations:

$$\frac{d^2}{dt^2}\,\omega_1 + \Omega_P^2\,\omega_1 = 0 \qquad \frac{d^2}{dt^2}\,\omega_2 + \Omega_P^2\,\omega_2 = 0 \qquad \Omega_P = \omega_3\left(\frac{I_3}{I_1} - 1\right)$$

The solutions are as usual sinusoidal, but one must be careful to ensure the initial conditions are applied in such a way as to satisfy the coupled first order equations, giving

$$\omega_1 = A\,\sin\left(\Omega_P\,t + \phi\right) \qquad \omega_2 = A\,\cos\left(\Omega_P\,t + \phi\right)$$

The complementary variations in ω_1 and ω_2 indicate that the xy components of $\mathbf{R}^T\vec{\underline{\omega}}'$ rotate about the z axis at angular velocity Ω_P.

Let's look at the body-frame representation of the angular momentum vector:

$$\left[\mathbf{R}^T\vec{\underline{L}}'\right] = \underline{\mathcal{I}}\,\left[\mathbf{R}^T\vec{\underline{\omega}}'\right] = \begin{pmatrix} I_1\,A\,\sin\left(\Omega_P\,t+\phi\right) \\ I_1\,A\,\cos\left(\Omega_P\,t+\phi\right) \\ I_3\,\omega_3 \end{pmatrix}$$

Now, because our equation of motion is $d\left[\mathbf{R}^T\vec{\underline{L}}'\right]/dt = -\left[\mathbf{R}^T\vec{\underline{\omega}}'\right]\times\left[\mathbf{R}^T\vec{\underline{L}}'\right]$, $\mathbf{R}^T\vec{\underline{L}}'$ only changes direction, not magnitude. So $L = \left|\mathbf{R}^T\vec{\underline{L}}'\right| = \left|\vec{\underline{L}}'\right|$ is set by initial conditions.[14] Thus, the space-frame angular momentum magnitude provides one of the initial conditions for the solution to our differential equations. Writing the magnitude, we have

$$L^2 = I_1^2\,A^2 + I_3^2\,\omega_3^2$$

The other initial condition is the total kinetic energy, which is a scalar and is given by

$$T = \frac{1}{2}\left[\mathbf{R}^T\vec{\underline{\omega}}'\right]^T\underline{\mathcal{I}}\left[\mathbf{R}^T\vec{\underline{\omega}}'\right] = \frac{1}{2}\left(I_1\,A^2 + I_3\,\omega_3^2\right)$$

Rewriting the above two relations, we have

$$\begin{aligned} |A| &= \sqrt{\frac{L^2 - 2\,I_3\,T}{I_1\,(I_1 - I_3)}} = \sqrt{\frac{I_3}{I_1 - I_3}\left(\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_1}\right)} \\ \omega_3 &= \sqrt{\frac{L^2 - 2\,I_1\,T}{I_3\,(I_3 - I_1)}} = \sqrt{\frac{I_1}{I_3 - I_1}\left(\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_3}\right)} \\ \Omega_P &= \omega_3\left(\frac{I_3}{I_1} - 1\right) = \frac{I_3 - I_1}{|I_3 - I_1|}\sqrt{\frac{I_3 - I_1}{I_1}\left(\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_3}\right)} \end{aligned}$$

(The prefactor for Ω_P ensures the correct sign.) So, with the initial conditions L and T determining the above three quantities, we have a full solution. (The phase is somewhat arbitrary, depending on how we choose to relate the F and F' axes at $t = 0$; a reasonable choice would be $\phi = 0$.) Clearly, $\mathbf{R}^T\vec{\underline{\omega}}'$ and $\mathbf{R}^T\vec{\underline{L}}'$ precess about the 3 axis of the top with angular speed Ω_P. The diameters of the circles they trace out are A and $I_1\,A$, respectively. Their constant projections onto the 3 axis are ω_3 and $I_3\,\omega_3$, respectively. We denote the

[14] Remember, rotations don't change the length of a vector

angles between the angular velocity and the 3 axis and between the angular momentum and the 3 axis by γ and θ, respectively, so we have

$$\tan\gamma = \frac{A}{\omega_3} \qquad \tan\theta = \frac{I_1\,A}{I_3\,\omega_3}$$

The fact that $\vec{\omega}$ is not fixed to the 3 axis implies that the instantaneous axis of rotation of the body frame is in fact continually changing, precessing with the same speed Ω_P. This is an important point: the body frame is related to the space frame by rotation about $\vec{\omega}$, not about the 3 axis! Note that Ω_P is *not* the angular speed of the body frame relative to the space frame; Ω_P is only the speed with which the axis that defines that relationship is changing *relative to the body frame axes* (not the space frame!). We shall see that the angular speed of this instantaneous axis of rotation precesses at a different speed $\omega_P \neq \Omega_P$ and $\omega_P \neq \omega$ relative to the space frame axes. The total relative angular speed of the two frames is given by $\omega \neq \Omega_P$.

Torque-Free Motion of a Symmetric Top in the Space Frame

While obtaining the above solution was easy, it is not at all clear what this motion looks like from the space frame. In particular, we would like to know, from the space-frame point of view, what is the angular velocity $\vec{\omega}_P$ at which the 3 axis of the top precesses about $\vec{L}$ and what is the "spin" $\vec{\Omega}$ of the top about its 3 axis. One might think that the spin of the top is just ω_3, the component of the angular velocity along the 3 axis. That is incorrect. The precession corresponds to an angular velocity $\vec{\omega}_P$ along $\vec{L}$. Because the 3 axis and $\vec{L}$ are not (in general) perpendicular, $\vec{\omega}_P$ has a projection along the 3 axis. So ω_3 has contributions from both $\vec{\Omega}$ and $\vec{\omega}_P$. We shall see, in fact, that Ω_P is the magnitude of the spin angular velocity.

Our tool to separate precession and spin will be to define a rotating frame F that does *not* spin with the top as the body frame does, but which only incorporates the precessional motion $\vec{\omega}_P$. The top's 3 axis is fixed in this frame, but its 1 and 2 axes are not. This F frame is therefore not the body frame. However, because the top is symmetric, the representation $\underline{\mathcal{I}}$ is the same as the body-frame representation, $\underline{\mathcal{I}} = \mathrm{diag}(I_1, I_1, I_3)$, so we gain the major advantage of the body frame, diagonalization of the inertia tensor.

As our earlier short sketch indicated, we will obtain the solution by making use of conservation of angular momentum and kinetic energy. We will start from scratch in defining the frames and angular velocities for the sake of clarity.

- **Definition of nonrotating F' frame**

  Since the space-frame angular momentum vector is constant due to the lack of torques, let it define the nonrotating F' system, with the angular momentum vector along $\hat{z}'$. Let the $y'z'$ plane of the F' frame be the plane in which the top's 3 axis resides at $t = 0$. Let the angle between the top's 3 axis and $\hat{z}'$ be θ at $t = 0$. These are absolutely general initial conditions. The definition of θ is the same as we used in the body frame, and it is valid here because the 3-axis of F and of the true body frame are coincident.

- **Definition of rotating F frame and its angular velocity $\vec{\omega}_P$**

  Define the F frame to be such that its z axis is always aligned with the top's 3 axis and the $y\,z$ plane coincides with the $y'z'$ plane at $t = 0$. Specifically, at $t = 0$, the two

frames are related by

$$\begin{aligned}\hat{x}' &= \hat{x}\\ \hat{y}' &= \hat{y}\cos\theta + \hat{z}\sin\theta\\ \hat{z}' &= -\hat{y}\sin\theta + \hat{z}\cos\theta\end{aligned}$$

where θ is the appropriate angle because it is the angle between the angular momentum ($\hat{z}'$) and 3 axis ($\hat{z}$). As we explained above, the representation $\underline{\mathcal{I}}$ of the inertia tensor will be diagonal in such a frame because the top is symmetric. Let the angular velocity of rotation of F with respect to F' be denoted by $\vec{\omega}_P$, with representations $\mathbf{R}^T\underline{\vec{\omega}}'_P$ and $\underline{\vec{\omega}}'_P$ in the rotating and nonrotating frames, respectively. By definition, $\vec{\omega}_P$ describes the angular velocity of the top's 3 axis, which we shall see is simply precession.

- **Definition of the spin angular velocity $\vec{\Omega}$ and its contribution to the total angular velocity $\vec{\omega}$**

  By definition of the frame F, the top spins in the frame F around its 3 axis, so there is additional angular velocity along that axis. Let us show that this angular velocity does indeed add to $\vec{\omega}_P$ in the obvious way to give the total angular velocity $\vec{\omega}$. This may seem an obvious point, but let's be careful about it. Let us consider the motion in the space frame of a point that is fixed to the body. First, to go from the body frame to F, we have

$$\frac{d}{dt}\underline{\vec{r}} = \underline{\vec{\Omega}} \times \underline{\vec{r}} + \mathbf{R}_{(body\to F)}\frac{d}{dt}\underline{\vec{r}}_{body} = \underline{\vec{\Omega}} \times \underline{\vec{r}}$$

  The notation $\underline{\vec{r}}_{body}$ indicates the body-frame representation of the position vector of the point $\vec{r}$; previously, we would have just used the notation $\underline{\vec{r}}$, but now our F frame is not the body frame and so we must distinguish $\underline{\vec{r}}$ and $\underline{\vec{r}}_{body}$. The rotation matrix $\mathbf{R}_{(body\to F)}$ denotes the rotation matrix to get from the body frame to F; it corresponds to the $\vec{\Omega}$ angular velocity. $\underline{\vec{\Omega}}$ is the representation of the $\vec{\Omega}$ angular velocity in the F frame. And the last term vanishes because we consider a point fixed to the body. Next, to go from F to F', we have a similar expression

$$\frac{d}{dt}\underline{\vec{r}}' = \underline{\vec{\omega}}'_P \times \underline{\vec{r}}' + \mathbf{R}\frac{d}{dt}\underline{\vec{r}} = \underline{\vec{\omega}}'_P \times \underline{\vec{r}}' + \mathbf{R}\left[\underline{\vec{\Omega}} \times \underline{\vec{r}}\right]$$

  As we saw in Section 5.2, it holds that $\mathbf{R}\left[\underline{\vec{\Omega}} \times \underline{\vec{r}}\right] = \underline{\vec{\Omega}}' \times \underline{\vec{r}}'$.[15] So we have

$$\vec{\omega} = \vec{\omega}_P + \vec{\Omega} \qquad \frac{d}{dt}\underline{\vec{r}}' = \underline{\vec{\omega}}' \times \underline{\vec{r}}'$$

  So, we see that, if we define $\vec{\Omega}$ in the intuitive way – as the angular velocity of the body frame relative to the precessing frame F – then it indeed adds linearly with $\vec{\omega}_P$ and together the two yield the total angular velocity. Explicitly, we have

$$\begin{aligned}\underline{\vec{\omega}}' &= \Omega\,\mathbf{R}\,\hat{z} + \underline{\vec{\omega}}'_P\\ \left[\mathbf{R}^T\underline{\vec{\omega}}'\right] &= \Omega\,\hat{z} + \left[\mathbf{R}^T\underline{\vec{\omega}}'_P\right]\end{aligned}$$

[15] Quickly: $\mathbf{R}\left[\underline{\vec{\Omega}} \times \underline{\vec{r}}\right] = \mathbf{R}\left(\underline{\vec{\Omega}} \cdot \vec{\mathbf{M}}\right)\underline{\vec{r}} = \mathbf{R}\,\mathbf{R}^T\left(\underline{\vec{\Omega}}' \cdot \vec{\mathbf{M}}\right)\mathbf{R}\,\underline{\vec{r}} = \underline{\vec{\Omega}}' \times \underline{\vec{r}}'$. We would have referred to $\underline{\vec{\Omega}}$ by $\mathbf{R}^T\underline{\vec{\Omega}}'$ in Section 5.2 because of the issue of writing the angular velocity of a rotating frame in that frame. Here, since F is not the body frame, it makes sense to write $\underline{\vec{\Omega}}$. Note also that our proof of the fact $\underline{\vec{\Omega}} \cdot \vec{\mathbf{M}} = \mathbf{R}^T\left(\underline{\vec{\Omega}}' \cdot \vec{\mathbf{M}}\right)\mathbf{R}$ is valid for any vector $\vec{\Omega}$.

where the top's 3 axis $\mathbf{R}\,\hat{z}$ is a function of time via $\mathbf{R}$ when viewed from the space frame.

- **$\mathbf{R}^T\vec{\underline{L}}'$ makes a fixed angle θ with the spin axis and F precesses about F' with fixed polar angle θ**

  $\mathbf{R}^T\vec{\underline{L}}'$ is the representation of $\vec{L}$ in the precessing frame F. Its dynamics in F tell us the dynamics of the top's 3 axis in the space frame. First, we know $\left|\mathbf{R}^T\vec{\underline{L}}'\right|$ is constant because $\mathbf{R}^T\vec{\underline{L}}'$ is related to $\vec{\underline{L}}'$ by a rotation, rotations leave the length of vectors unchanged, and $\vec{\underline{L}}'$ is constant. To determine the time dependence of the direction of $\mathbf{R}^T\vec{\underline{L}}'$, let's calculate the angular momentum in the F frame from $\mathbf{R}^T\vec{\underline{\omega}}'$:

$$\mathbf{R}^T\vec{\underline{L}}' = \underline{\mathcal{I}}\left[\mathbf{R}^T\vec{\underline{\omega}}'\right] = I_1\left(\omega_{P,1}\,\hat{x} + \omega_{P,2}\,\hat{y}\right) + I_3\left(\omega_{P,3} + \Omega\right)\hat{z}$$

  (Remember that $\underline{\mathcal{I}} = \mathrm{diag}(I_1, I_1, I_3)$ in F because the top is symmetric and its 3 axis is fixed in F. Also, we write $\omega_{P,i}$ for the components of $\mathbf{R}^T\vec{\underline{\omega}}'_P$.) Now consider the kinetic energy. Recall that the kinetic energy is a scalar under rotational transformations, so we may calculate it using F frame quantities:

$$\begin{aligned} T &= \frac{1}{2}\left[\mathbf{R}^T\vec{\underline{\omega}}'\right]^T\underline{\mathcal{I}}\left[\mathbf{R}^T\vec{\underline{\omega}}'\right] = \frac{1}{2}\,I_1\left(\omega_{P,1}^2 + \omega_{P,2}^2\right) + \frac{1}{2}\,I_3\left[\omega_{P,3} + \Omega\right]^2 \\ &= \frac{\left|\mathbf{R}^T\vec{\underline{L}}'\right|^2}{2}\left(\frac{\sin^2\theta}{I_1} + \frac{\cos^2\theta}{I_3}\right) \end{aligned}$$

  where we have made use of our definition of θ as the angle between $\vec{L}$ and the top's 3 axis. θ may, *a priori*, change with time. But, since $\left|\mathbf{R}^T\vec{\underline{L}}'\right|$ is constant, the only way to maintain constant T when $I_1 \neq I_3$ is to fix θ. Hence, the angle between $\mathbf{R}^T\vec{\underline{L}}'$ and the top's 3 axis is fixed. In the F frame, this implies the angular momentum vector sits on a cone of constant half-angle θ centered on the top's 3 axis. Viewed in the F' frame, the top's 3 axis is on a cone of constant half-angle θ centered on the angular momentum vector. Since the angular momentum vector defines the F' system and the top's 3 axis defines the F system, the angle between F' and F is the constant θ.

- **The angular velocity of F relative to F' is $\vec{\underline{\omega}}'_P = \omega_P\,\hat{z}'$**

  Remember that $\vec{\omega}_P = \vec{\omega} - \vec{\Omega}$ describes, by definition, the rotation of F relative to F'. We showed above that the rotation of F relative to F' due to $\vec{\underline{\omega}}'_P$ maintains the angle θ between $\hat{z}'$ and $\hat{z}$ (*i.e.*, between the angular momentum and the top's 3 axis). If, $\vec{\underline{\omega}}'_P$ had any $\hat{x}'$ or $\hat{y}'$ component, there would be rotation of F around x' or y', which would change θ. So, it is only allowed to have $\vec{\underline{\omega}}'_P = \omega_P\,\hat{z}'$.

- **The transformation between F and F' is now fully specified**

  Knowing the direction of $\vec{\underline{\omega}}'_P$ now fixes the coordinate transformation relating F and F'. Because the polar angle θ between $\hat{z}'$ and $\hat{z}$ is fixed, and because at $t = 0$ we had $\hat{z}'$ in the yz plane (by definition), we may conclude that F rotates such that $\hat{z}'$ (and therefore $\vec{L}$) is *always* in the yz plane (and makes an angle θ with $\hat{z}$). Thus, the decomposition

$$\hat{z}' = -\hat{y}\,\sin\theta + \hat{z}\,\cos\theta$$

holds for all time. This lets us further specify the decompositions of $\mathbf{R}^T\underline{\vec{L}}'$, $\mathbf{R}^T\underline{\vec{\omega}}_P'$, and $\mathbf{R}^T\underline{\vec{\omega}}'$. We also write down $\underline{\vec{\Omega}}$ for completeness:

$$\begin{aligned}
\mathbf{R}^T\underline{\vec{L}}' &= -\hat{y}\, I_1\, \omega_P\, \sin\theta + \hat{z}\, I_3\, (\omega_P\, \cos\theta + \Omega) \\
&= -\hat{y}\, L\, \sin\theta + \hat{z}\, L\, \cos\theta \\
\mathbf{R}^T\underline{\vec{\omega}}_P' &= -\hat{y}\, \omega_P\, \sin\theta + \hat{z}\, \omega_P\, \cos\theta \\
\mathbf{R}^T\underline{\vec{\omega}}' &= -\hat{y}\, \omega_P\, \sin\theta\, \hat{y} + \hat{z}\, (\omega_P\, \cos\theta + \Omega) \\
\underline{\vec{\Omega}} &= \Omega\, \hat{z}
\end{aligned}$$

where the value and time dependence of the magnitude ω_P has not yet been determined. (We know rotations preserve lengths so using ω_P for the length of any representation of $\vec{\omega}_P$ is ok.) We have not written equations giving x' and y' in terms of x, y, and z because we don't need them, but clearly they could be given.

- **Value of $\omega_P = |\mathbf{R}^T\underline{\vec{\omega}}_P'|$**

  To determine the value of ω_P, we just need to use the above equation for $\mathbf{R}^T\underline{\vec{L}}'$:

$$\begin{aligned}
L\,(-\hat{y}\, \sin\theta + \hat{z}\, \cos\theta) &= -\hat{y}\, I_1\, \omega_P \sin\theta + \hat{z}\, I_3\, (\omega_P\, \cos\theta + \Omega) \\
\Longrightarrow \quad \omega_P &= \frac{L}{I_1} = \frac{1}{\cos\theta}\left(\frac{L}{I_3}\, \cos\theta - \Omega\right)
\end{aligned}$$

  Since L is constant, we conclude ω_P is constant. Note that we had to go to the F frame to relate $\mathbf{R}^T\underline{\vec{L}}'$ and $\mathbf{R}^T\underline{\vec{\omega}}_P'$ because the inertia tensor is diagonal there.

- **The full solution**

  The above two equations can be used to find ω_P and θ once L and Ω are specified. Or, if combined with the kinetic energy equation

$$T = \frac{L^2}{2}\left(\frac{\sin^2\theta}{I_1} + \frac{\cos^2\theta}{I_3}\right)$$

  ω_P, θ, and Ω can be determined from L and T. A small bit of algebra yields

$$\begin{aligned}
\omega_P &= \frac{L}{I_1} \\
\frac{\sin^2\theta}{I_1} + \frac{\cos^2\theta}{I_3} &= \frac{2\,T}{L^2} \\
\Omega &= L\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\cos\theta
\end{aligned}$$

  Thus, we have a complete solution, in terms of the initial conditions L and T, for the precession angular velocity ω_P, the polar angle θ, and the spin angular velocity Ω. The vectors $\vec{L}$ and $\vec{\omega}_P$ are constant and lie along the $\hat{z}'$ axis of the fixed frame. The top's 3 (spin) axis makes an angle θ with $\hat{z}'$ and precesses about $\hat{z}'$ with angular speed ω_P. The spin angular velocity vector $\vec{\Omega}$ lies along the top's 3 axis, and the total angular velocity vector $\vec{\omega}$ lies in the plane formed by $\vec{L}$ and the top's 3 axis and also precesses about $\vec{L}$ at speed ω_P.

  For completeness, we note that the angle between $\vec{\omega}$ and the top's 3 axis ($\hat{z}$) is given by

$$\tan\gamma = \frac{\omega_{P,2}}{\omega_{P,3} + \Omega} = \frac{\omega_P\, \sin\theta}{\omega_P\, \cos\theta + I_1\, \omega_P\, \left(I_3^{-1} - I_1^{-1}\right)\cos\theta} = \frac{I_3}{I_1}\, \tan\theta$$

The angle between $\vec{\omega}$ and the angular momentum vector ($\hat{z}'$) is given by $\theta - \gamma$.

Inspired by the body-frame solution, we can write all the parameters explicitly in terms of L and T:

$$\begin{aligned}
\omega_P &= \frac{L}{I_1} \\
\cos\theta &= \frac{I_3}{L}\sqrt{\frac{I_1}{I_3 - I_1}\left(\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_3}\right)} \\
\sin\theta &= \frac{I_1}{L}\sqrt{\frac{I_3}{I_1 - I_3}\left(\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_1}\right)} \\
\tan\theta &= \sqrt{\frac{I_1}{I_3}\left|\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_1}\right|\left|\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_3}\right|^{-1}} \\
\tan\gamma &= \sqrt{\frac{I_3}{I_1}\left|\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_1}\right|\left|\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_3}\right|^{-1}} \\
\Omega &= -\frac{I_3 - I_1}{|I_3 - I_1|}\sqrt{\frac{I_3 - I_1}{I_1}\left(\frac{L^2}{I_1\,I_3} - \frac{2\,T}{I_3}\right)}
\end{aligned}$$

(once again, the prefactor provides the correct sign for ω).

- **Relation to body-frame solution**

  We can use the above explicit formulae to see the relation between the space-frame and body-frame solution parameters:

  $$I_3\,\omega_3 = I_1\,\omega_P\,\cos\theta = L\,\cos\theta \qquad A = -\omega_P\,\sin\theta \qquad \Omega = -\Omega_P$$

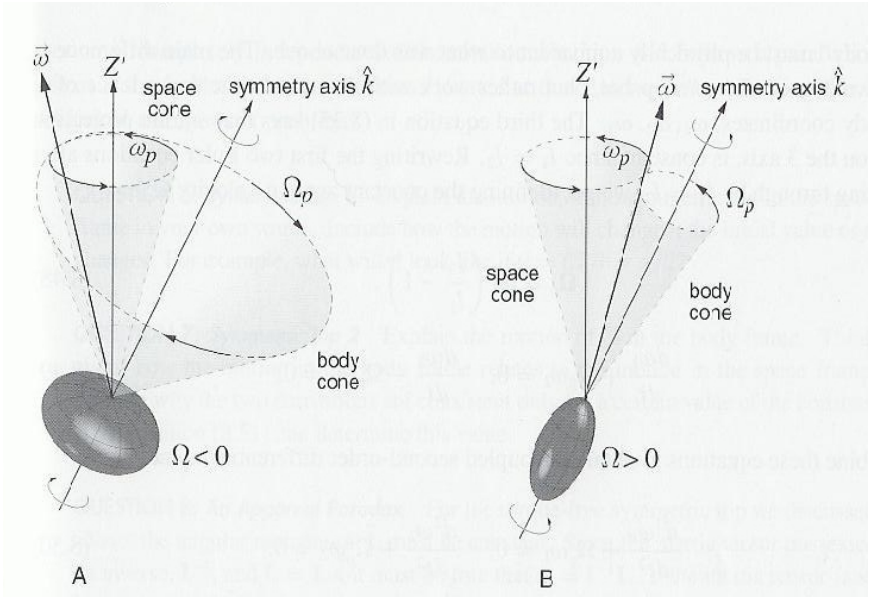
  Note that $I_1\,A = -I_1\,\omega_P\,\sin\theta = L\,\sin\theta$, so θ gives the angle between the body z axis and the angular momentum in the body frame also. We can check γ too:

  $$\frac{|A|}{\omega_3} = \frac{\omega_P\,\sin\theta}{I_3^{-1}\,I_1\,\omega_P\,\cos\theta} = \frac{I_3}{I_1}\,\tan\theta = \tan\gamma$$

  So, the angular velocity vector makes the same angle with the top's 3 axis as in the space-frame solution.

  We may also explain these relations more intuitively:

  - The relation between $I_3\,\omega_3$ and L is found simply by projecting $\vec{L}$ along the top's 3 axis in F and noting that, because the top's 3 axis is fixed in both F and its body frame, that is indeed the component of $\vec{L}$ along the top's 3 axis in the body frame representation, which is $I_3\,\omega_3$.
  - The relation between A and ω_P can be seen as follows. A describes the component of the total angular velocity $\vec{\omega}$ in the top's 1-2 body plane. This body plane coincides with the F system's xy plane, and so the the magnitude of the component of $\vec{\omega}$ perpendicular to the top's 3 axis, which is $-\omega_P\sin\theta$, must equal A.
  - The same argument provides the relation between Ω and Ω_P: Ω describes how the top's 1-2 plane rotates relative to F, a frame in which $\vec{\omega}$ is fixed, while Ω_P tells how $\vec{\omega}$ rotates in the top's body frame (in which its 1-2 plane is fixed).



(Hand and Finch 8.7)

Analytical Mechanics

Figure 5.3: Left: oblate ($I_3 > I_1$) case. Right: prolate ($I_1 > I_3$) case.

One might still be confused by the following: In the space frame, where $\vec{L}$ is constant, the top's 3 axis precesses with angular speed ω_P, while in the top's body frame, where its 3 axis is constant, $\vec{L}$ precesses with angular speed $\Omega_P = -\Omega$. Shouldn't the two precession angular velocities just be the negatives of each other? No, because in the former instance, the spin of the body frame doesn't really enter: ω_P describes the precession of the frame F in which the top's 3 axis is fixed but its 1-2 plane is moving. Especially in cases where $\Omega \gg \omega_P$, the two angular velocities will be very different.

Interpretation of Symmetric Top Motion

The reader is no doubt very confused at this point about what is actually happening! Let's first consider the physical interpretation of the space-frame motion, and then relate the body-frame motion to it. Here F' implies space frame, F implies the frame rotating with the precessing spin axis but not with the spin, and "body" is the frame that also spins with the spin of the top. $\underline{\vec{L}}'$ is always along the $+\hat{z}'$ direction and has constant length.

Refer to Figure 5.3 for the following discussion.

- **Prolate ($I_3 < I_1$) case**
  This is the case you probably have a picture of in your mind. This consists of a top that is spinning around its longest axis The top has the smallest transverse extent perpendicular to this axis, so this is the axis with the smallest moment of inertia.
  - **Space Frame**
    In this case, $\Omega > 0$ and the top is spinning counterclockwise about its 3 axis. $\omega_P > 0$, so the precession is also counterclockwise about the $\hat{z}'$ axis. $\underline{\vec{\omega}}'_P$ is constant in length and points along the $\hat{z}'$ axis. We have demonstrated that $\vec{\Omega}$ and $\underline{\vec{\omega}}'$ both

lie in a plane that precesses about $\vec{\underline{L}}'$ ($= \hat{z}'$) with angular speed ω_P. The total angular velocity $\vec{\underline{\omega}}'$ thus sits on a cone of half-angle $\theta - \gamma$ and the spin axis $\vec{\Omega}$ sits on a cone of half-angle θ and both precess about the angular momentum vector with the same angular speed ω_P. The spin of the top about its 3 axis is included in $\vec{\underline{\omega}}'$.

- **Body Frame**
  We have demonstrated in the comparison of solutions that the same angles θ and γ arise in the F frame and describe the same angles as in the F' frame – between spin and angular momentum and between spin and angular velocity, respectively. Rather than the spin axis precessing with angular velocity ω_P, though, we have the angular momentum and angular velocity precessing with angular speed $\Omega_P = -\Omega$. In order to make a circuit about the top's 3 axis at angular speed Ω, $\mathbf{R}^T \vec{\underline{L}}'$ must also make a circuit around $\mathbf{R}^T \vec{\underline{\omega}}'$ at angular speed Ω. How do we reconcile this with the fact that $\vec{\underline{\omega}}'$ precesses with angular speed ω_P about $\vec{\underline{L}}'$? The difference occurs because the precession period is referenced to the coordinate system, so a vector may precess at one speed in one coordinate system and another speed in another system. Ω is the precession speed in the body-frame system while ω_P is the precession speed in the space frame and F frame. We will demonstrate how the two precession speeds are related using the rolling cone picture below.
- **Rolling Cones**
  The vector $\vec{\omega}$ now precesses on two cones, depending on which frame you consider:
  * In frame F', at angular speed ω_P on a cone of half-angle $\theta - \gamma$ about $\vec{L}$.
  * In frame F, at angular speed Ω on a cone of half-angle γ about $\vec{\Omega}$.

  These two pictures can be combined into one picture of the two cones rolling while remaining tangent at their line of contact, which is the $\vec{\omega}$ vector. We can see this by calculating the relation between the speeds at which $\vec{\omega}$ would have to move on each cone for the cones to roll without slipping. On the $\theta - \gamma$ cone, the angular velocity vector $\vec{\omega}$ traces out a circle of radius $\omega \sin(\theta - \gamma)$. Since $\vec{\omega}$ moves at angular velocity ω_P, its "linear speed" on the cone is $\omega\, \omega_P \sin(\theta - \gamma)$. By a similar argument, ω moves with linear speed $\omega\, \Omega \sin\gamma$ on the cone about $\vec{\Omega}$. These two speeds must be the same for the two cones to roll without slipping. We can see that they are, starting from the relation between Ω and ω_P:

$$\Omega = L\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\cos\theta = \frac{L}{I_1}\left(\frac{I_1}{I_3} - 1\right) = \omega_P\left(\frac{\tan\theta}{\tan\gamma} - 1\right)\cos\theta = \omega_p \frac{\sin(\theta - \gamma)}{\sin\gamma}$$

- **Oblate ($I_3 > I_1$) case**
  Obviously, this is the opposite of the prolate case. Oblate objects look more like squashed spheres.
  - **Space Frame**
    Because $I_3 > I_1$, two main changes arise. First, $\gamma > \theta$ now. This means that the $\vec{\omega}$ vector is now on the opposite side of the angular momentum vector as the spin axis (remember, all three always lie in one plane). The precession frequency is still positive so the precession is counterclockwise. The other main difference is that Ω changes sign – the top now spins clockwise. Probably a better way to think of the whole thing is that, if initial conditions make the oblate top spin counterclockwise, then everything else gets reversed – the spin axis precession becomes clockwise, etc. Other than these sign changes, everything else remains the same.

– **Body Frame**
Again, the angles are compatible between the body and space frame solutions, so the cone on which the angular velocity vector precesses is now larger than the one on which the angular momentum vector precesses. But, as before, they must be tangent to each other at $\vec{\omega}$. Since the precession of $\vec{\omega}$ around $\vec{L}$ has the same sign as in the prolate case, the rolling cone implies that the body frame cone now rolls in the opposite direction as the angular velocity vector precesses about the angular momentum.

Plots

Here we make some summary plots indicating how the various quantites in the solution depend on the initial conditions and the ratio I_1/I_3. We take the point of view that it is most useful to make plots as a function of the initial parameters ω_0 and γ_0 rather than as a function of L and T because the former pair are things the experimenter has direct control over, while L and T are only derived quantities. γ_0 is the the γ in our full solution, so we write γ for it here.

Here we first summarize the relations we use to make the plots:

$$
\begin{aligned}
L_* &= I_3\,\omega_0 & L &= L_*\sqrt{1+\left(\frac{I_1^2}{I_3^2}-1\right)\sin^2\gamma}\\
T_* &= \frac{1}{2}\,I_3\,\omega_0^2 = \frac{1}{2}\,\frac{L_*^2}{I_3} & T &= T_*\left(1+\left(\frac{I_1}{I_3}-1\right)\sin^2\gamma\right)\\
\omega_p &= \frac{L}{I_1} & \Omega &= \omega_p\left(\frac{I_1}{I_3}-1\right)\cos\theta = \omega_0\left(1-\frac{I_3}{I_1}\right)\cos\gamma\\
\frac{\tan\theta}{\tan\gamma} &= \frac{I_1}{I_3}
\end{aligned}
$$

We parameterize in terms of γ rather than θ because γ is the quantity more easily related to the initial conditions.

In the plots, we set $I_3 = 1$ and $\omega_0 = 1$ so $L_* = 1$ and $T_* = 1/2$. All frequencies will scale with ω_0, all angular momenta scale with L_*, and all kinetic energies scale with T_*.

We may note a few things about the plots:

- The precession frequency becomes very small and the spin frequency dominates for small γ and $I_1 > I_3$ (prolate). For oblate cases, the precession frequency and the spin frequency are almost the same except for very large γ. Since $\Omega < 0$ for $I_1 < I_3$, this means that spin and precession almost cancel each other to recover $\omega_0 = 1$.
- In all cases, as $\gamma \to \pi/2$, the spin frequency vanishes and the precession frequency goes to 1. This occurs because $\vec{L}$ is along the $\hat{z}'$ axis, so angular velocity perpendicular to $\hat{z}'$ is suppressed.
- As noted above, the characteristic values L_* and T_* become minimum values of angular momentum and kinetic energy for oblate ($I_1 < I_3$) and maximum values for prolate cases ($I_1 > I_3$). Both become large for prolate cases as $\gamma \to \pi/2$ because all the angular velocity into the larger I_1 moment, which thus yields large L and T. The opposite occurs for $I_1 < I_3$. Remember, L_* and T_* are essentially the angular momentum and kinetic energy when all the angular velocity is along the 3 axis.

- The ratio θ/γ behaves as discussed in the text, being > 1 for prolate cases and < 1 for oblate cases.

You may find other insights.

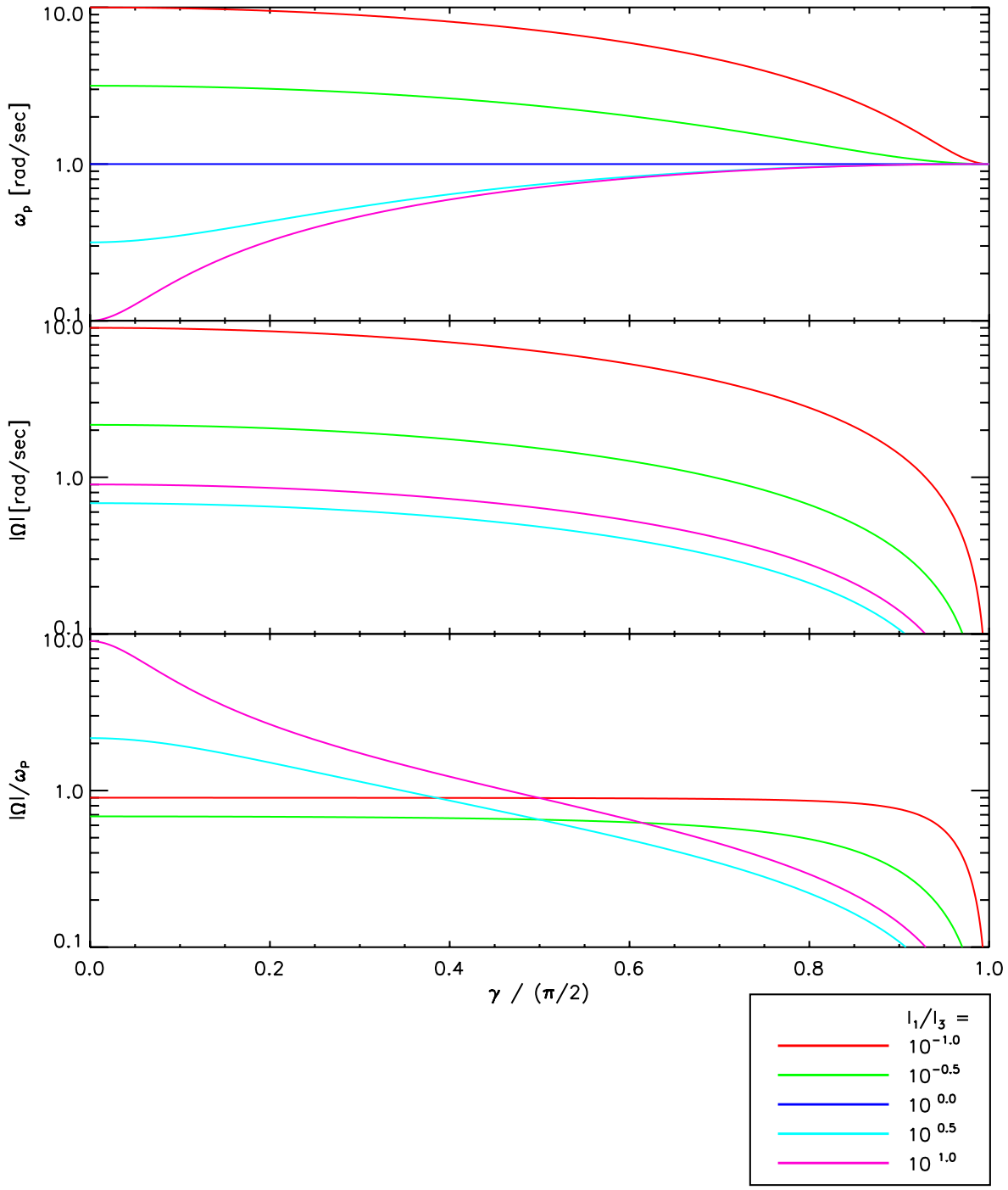


Figure 5.4: ω_p and Ω vs. γ for various values of I_1/I_3. Ω and $|\Omega|/\omega_P$ vanish for $I_1/I_3 = 1$. The sign of Ω is positive for $I_1/I_3 > 1$ and negative for $I_1/I_3 < 1$.

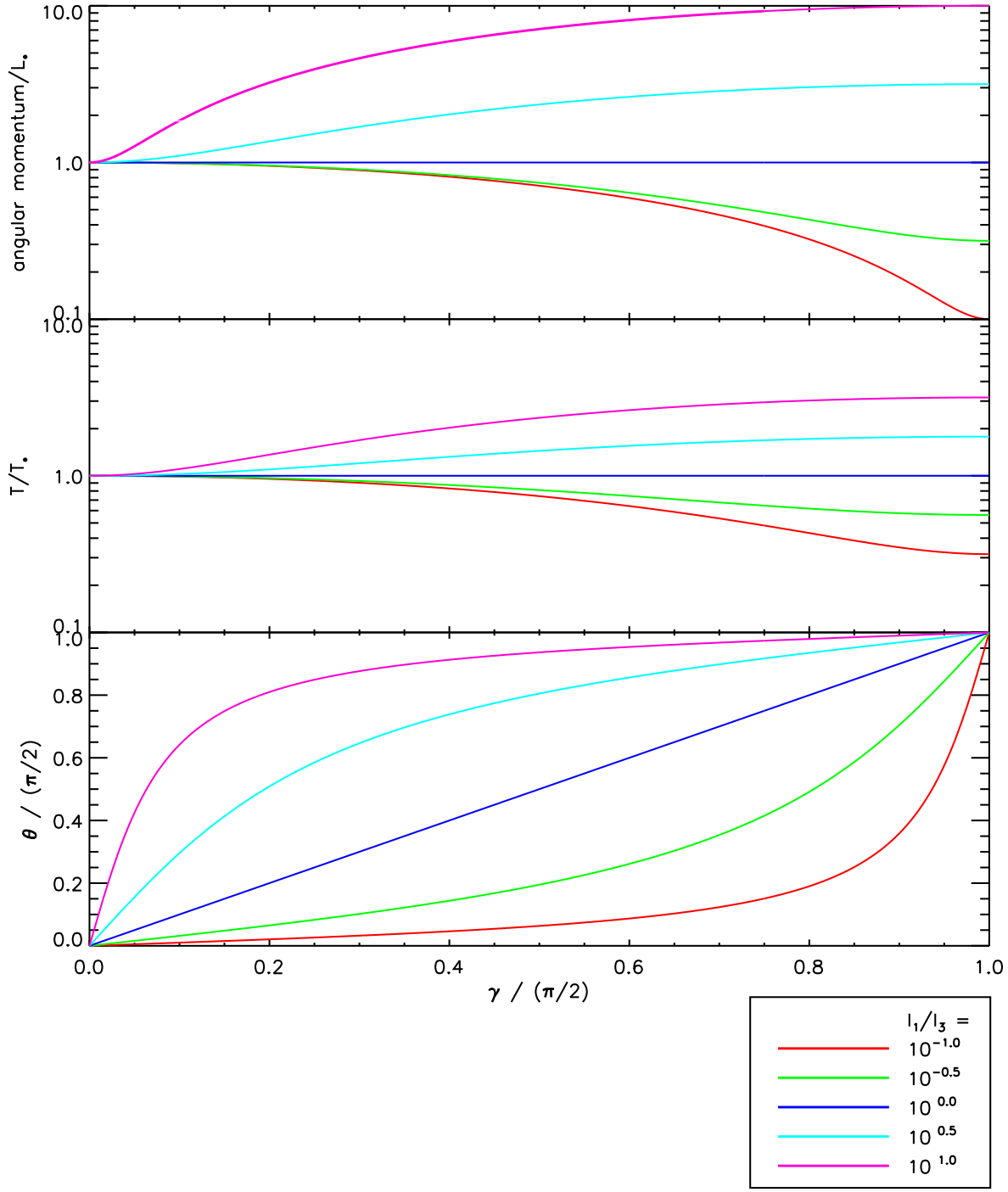
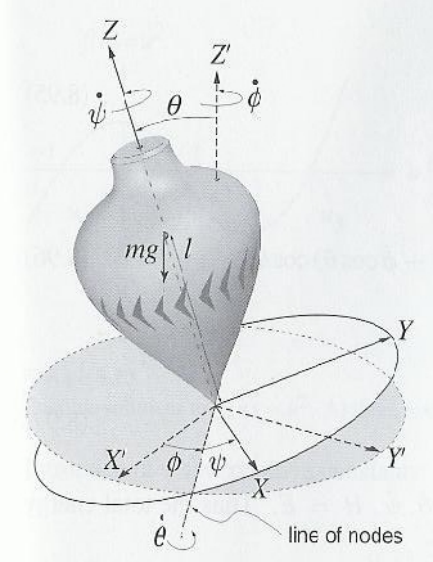


Figure 5.5: L/L_*, T/T_*, and θ vs. γ for various values of I_1/I_3. Notice how L_* and T_* are maximum values for $I_1/I_3 < 1$ and minimum values for $I_1/I_3 > 1$. Note also how the ratio $\theta/\gamma < 1$ for $I_1/I_3 < 1$ and vice versa, as discussed in the text.

5.3.3 Motion under the Influence of External Torques

Now, let us consider motion under the influence of external torques. We will focus on the example of the heavy symmetric top. It now becomes useful to use Euler angles because we may not assume $\vec{L}$ is constant, and thus there are more non-constant degrees of freedom. Unfortunately, we will not have time to consider any further applications, but you can refer to Hand and Finch and Goldstein, which have discussions of the precession of the earth's spin axis due to torques from the Sun and Moon and Larmor precession of the orbits of charged particles in a magnetic field.

We consider the motion of a symmetric top held fixed at some point along its symmetry axis and subject to gravity. Let gravity act along the $-\hat{z}'$ space frame axis. Refer to the following figure:



(Hand and Finch 8.14)

Analytical Mechanics

Euler's Equations with Torque

It is straightforward to rederive Euler's equations including external torques – we simply set $\frac{d}{dt}\vec{\underline{L}}' = \vec{\underline{\tau}}'$, where $\vec{\underline{\tau}}'$ is the space-frame representation of the sum of all external torques acting on the body. (We shall see in practice that specifying $\mathbf{R}^T\vec{\underline{\tau}}'$, the body frame representation, is in general easier because the torque direction tends to change with the top's motion, but we will deal with that naturally later.) So we begin with

$$\vec{\underline{\tau}}' = \vec{\underline{\omega}}' \times \vec{\underline{L}}' + \mathbf{R}\,\frac{d}{dt}\left[\mathbf{R}^T\vec{\underline{L}}'\right]$$

Moving the $\vec{\omega} \times \vec{L}$ term to the left and applying $\mathbf{R}^T$ as before, we find

$$\frac{d}{dt}\left[\mathbf{R}^T\vec{\underline{L}}'\right] = -\left[\mathbf{R}^T\vec{\underline{\omega}}'\right] \times \left[\mathbf{R}^T\vec{\underline{L}}'\right] + \mathbf{R}^T\vec{\underline{\tau}}'$$

That is, the total apparent torque in the body frame is the sum of the fictitious Coriolis-like term present in the absence of external torque and the external torque transformed into the body frame.

We must first write down the torque. This is a bit tricky because the torque is rotating in both the space and the body frames. First, to be clear about the frames: our nonrotating and rotating frames F' and F will both have their origins at the bottom of the top, which we assume is held fixed. The F' system is inertial. The F system is fixed to the top body and so precesses, tilts, and spins with it.

For the torque, certainly we know $\vec{\tau} = \vec{r} \times \vec{F}$ regardless of frame. The center of mass is a distance l from the origin of both coordinate systems. It holds that $\underline{\vec{r}} = l\,\hat{z}$ because the 3-axis of the top coincides with the z axis of the F system. We also know $\underline{\vec{F}}' = -M\,g\,\hat{z}'$ because gravity points downward in the space frame. The unit vectors we use are in different frames, though, so we need to convert one representation. We could use the explicit form of the rotation matrix in terms of Euler angles, Equation 5.9, to directly convert $\underline{\vec{F}}'$ to the body frame or $\underline{\vec{r}}$ to the space frame. But we can use a small trick based on already having done that to the angular velocity vector. Suppose we have an angular velocity vector $\vec{\omega}$ with $\underline{\vec{\omega}}' = a\,\hat{z}'$. Then, based on Equation 5.10, we know $\dot{\phi} = a$ and $\dot{\psi} = 0$ and $\dot{\theta} = 0$. Using the expression for $\mathbf{R}^T\,\underline{\vec{\omega}}'$ also given in Equation 5.10, we therefore know $\mathbf{R}^T\,\underline{\vec{\omega}}' = a\,\sin\theta\,(\hat{x}\,\sin\psi + \hat{y}\cos\psi) + \hat{z}\,a\,\cos\theta$. Dropping out the arbitrary a factor, we thus see that $\hat{z}' = \sin\theta\,(\hat{x}\,\sin\psi + \hat{y}\cos\psi) + \hat{z}\,a\,\cos\theta$. We use this to trivially do the cross product, obtaining $\mathbf{R}^T\underline{\vec{\tau}}' = M\,g\,l\,\sin\theta\,(\hat{x}\cos\psi - \hat{y}\sin\psi)$. With the torque in hand, we may write down Euler's equations with torque:

$$
\begin{aligned}
I_1\,\frac{d}{dt}\,\omega_1 &= \omega_2\,\omega_3\,(I_1 - I_3) + M\,g\,l\,\sin\theta\,\cos\psi \\
I_1\,\frac{d}{dt}\,\omega_2 &= \omega_1\,\omega_3\,(I_3 - I_1) - M\,g\,l\,\sin\theta\,\sin\psi \\
I_3\,\frac{d}{dt}\,\omega_3 &= 0
\end{aligned}
$$

We see that $I_3\,\omega_3$ is constant. But the behavior in the other two axes is complicated by the presence of the torque term so that ω_1 and ω_2 cannot be easily obtained. In particular, note that the form for $\vec{\omega}$ in terms of Euler angles is nontrivial (Equation 5.10), so the presence of functions of θ and ψ on the right side is a very significant complication.

Another difficulty with Euler's equations is that they hide a second conserved momentum, p_ϕ. We see $p_\psi = I_3\omega_3$ is trivially conserved because the top is symmetric, but we shall see p_ϕ is a strange linear combination of the angular momenta along the three Cartesian body axes. Once we have determined p_ψ and p_ϕ from the Lagrangian in terms of Euler angles, we will rewrite them in terms of the body-frame angular velocity components to illustrate how hard it would be to obtain conservation of p_ϕ from Euler's equations.

We note that we could have circumvented part of the complication above by working in an intermediate frame F_P as we did for the torque-free top problem, one that precesses and tilts with the top but does not spin. In this case, we can set $\psi = 0$, so the torque is simply $\mathbf{R}^T\underline{\vec{\tau}}' = \hat{x}\,M\,g\,l\,\sin\theta$, yet $\mathcal{I}$ is still diagonal in this frame so Euler's equations are relatively simple. There is an added complication, though, in working in this intermediate frame. It is only separated from the inertial frame by the precession angular velocity $\vec{\omega}_P$, yet we want to write down the rate of change of the total angular momentum $\vec{L} = \mathcal{I}\,\vec{\omega}$, where $\vec{\omega} = \vec{\omega}_P + \vec{\Omega}$ with $\vec{\Omega}$ being the spin angular velocity. So, Euler's equations become

$$
\frac{d}{dt}\,\underline{\vec{L}}' = \underline{\vec{\omega}}'_P \times \underline{\vec{L}}' + \mathbf{R}_{F_P \to F'}\,\frac{d}{dt}\left[\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{L}}'\right]
$$

where $\mathbf{R}_{F_P \to F'}$ is the rotation matrix to go from F_P to F' as distinguished from the matrix needed to go from the body frame F to F'. So, we consider the time derivatives of the total angular momentum vector, but the relative angular velocity of the two frames is only $\vec{\omega}_P$, not $\vec{\omega}$. Rearranging terms and applying $\mathbf{R}^T_{F_P \to F'}$ to obtain the F_P frame representation:

$$\frac{d}{dt}\left[\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{L}}\,'\right] = -\left[\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{\omega}}\,'_P\right] \times \left[\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{L}}\,'\right] + \mathbf{R}^T_{F_P \to F'}\,\underline{\vec{\tau}}\,'$$

and, then, using $\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{L}}\,' = \underline{\mathcal{I}}^P\left[\mathbf{R}^T\underline{\vec{\omega}}\,'\right]$, we find

$$\underline{\mathcal{I}}^P\,\frac{d}{dt}\left[\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{\omega}}\,'\right] = -\left[\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{\omega}}\,'_P\right] \times \underline{\mathcal{I}}^P\left[\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{\omega}}\,'\right] + \mathbf{R}^T_{F_P \to F'}\,\underline{\vec{\tau}}\,'$$

Finally, we write the above in component form, writing ω_i^P for $\left[\mathbf{R}^T_{F_P \to F'}\,\underline{\vec{\omega}}\,'\right]_i$ and similarly for $\vec{\omega}_P$:

$$\begin{aligned}
I_1\,\frac{d}{dt}\,\omega_1^P &= I_1\,\omega_{P,3}^P\,\omega_2^P - I_3\,\omega_{P,2}^P\,\omega_3^P + M\,g\,l\,\sin\theta \\
I_1\,\frac{d}{dt}\,\omega_2^P &= I_3\,\omega_{P,1}^P\,\omega_3^P - I_1\,\omega_{P,3}^P\,\omega_1^P \\
I_3\,\frac{d}{dt}\,\omega_3^P &= I_1\left(\omega_{P,2}^P\,\omega_1^P - \omega_{P,1}^P\,\omega_2^P\right)
\end{aligned}$$

The torque is more easily written, but the rest of it becomes very complex because the symmetry of the $\vec{\omega} \times \vec{L}$ expression is broken. And the components ω_i^P are no simpler than the ω_i, though the $\omega_{P,i}^P$ are simpler because $\dot{\psi} = 0$ for that angular velocity. Overall, still quite a mess.

Calculating the Lagrangian

For our analysis here of the symmetric top problem with torque, we will not be considering motion about the center of mass, but relative to some pivot point along the 3 (symmetry) axis of the top. That is, we fix one point of the top in the space frame and determine how the top moves about that point. One immediately has to ask the question – if the fixed point is not the center of mass, don't we end up having to worry about translational motion and displaced axes? Doesn't it get very complicated? We can deal with the complications by carefully making use of the different frames at the appropriate points in the analysis. Working step-by-step:

1. Begin in the body frame, whose origin is the center of mass and whose axes are aligned with the principal axes of the top. The inertia tensor in this frame is $\underline{\mathcal{I}} = \mathrm{diag}(I_1,I_1,I_3)$.
2. Define the frame F_d to be the body frame displaced along the 3 axis so its origin is at the pivot point. Let the distance from the new origin to the center of mass be l. The new moment of inertia tensor is

$$\begin{aligned}
\underline{\mathcal{I}}_d &= \underline{\mathcal{I}} + M\left(l^2\,\mathbf{1} - \vec{l}\,\vec{l}^{\,T}\right) \\
&= \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} + M\begin{pmatrix} l^2 & 0 & 0 \\ 0 & l^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} I_{1d} & 0 & 0 \\ 0 & I_{1d} & 0 \\ 0 & 0 & I_3 \end{pmatrix}
\end{aligned}$$

where $I_{1d} = I_1 + M\,l^2$. We see that displacing along the 3 axis keeps the inertia tensor diagonal and the top symmetric, both of which are critical. Now, we may write the kinetic energy, using Equation 5.11:

$$T = \frac{1}{2}\,I_{1d}\left(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\theta\right) + \frac{1}{2}\,I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$$

Note that, in many texts, the fact that it is I_{1d} that appears, rather than I_1, usually is barely noted or ignored completely. Symbolically, there is nothing wrong with the usual derivations as long as one is careful to interpret I_1 as I_{1d}. It would matter if one wanted to do numerical work.

3. Calculate the torque relative to the pivot point in the fixed frame F'. We must do this in the fixed frame because it is inertial. Assume the top is oriented with Euler angles ϕ, θ, and ψ. The gravitational force acts along $-\hat{z}\,'$. Therefore, the magnitude of the torque is $\tau = M\,g\,l\,\sin\theta$. The torque arises from gravity, so it is conservative. We may therefore calculate a potential energy. The "line element" is $d\vec{\theta}$, which is always along $\vec{\tau}$, so we have

$$U(\theta) - U(\theta = 0) = -\int_0^\theta \vec{\tau}\cdot d\vec{\theta}\,' = -\int_0^\theta d\theta'\,Mgl\,\sin\theta = M\,g\,l\,\left(\cos\theta - 1\right)$$

4. Since T is a scalar, and F' and F_d are related only by rotation, T has the same form in the F frame. We already have U in the F' frame. So we have

$$L = T - U = \frac{1}{2}\,I_{1d}\left(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\theta\right) + \frac{1}{2}\,I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - M\,g\,l\,\cos\theta$$

where we have dropped the constant term in U.

Conserved Quantities

The Lagrangian is obviously cyclic in ψ and ϕ because the torque is both only in the θ direction and only dependent on θ, so the corresponding canonical momenta are conserved. They are

$$\begin{aligned} p_\psi &= \frac{\partial L}{\partial\dot{\psi}} = I_3\left(\dot{\psi} + \dot{\phi}\,\cos\theta\right) \\ p_\phi &= \frac{\partial L}{\partial\dot{\phi}} = I_{1d}\,\dot{\phi}\,\sin^2\theta + I_3\left(\dot{\psi} + \dot{\phi}\,\cos\theta\right)\cos\theta \\ &= \dot{\phi}\left(I_{1d}\,\sin^2\theta + I_3\,\cos^2\theta\right) + I_3\,\dot{\psi}\,\cos\theta = I_{1d}\,\dot{\phi}\,\sin^2\theta + p_\psi\,\cos\theta \end{aligned}$$

With the canonical momenta in hand, it is useful to compare the results of the Lagrangian technique to our previous solution for the torque-free case. In the absence of torque, we obtain the equation of motion in θ:

$$I_{1d}\,\ddot{\theta} = I_{1d}\,\dot{\phi}^2\sin\theta\,\cos\theta - I_3\left(\dot{\psi} + \dot{\phi}\,\cos\theta\right)\dot{\phi}\,\sin\theta$$

($p_\theta = I_{1d}\ddot{\theta}$ is easy to see). Requiring stability in θ (as we indeed had for the torque-free top) would yield

$$I_{1d}\,\dot{\phi}\,\cos\theta = p_\psi$$

Inserting this relation in the expression for p_ϕ gives

$$p_\phi = I_{1d}\,\dot{\phi}$$

We can therefore make the following correspondences:

$$\begin{aligned} \dot{\phi} &\iff \omega_P \\ p_\phi = I_{1d}\,\dot{\phi} &\iff L = |\vec{\underline{L}}'| \\ p_\psi &\iff I_3\,\omega_3 = L\,\cos\theta \\ \dot{\psi} &\iff \Omega \end{aligned}$$

(Don't confuse the angular momentum L with the Lagrangian L!) The correspondences between $\dot{\phi}$ and ω_P and $\dot{\psi}$ and Ω are not surprising, but the relation between L and the canonical momenta is interesting. In the torque-free case, p_ϕ is the total angular momentum and p_ψ is the projection of the total angular momentum along the body 3 axis. Since both p_ψ and p_ϕ are conserved *and* $L = p_\phi$ when there is no torque, we find that $\cos\theta$ is constant, confirming the earlier assumption of stability in θ and also matching our original solution. Note that the simple relationship $L = p_\phi$ only holds when we can assume θ is constant. Thus, when torque is applied, θ may change even though p_ϕ and p_ψ are conserved. Note also that p_ϕ and p_ψ are constant only when the torque is a function of θ only. A counterexample would be a frictional torque acting at the pivot point or via air resistance.

We may also now write the correspondence between p_ψ and p_ϕ and the Cartesian components of angular momentum. We make use of the expressions for angular velocity in the body frame in terms of Euler angles, Equation 5.10. We obtain

$$\begin{aligned} p_\psi &= I_3\,\omega_3 = L_3 \\ p_\phi &= (I_1\omega_1\sin\psi + I_1\,\omega_2\cos\psi)\sin\theta + I_3\,\omega_3\cos\theta \\ &= (L_1\sin\psi + L_2\cos\psi)\sin\theta + L_3\,\cos\theta \end{aligned}$$

So, indeed, p_ϕ is a complicated combination of angular momenta along the different Cartesian axes.

When we do consider torques below, total angular momentum is both not conserved and also has a contribution from p_θ that is not present in the torque-free case. But the conservation of p_ϕ and p_ψ will remain useful.

Effective Potential and 1D Equation of Motion

With the gravitational torque, the equation of motion of θ is

$$\begin{aligned} I_{1d}\,\ddot{\theta} &= I_{1d}\,\dot{\phi}^2\,\sin\theta\,\cos\theta - I_3\left(\dot{\psi} + \dot{\phi}\,\cos\theta\right)\dot{\phi}\,\sin\theta + Mg\,l\,\sin\theta \\ &= -\frac{\partial}{\partial\theta}\left(\frac{1}{2\,I_{1d}}\frac{(p_\phi - p_\psi\,\cos\theta)^2}{\sin^2\theta}\right) + Mg\,l\,\sin\theta \end{aligned}$$

Some of the kinetic terms have generated a "centrifugal" potential. We may sum the centrifugal and true potentials to obtain an effective potential

$$V_{eff}(\theta) = \frac{1}{2\,I_{1d}}\frac{(p_\phi - p_\psi\,\cos\theta)^2}{\sin^2\theta} + M\,g\,l\,\cos\theta$$

(It takes some algebra to see that the effective potential version is equivalent to the original equation of motion.) We may use the effective potential to write an effective one-dimensional Lagrangian that would yield the above equation of motion:

$$\begin{aligned} L_{1D} &= \frac{I_{1d}}{2}\,\dot{\theta}^2 + \frac{1}{2\,I_3}\,p_\psi^2 - V_{eff}(\theta) \\ &= \frac{I_{1d}}{2}\,\dot{\theta}^2 + \frac{1}{2\,I_3}\,p_\psi^2 - \frac{1}{2\,I_{1d}}\frac{(p_\phi - p_\psi\,\cos\theta)^2}{\sin^2\theta} - M\,g\,l\cos\theta \end{aligned}$$

(One could have more directly obtained L_{1D} and V_{eff} via the Routhian procedure – see Hand and Finch Problems 5-3 and 8-19.) One important point to make is that this is **not** just the original Lagrangian with p_ϕ and p_ψ substituted in – there is a sign flip in the kinetic terms that give rise to the effective potential. Recall that we ran into a similar issue when defining the effective potential in the case of central forces. You can never just substitute back into the Lagrangian because it violates the formalism wherein different coordinates are assumed to vary independently. We are basically using the full Lagrangian to "inspire" a one-dimensional Lagrangian that yields the correct one-dimensional equation of motion. We may rewrite the total energy using the effective potential energy, giving

$$\left(E - \frac{1}{2\,I_3}\,p_\psi^2\right) = \frac{I_{1d}}{2}\,\dot{\theta}^2 + \frac{1}{2\,I_{1d}}\frac{(p_\phi - p_\psi\,\cos\theta)^2}{\sin^2\theta} + M\,g\,l\,\cos\theta$$

(The total energy does not suffer the sign flip confusion.) We thus have a one-dimensional differential equation in θ and the constants of the motion E, p_ϕ, and p_ψ.

Solving for the Motion

Rather than using the rather difficult to integrate and analyze equation for $\ddot{\theta}$, we will make use of the energy equation to gain some insight into the form of solutions. It is standard to rewrite it as follows:

$$E' = E - \frac{1}{2\,I_3}\,p_\psi^2 \qquad u = \cos\theta \Rightarrow \dot{u} = -\sin\theta\,\dot{\theta} \qquad a = \frac{p_\psi}{I_{1d}} \qquad b = \frac{p_\phi}{I_{1d}}$$

$$E'\left(1-u^2\right) = \frac{1}{2}\,I_1\,\dot{u}^2 + \frac{1}{2}\,I_1\,(b - a\,u)^2 + M\,g\,l\,u\left(1-u^2\right)$$

We may further rewrite:

$$\alpha \equiv \frac{2\,E'}{I_{1d}} \qquad \beta \equiv \frac{2\,M\,g\,l}{I_{1d}}$$

$$\dot{u}^2 = \left(1-u^2\right)(\alpha - \beta\,u) - (b - a\,u)^2 \equiv f(u)$$

The function $f(u)$ is a cubic polynomial. We can convert the above into an integral relation between t and u:

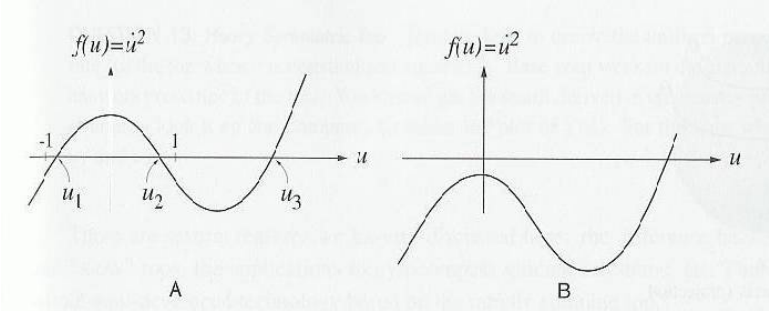
$$t = \int_{u(0)}^{u(1)} \frac{du}{\sqrt{f(u)}}$$

but, because $f(u)$ is cubic, we have an elliptic integral that cannot be done analytically.

General Characteristics of the Motion

We may obtain some qualitative information about the motion without dealing with the elliptic integral. Since $\dot{u}^2$ must be positive, valid values of u are in the regions where $f(u)$ is positive, and turning points of the motion occur where $f(u) = 0$. The following facts hold:

- Generically, for any third-order polynomial, there are either three real, distinct roots, three real roots with two of them equal, or one real and two complex conjugate roots.
- $f(u) \to +\infty$ as $u \to +\infty$ and $f(u) \to -\infty$ as $u \to -\infty$, so the polynomial generically has one of the two shapes shown in the figure below, where the curve may also have the second shape but translated vertically so the one root is on the left side. There is always at least one real root because of the limiting behaviors.
- At $u = \pm 1$ (the physical limits of the motion because $-1 \leq \cos\theta \leq 1$), we have $f(u) = -(b \mp a)^2 < 0$. Therefore, the curve must always be negative at these two points. Thus, the third possible form, with the local minimum of the function above zero, is not allowed.
- We may see that the second possible form is not allowed either. Clearly, we are able to start the top at some value of u and $\dot{u}$. Therefore, for physically allowed values of the constants, there must be a region between $u = -1$ and $u = +1$ with $f(u) \geq 0$. Thus, either the central "bump" in the polynomial or the final rising region must overlap the interval $[-1, 1]$ in u. The latter is not allowed because it would not satisfy $f(u) < 0$ at $u = \pm 1$. So only the central "bump" type of polynomial is allowed. Moreover, because of the condition $f(u) < 0$ at $u = \pm 1$, we also have the condition that the bump must be entirely contained in the interval $[-1, 1]$; the curve must go negative again before $u = \pm 1$.
- Thus, to sum up, we have two turning points at the two roots $\cos\theta_1 = u_1$ and $\cos\theta_2 = u_2$, and the motion is isolated to this range of angles. This bounded motion between the two turning points is called **nutation**.



(Hand and Finch 8.15)

Analytical Mechanics

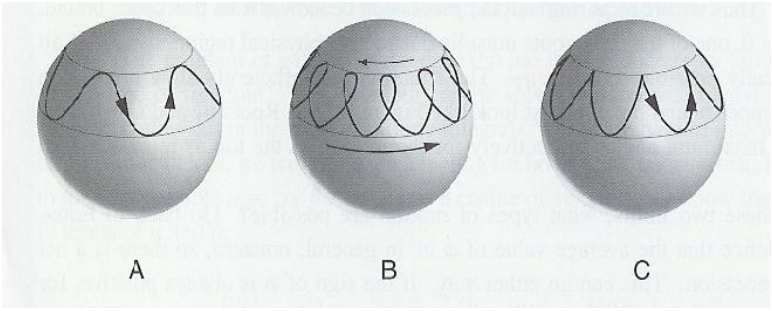
The precessional motion is specified by $\dot{\phi}$, which is given by (from the equation for p_ϕ):

$$\dot{\phi} = \frac{1}{I_{1d}} \frac{p_\phi - p_\psi \cos\theta}{\sin^2\theta}$$

The denominator is always positive, but the numerator may carry either sign and may in fact carry both during different parts of the motion. There are three different regimes:

- $\dot{\phi} > 0$ or $\dot{\phi} < 0$ for all θ. There is always precession in one direction. The $\dot{\phi} > 0$ case is illustrated in (A) below.
- $\dot{\phi} = 0$ at one of the turning points in θ. This value must be at the upper (smaller θ) turning point if $\dot{\phi} > 0$ and at the lower (larger θ) turning point if $\dot{\phi} < 0$. The first case is illustrated in (C) below.

- $\dot{\phi} = 0$ for some value not equal to the turning points. Then there is a range of θ with $\dot{\phi} > 0$ and a range with $\dot{\phi} < 0$, like (B) below.



(Hand and Finch 8.16)

Pure Precession Solutions

In addition to the above nutation behavior, we can obtain solutions that correspond to pure precession if we just find the value of θ for which the effective potential is minimized (or, equivalently, for which $f(u)$ has a double real root). We saw earlier that the effective potential is

$$V_{eff}(\theta) = \frac{1}{2\,I_{1d}} \frac{(p_\phi - p_\psi\,\cos\theta)^2}{\sin^2\theta} + M\,g\,l\,\cos\theta$$

Taking the derivative with respect to θ gives

$$\left.\frac{\partial V_{eff}}{\partial\theta}\right|_{\theta_0} = \frac{(p_\phi - p_\psi\,\cos\theta_0)^2}{I_{1d}} \frac{-\cos\theta_0}{\sin^3\theta_0} + \frac{(p_\phi - p_\psi\,\cos\theta_0)\,p_\psi\,\sin\theta_0}{I_{1d}\,\sin^2\theta_0} - Mgl\,\sin\theta_0 = 0$$

$$\cos\theta_0\,(p_\phi - p_\psi\,\cos\theta_0)^2 - (p_\phi - p_\psi\,\cos\theta_0)\,p_\psi\,\sin^2\theta_0 + Mgl\,I_{1d}\sin^4\theta_0 = 0$$

The above equation is a quadratic in $(p_\phi - p_\psi\,\cos\theta_0)$, which can be solved to find

$$p_\phi - p_\psi\,\cos\theta_0 = \frac{p_\psi \sin^2\theta_0}{2\,\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4\,Mgl\,I_{1d}\,\cos\theta_0}{p_\psi^2}}\right)$$

This does not yield an analytic formula for θ_0, but we can certainly investigate under what conditions a solution exists and what happens in those cases.

First, if $\theta_0 < \pi/2$, the quantity under the radical can be negative, so we require it be positive to obtain a reasonable solution. That gives the condition

$$p_\psi^2 \geq 4Mgl\,I_{1d}\cos\theta_0 \quad\Longleftrightarrow\quad \omega_3 \geq \frac{2}{I_3}\sqrt{Mgl\,I_{1d}\cos\theta_0}$$

That is, the top must be spinning faster than a particular threshold value in order for there to be a steady precession solution. When there is such a solution, we can use our expression for $\dot{\phi}$,

$$\dot{\phi} = \frac{1}{I_{1d}} \frac{p_\phi - p_\psi\cos\theta_0}{\sin^2\theta_0}$$

to see that the existence of two solutions for $p_\phi - p_\psi \cos\theta_0$ implies there are two possible precession velocities, a fast and a slow one, depending on the sign chosen. When p_ψ greatly exceeds the precession threshold and the second term under the radical is small, we can Taylor expand to find

$$\dot{\phi}_{fast} = \frac{p_\psi}{I_{1d}\cos\theta_0} \qquad \dot{\phi}_{slow} = \frac{M\,g\,l}{p_\psi}$$

If $\pi/2 < \theta_0 < \pi$, then the quantity under the radical is always positive and there is no minimum value of ω_3. In fact, the quantity under the radical is larger than 1, so the fast precession occurs in the same direction as before but the slow precession speed becomes negative. This is confirmed in the above formulae for large p_ψ by recognizing, if $\dot{\psi} > 0$ and $\dot{\phi} > 0$, then $p_\psi < 0$ and $\cos\theta_0 < 0$ if $\pi/2 < \theta_0 < \pi$.

A final interesting case is that of the "sleeping top", for which $\theta = 0$ for all time. Since $\sin^2\theta$ would result in an infinite effective potential, this case only occurs when $p_\phi = p_\psi$, which, if we refer back to the expressions for these conserved momenta, can only occur if $\dot{\phi} = 0$ and $\theta = 0$. This case thus becomes possible because the effective potential reduces to the true potential and the true potential has an unstable equilibrium at $\theta = 0$. In terms of $f(u)$, this occurs when two of the roots are equal and coincide with $u = 1$.

Chapter 6

Special Relativity

We present the special theory of relativity, derive the transformation rules for various physical quantities, display the analogy between these transformations and rotational transformations, and explore dynamics in relativistic situations.

6.1 Special Relativity

We follow Hand and Finch for much of this section, though we do insert some additional material relating transformations between inertial frames to rotational transformations.

6.1.1 The Postulates

The two basic postulates of special relativity are

1. Physics is the same in all inertial frames.

2. The speed of light is the same in all inertial frames.

The first postulate is just the principle of Galilean relativity that we stated at the start of the course. At that point, we defined an inertial frame to be a frame in which Newton's second law holds. The implication of Galilean relativity is that there is no absolute frame of reference; every inertial reference frame is as good as every other one.

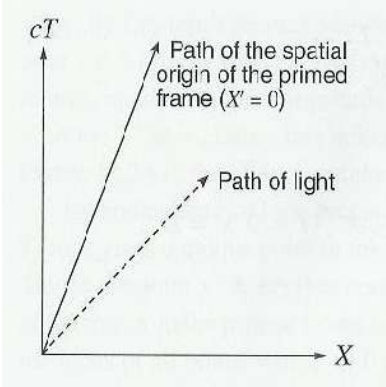
It is the second postulate that was Einstein's brilliant leap and that leads to all the nonintuitive implications of special relativity. To some extent, the second postulate is a corollary of the first once one realizes that electromagnetic waves do not travel in a medium. If the laws of electromagnetism, which give rise to the speed of light, are to be the same in all frames, then the speed of light must of necessity be the same in all frames. But it took the Michelson-Morley experiment to kill the concept of the ether, a medium in which light propagates.

6.1.2 Transformation Laws

The basic content of special relativity is in the transformation laws it implies for physical quantities. From these transformation laws we may derive all the strange implications of special relativity – time dilation, length contraction, etc. We will begin with transformation rules for spatial coordinates – the Lorentz transformation – and from there develop transformation rules for velocities, momentum, energy, etc.

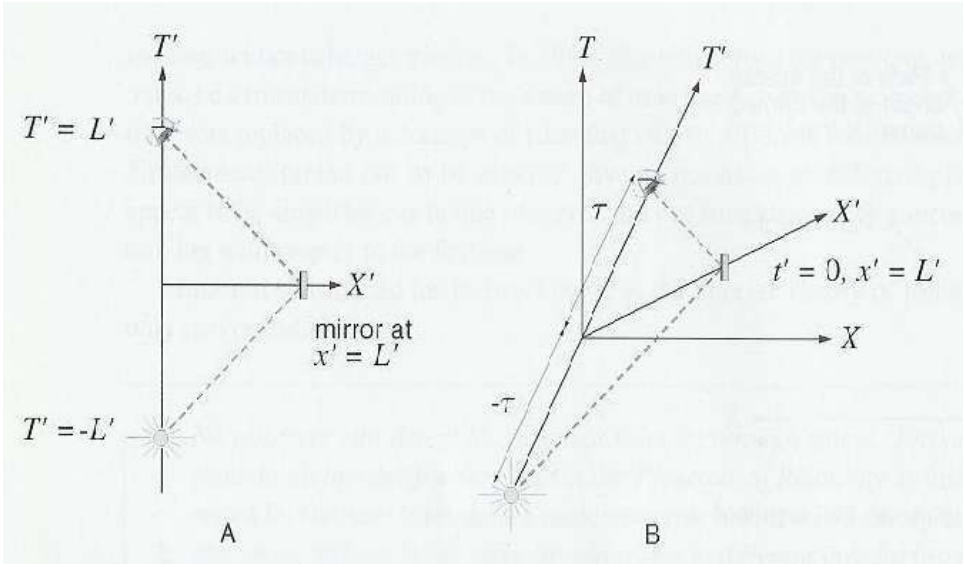
Developing the Lorentz Tranformation Rules for Space-Time Coordinates

Consider two inertial reference frames F and F'. Let them have coordinate axes (x, y, z, t) and (x', y', z', t'). We include t as a coordinate and allow it to be different in the F' frame because it will be necessary to avoid a contradiction with the second postulate. All times are scaled by the speed of light (*i.e.*, t is actually $c\,t$) for reasons that will become apparent. With this scaling, light travels with speed 1. Let the two systems' axes and origins coincide at $t = t' = 0$. Let the F' frame be moving at speed β (in $c = 1$ units) along the $+x$ axis with respect to F, which means that the position of the F' origin obeys $x = \beta\,t$ in the F frame. This information is summarized in our **space-time diagram** in the F system, a plot of t vs. x with the origin of the F' system represented by the solid line of slope β^{-1} and the path of a light ray emitted from the origin shown by the dashed line of slope 1. Points in the space-time diagram are referred to as **events** because they are not just points in space, but in time also. Obviously, one can generalize space-time to more than one spatial dimension, it just becomes hard to visualize.



(Hand and Finch 12.1)

Let us consider the path of a light ray in the two frames, requiring that the second postulate hold. Suppose the light ray is emitted from the F' origin at time $t'_e = -L'$ in the $+x$ direction, it hits a mirror at $x'_r = L'$ at $t'_r = 0$ and is reflected, returning to the F' origin at $t'_a = L'$. The light ray has $y' = z' = 0$ for all time. In a space-time diagram of the F' frame, the reflection event (x'_r, t'_r) is obtained by the intersection of light rays propagating forward in time from the emission event $(x'_e, t'_e) = (0, -L')$ and backward in time from the absorption event $(x'_a, t'_a) = (0, L')$. The intersection is at $(x'_r, t'_r) = (L', 0)$.



Left: F' system. Right: F system.
(Hand and Finch 12.2,
note the correction of the emission and absorption times.)

Let's examine the path in the F system, assuming we do not yet know the transformation law for coordinates between reference frames. The light ray travels only along the x direction, so it satifies $y = z = 0$ for all time also. Let (x_e, t_e), (x_r, t_r) and (x_a, t_a) indicate the coordinates of the three events in the F frame. The emission and absorption events must

occur on the solid line representing the position of the F' origin, which means their space and time coordinates are related by the slope β^{-1}. The line of this slope through the origin thus gives the space-time location of the t' axis in F. Time reversal invariance says that these events' symmetric occurence in F' implies they occur at symmetric times in F. So the two events must be at $(-t_a\,\beta, -t_a)$ and $(t_a\,\beta, t_a)$. Let's calculate where light rays from these two events would intersect, which will give us the position of the reflection event in F (and also the space-time location of the x' axis in F). In the following, r and s begin as undetermined parameters indicating how much time passes in the F frame between the emission or absorption event and the reflection event, respectively:

$$\begin{gathered}(-t_a\,\beta, -t_a) + r\,(1,1) = (t_a\,\beta, t_a) + s\,(1,-1)\\ r - s = 2\,\beta\,t_a \qquad r + s = 2\,t_a\\ r = (1+\beta)\,t_a \qquad s = (1-\beta)\,t_a\\ (x_r, t_r) = (-t_a\,\beta, -t_a) + r\,(1,1) = (t_a, \beta\,t_a)\end{gathered}$$

The last line implies that the reflection event (x_r, t_r) sweeps out the line through the origin with slope β. Its position on that line depends on the value of t_a, the time of the absorption event – that is to say, on L' and β. Since the reflection event (x'_r, t'_r) is always at $t' = 0$, the line thus tells us where the x' axis sits in the F space-time diagram. So, in sum, we have that the t' axis is a line of slope β^{-1} and the x' axis is a line of slope β, both going through the F origin. The angle between the lines is α, where

$$\tan\frac{\alpha}{2} = \frac{1-\beta}{1+\beta}$$

We may construct the general form for the transformation of coordinates from F' to F by using the above information along with expected properties of the transformation:

1. **Linearity**: We have seen that events on the x' axis lie on a line of slope β and events on the t' axis lie on a line of slope β^{-1} in F. If we assume the transformation is linear in the space-time coordinates then the transformation for an arbitrary space time event with F' frame coordinates (x', t') to the F frame can be written

$$(x,t) = \gamma(\beta)\,(1,\beta)\;x' + \tilde{\gamma}(\beta)\,(\beta,1)\;t'$$

2. **Equivalence of the two frames:** Since one of our postulates is that physics is the same in any reference frame, the transformation for going from F to F' must have the same form as the transformation from F' to F, modulo the change in sign of β, so we also have

$$(x',t') = \gamma(-\beta)\,(1,-\beta)\;x + \tilde{\gamma}(-\beta)\,(-\beta,1)\;t$$

3. **Symmetry of the two frames:** By symmetry, the slope of the x and t axes in the F' space-time diagram must be the same as the slope of the x' and t' axes in the F space-time diagram, so $\gamma(\beta) = \gamma(-\beta)$ and $\tilde{\gamma}(\beta) = \tilde{\gamma}(-\beta)$ is required. So we have

$$\begin{aligned} x' &= \gamma\,x - \tilde{\gamma}\,\beta\,t & \qquad t' &= -\gamma\,\beta\,x + \tilde{\gamma}\,t\\ x &= \gamma\,x' + \tilde{\gamma}\,\beta\,t' & \qquad t &= \gamma\,\beta\,x' + \tilde{\gamma}\,t'\end{aligned}$$

4. **The transformation be invertible by the reverse transformation:** We require that if we transform from F' to F and then from F to F', the overall transformation should return the original (x', t'). We apply this by using the first pair of formulae for x and t in the second pair of formulae and requiring that we recover x' and t':

$$\gamma^2 - \tilde{\gamma}\,\gamma = 0 \qquad \gamma^2 - \tilde{\gamma}\,\gamma\,\beta^2 = 1$$

which is solved by

$$\tilde{\gamma} = \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

With these conditions, the transformation law is

$$\begin{aligned} x' &= \gamma\,(x - \beta\,t) & t' &= -\gamma\,(\beta\,x - t) \\ x &= \gamma\left(x' + \beta\,t'\right) & t &= \gamma\left(\beta x' + t'\right) \end{aligned} \tag{6.1}$$

This is known as the **Lorentz transformation** for historical reasons.

Implications of the Lorentz Transformation

We can derive a number of the most shocking implications of relativity from the simple Lorentz transformation laws:

- **Time dilation**
  Consider two events occuring at a fixed point in space in the frame F'; for example, two ticks of a clock. They are separated by the vector $(x', t') = (0, \tau)$. What is the separation of the two events in the frame F, relative to which F' is moving at speed β? The Lorentz transformation tells us

  $$x = \gamma\,\beta\,\tau \qquad t = \gamma\,\tau$$

  The time between the events in the F frame is larger than in F'. Hence, the term "time dilation" – time "slows down" in the moving frame, the two events have a smaller time separation in their rest frame than in any other frame. One is not obtaining something for nothing, though, because the spatial separation of the two events has become nonzero. That is, we are no longer measuring just the time separation of two events that occur at the same point in space; we are measuring a separation with both time and space components.
- **Length contraction**
  The length L' of an object at rest in F' can be viewed as two events with separation $(\Delta x', \Delta t') = (L', 0)$, representing the left and right ends of the object at some common time t'. The time separation between these two events will become nonzero in the F frame because of their nonzero spatial separation, so these two events are not valid as a length measurement in F. Explicitly, the ends of the object are at space-time coordinates

  $$(x'_1, t'_1) = (0, \tau) \qquad (x'_2, t'_2) = (L', \tau)$$

  Without lack of generality, we let the two events corresponding to the F' length measurement be

  $$(x'_1, t'_1) = (0, 0) \qquad (x'_2, t'_2) = (L', 0)$$

The Lorentz transformation of the trajectories is

$$\begin{array}{lll} \text{in general:} & (x_1,t_1) = (\gamma\,\beta\,\tau, \gamma\,\tau) & (x_2,t_2) = (\gamma\,L' + \gamma\,\beta\,\tau, \gamma\,\beta\,L' + \gamma\,\tau) \\ \text{with } \tau = 0\text{:} & (x_1,t_1) = (0,0) & (x_2,t_2) = (\gamma\,L', \gamma\,\beta\,L') \end{array}$$

$$\Longrightarrow (\Delta x, \Delta t) = (\gamma\,L', \gamma\,\beta\,L')$$

confirming the expectation that the length measurement events in F' do not give a length measurement in F. To make a length measurement in F, we must pick points on the two trajectories that are separated by $\Delta t = 0$, which the above expression implies is not possible if $t'_1 = t'_2$. Let us allow $t'_1 \neq t'_2$ and apply the requirement $\Delta t = 0$, using the Lorentz transformation to write this condition in terms of F' coordinates:

$$0 = \Delta t = t_2 - t_1 = \gamma\left[\beta\left(x'_2 - x'_1\right) + \left(t'_2 - t'_1\right)\right]$$

Without lack of generality, we again assume event 1 occurs when the two origins intersect, so $x'_1 = x_1 = t'_1 = t_1 = 0$. $t_2 = 0$ also in order to obtain a length measurement in F. The assumption about the origins implies that $x'_2 = L'$ for all time because the object is at rest in F'. So the above reduces to

$$t'_2 = -\beta x'_2 = -\beta\,L'$$

The length as measured in F is simply the Lorentz transformation of event 2:

$$x_2 = \gamma\left(x'_2 + \beta\,t'_2\right) = \gamma\left(x'_2 - \beta^2 x'_2\right) = \gamma^{-1}x'_2 = \gamma^{-1}L'$$

We see a decrease in the apparent length. The F-frame length measurement has space-time coordinates

$$\begin{array}{ll} (x_1,t_1) = (0,0) & (x'_1,t'_1) = (0,0) \\ (x_2,t_2) = (\gamma^{-1}L', 0) & (x'_2,t'_2) = (L', -\beta\,L') \end{array}$$

We see that in order to be simultaneous in the F frame, the two events must occur with negative time separation $-\beta\,L'$ in the rest frame of the object. The object is moving to the right with the F' frame, so the right end of the object is not at $x' = L'$ yet in the F frame when the measurement event occurs in F, hence the apparent length contraction.

- **Relativity of Simultaneity**

  The length contraction example shows that two events that are temporally simultaneous but spatially separated in one frame may not be temporally simultaneous in another – *i.e.*, $t' = 0$ does not imply $t = 0$ unless $x' = 0$ also. Generically, then, simultaneity of physically separated events is no longer well defined.

- **Transformations of Areas**

  The matrix of partial derivatives of our transformation is

$$\mathbf{J}(\beta) = \begin{pmatrix} \gamma & -\beta\,\gamma \\ -\beta\,\gamma & \gamma \end{pmatrix}$$

  So, space-time areas are preserved:

$$dx'\,dt' = |\mathbf{J}|\,dx\,dt = dx\,dt$$

because $\gamma^2 - \beta^2\gamma^2 = 1$. This is a necessity, as there must be symmetry between the two directions for the Lorentz transformation, which would not hold if the Jacobian determinant were not unity.

Invariant Interval

From the Lorentz transformation, one can easily obtain the identity

$$t^2 - x^2 = (t')^2 - (x')^2 \equiv s^2$$

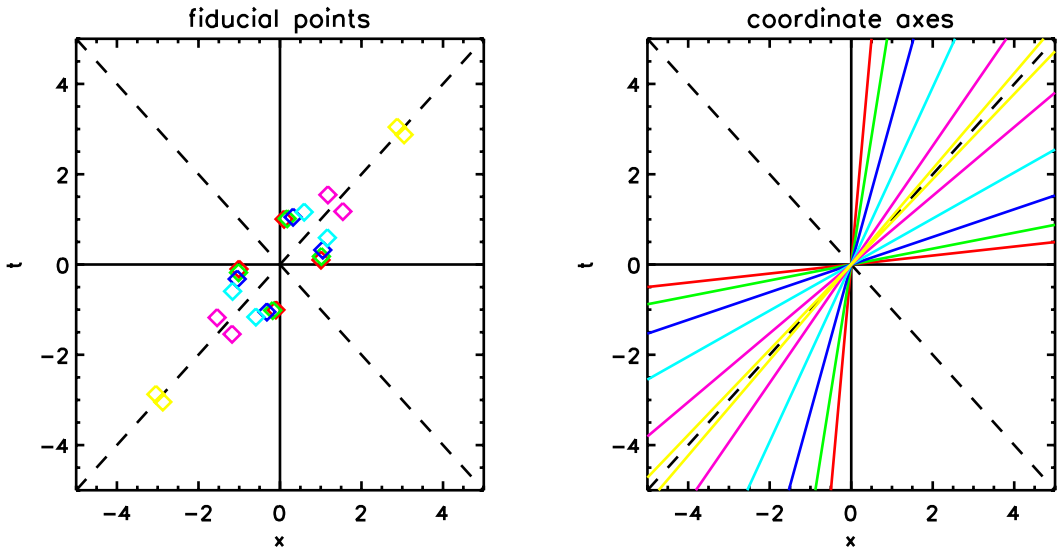
The quantity s^2 is the **invariant interval** associated with the space-time vector (x, t) and (x', t'). It can be thought of like the magnitude of a vector in space, which is invariant under spatial rotations. The invariant interval is invariant under Lorentz transformations, also known as **boosts**. In the geometrical interpretation given below, the invariant interval is a consequence of the identity $\cosh^2\eta - \sinh^2\eta = 1$.

Geometrical Interpretation of Lorentz Transformation

We may develop a geometrical interpretation of how the Lorentz transformation gives the shape of one set of space-time axes in another frame. Define the **rapidity** or **boost angle** or **boost parameter** η by $\tanh\eta = \beta$ (any η is possible because $\beta = \tanh\eta \to \pm 1$ as $\eta \to \pm\infty$. Then we have

$$\beta = \tanh\eta \qquad \gamma = \cosh\eta \qquad \beta\gamma = \sinh\eta$$
$$x = x'\cosh\eta + t'\sinh\eta \qquad t = x'\sinh\eta + t'\cosh\eta$$

So, the Lorentz transformation looks something like a coordinate rotation, except by an imaginary angle $i\,\eta$. More importantly, though, we see the contours of constant x' or t' form hyperbolic curves in the $x\,t$ plane. That is, the event $(x', 0)$ in the F' frame appears on the hyperbola $x'\,(\cosh\eta, \sinh\eta)$, with η increasing as β increases. Similarly, the event $(0, t')$ appears on the hyperbola $t'\,(\sinh\eta, \cosh\eta)$. This is illustrated in the following figure:



Left: position of $(x', t') = (1, 0), (0, 1), (-1, 0), (0, -1)$ in F for $\eta = 10^{-1}, 10^{-3/4}, 10^{-1/2}, 10^{-1/4}, 1, 10^{1/4}$ (moving sequentially outward). This shows how a particular event is seen in another frame as the relative speed β is increased. Right: F' space-time axes in F for same values of η (larger $\eta \Longrightarrow$ increasingly oblique). This plot shows how the F' space-time axes appear "squeezed together" when seen in the F frame. Both plots are for positive β and η. Negative β and η would occupy the other two quadrants.

NOTE: Even though we have made the above geometrical intepretation, one has to be a bit careful about overinterpreting it. The difficulty is that the quantity left constant by a Lorentz transformation, the invariant interval, does not correspond to a curve of fixed distance from the origin on the above plots. Rather, it corresponds to the hyperbolic curves that the fiducial points follow as β is changed. Another way of seeing this is that the entire first and third quadrants of the F' frame occupy the area between the two corresponding slanted space-time axes displayed in the right plot. Therefore, spatial distances and areas are not preserved by the mapping. Invariant interval is preserved.

Light Cones, Causality, and Simultaneity

The geometrical interpretation and invariant interval let us prove that the relativity of simultaneity can never produce causality problems. First, let us define the **light cone** as the region in space-time that can be reached from a given event; refer to Figure 6.1. The light cone of the event at the origin of a space-time diagram consists of all events with $|t| > |x|$. For $t < 0$, the events in the light cone are those that can have causal influence on the event at the origin; this is the **past light cone** of the event at the origin. For $t > 0$, the **future light cone**, the events in the light cone are those that the event at the origin can have causal influence on. Since we know that, under Lorentz transformations, an event slides along a hyperbolic curve in space-time as the speed β is varied, we are assured that events that are in the future light cone of the event at the origin in one reference frame are also in that event's future light cone in any other reference frame. Similarly for past light cones.

What about simultaneous events? Two events are simultaneous in a particular frame F' if they occur at the same value of t', the time in the frame F'. Let one event be at the origin $(0, 0)$ and the other event at $(x', 0)$, so the space-time vector separating them is $(x', 0)$. This event is outside the light cone of the origin, so, in this frame, the two events are out of causal contact. When we transform to a different frame F, but one whose origin coincides with that of F' at $t = t' = 0$, the space-time vector between the two events will slide on the hyperbolic curve $(\cosh\eta, \sinh\eta)$. The event at $(x', 0)$ in F' may move to $t < 0$ or $t > 0$ in other frames, raising causality questions. The causality worries are put to rest by the fact that the hyperbola is entirely outside the light cone of the first event, so the second event is always outside of causal contact with the first event.

Whether two events are causally connected is determined by the sign of the invariant interval of the space-time vector separating them. If $s^2 > 0$, then $|t| > |x|$ and the two events connected by the vector are in causal contact. The vector is called **time-like**. Our argument about hyperbolic curves ensures that the sign of the time component of the vector does not change, preserving causal relationships. If $s^2 < 0$, then $|t| < |x|$ and the two events are never in causal contact, regardless of frame. The vector is called **space-like**. The sign of the time component of the vector may depend on the reference frame. If $s^2 = 0$, the vector is called **null** or **light-like** since only light (or, as we shall see, any other massless particle) can travel on such a path in space-time.

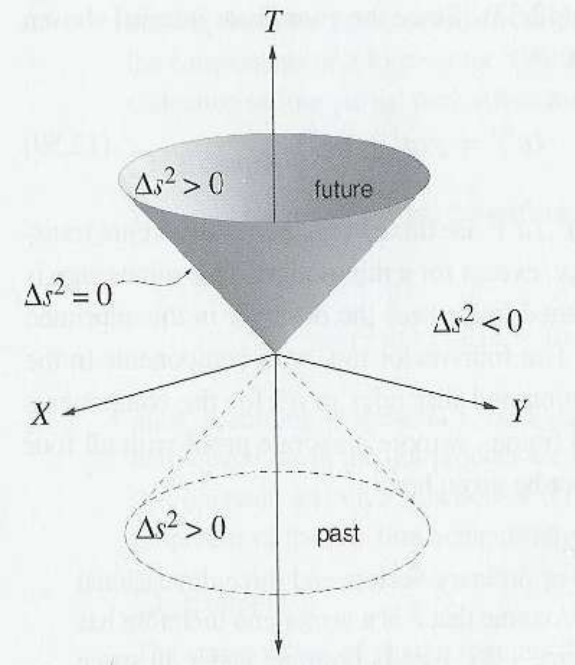


Figure 6.1: Illustration of light cones. (Hand and Finch 12.10)

Perpendicular Coordinates

What happens to the coordinates perpendicular to the boost direction, y and z or y' and z'? It turns out they are unaffected. An argument using clocks is presented in Hand and Finch to justify this. Here, we use an argument similar to the one that we used to deduce the Lorentz transformation for x and t.

Consider again two frames F and F', with F' moving at speed β in the $+x$ direction relative to F, and assume the origins of the two frames coincide at $t = t' = 0$. In this case, we will emit a light ray from the origin in the $+y$ direction and reflect it back to the origin. The emission, reflection, and absorption events are given by:

$$\begin{aligned}(x'_e, y'_e, t'_e) &= (0, 0, -L')\\ (x'_r, y'_r, t'_r) &= (0, L', 0)\\ (x'_a, y'_a, t'_a) &= (0, 0, L')\end{aligned}$$

Let's now repeat our argument regarding the coordinates of the three events in F, with the complication that our space-time now has three dimensions, though the motion is only in two of them. Walking through the argument:

1. The position of the origin of F' in F is described by the same line as in our original argument, except it is now a line in the $x\,t$ plane in three dimensions. Its coordinates in F are $(x_o, y_o, t) = (\beta\, t, 0, t)$ as a function of t. We know the origin's y coordinate does not change because there is no motion in that direction; whatever nonintuitive there may be about relativity, relative motion of two points in a given frame is always well-defined.

2. As before, the emission and absorption events both occur at the position of the origin of the F' system in F, and they must occur symmetrically about the time origin. So we have

$$(x_e, y_e, t_e) = -t_a(\beta, 0, 1)$$
$$(x_a, y_a, t_a) = t_a(\beta, 0, 1)$$

   Symmetry about the origin relates the time coordinates and the known velocity vector of F' in F relates the x and y coordinates.

3. Though the reflection event occurs at $y' \neq 0$ and $y \neq 0$, the emission and absorption events occur at the origin and thus are unaffected by the existence of the y dimension. They must thus obey the Lorentz transformation rule, which implies

$$(x_e, y_e, t_e) = -\gamma\, L'\,(\beta, 0, 1)$$
$$(x_a, y_a, t_a) = \gamma\, L'\,(\beta, 0, 1)$$

4. Let us now calculate the position of the intersection of light rays sent out from the emission and absorption events, again using the fact that we know the speed of light is 1:

$$\begin{aligned} -\gamma\, L'(\beta, 0, 1) + r(\sqrt{1-\delta^2}, \delta, 1) &= \gamma\, L'(\beta, 0, 1) + s(-\sqrt{1-\delta^2}, \delta, -1) \\ r + s &= 2\,\gamma\, L' \\ (r+s)\sqrt{1-\delta^2} &= 2\,\beta\,\gamma\, L' \\ (r-s)\,\delta &= 0 \end{aligned}$$

   The space-time displacement of a light ray in one unit of time is $(\sqrt{1-\delta^2}, \delta, 1)$, where δ allows for freedom in the direction (in the xy plane). δ carries the same sign as y_r. We may assume the velocity vector has the same components on the two sides, with just a sign flip in the y component because the light path must be symmetric about $t = t' = 0$ because the motion is along x, not y. The signs on the left-side velocity term are obtained by simple arguments: 1) if $y_e = 0$ and $y_r > 0$, then the y velocity must be positive between the two events; and 2) in the limit $\beta \ll 1$, we must recover the nonrelativistic limit, and in that limit we know that if the light ray always has $x' = 0$ and the x' origin is moving in the $+x$ direction, then $x \geq 0$ is required for the position of the light ray. The signs on the right-side are obtained by similar arguments: 1) if $y_r > 0$ and $y_a = 0$, then the y velocity must be negative between the two events; and 2) again, the x velocity must be nonnegative to obtain the nonrelativistic limit. Simplifying, we obtain

$$r = s = \gamma\, L'$$
$$\sqrt{1-\delta^2} = \beta$$
$$\delta = \sqrt{1-\beta^2} = \gamma^{-1}$$

   The reflection event thus satisfies

$$(x_r, y_r, t_r) = -\gamma\, L'\,(\beta, 0, 1) + \gamma\, L'(\beta, \gamma^{-1}, 1) = (0, L', 0)$$

   We thus see that the transverse coordinate is unchanged by the Lorentz transformation. This would hold for z also.

6.1.3 Mathematical Description of Lorentz Transformations

We proceed with special relativity by generalizing the transformation rules for the space-time coordinates to other physical quantities. The natural way to proceed is by analogy to spatial rotations. We have obtained rules for the transformation of the space-time coordinate and have discovered the invariant interval. These are analogous to the transformation rules for spatial coordinates under rotation and the invariance of the length of the position vector under rotations. We may thus proceed by generalizing the concepts of scalars, vectors, and tensors and use their transformation properties to obtain transformation properties for various physical quantities.

There are other ways to proceed. The treatment found in Hand and Finch uses physical examples to discover the transformation rules for various physical quantities, essentially working from the specific to the general. We instead will guess the general from one specific case, and then see how the general transformation rules are consistent with the typical specific examples. We follow this route for two reasons: 1) it is an alternative to the usual treatment; and 2) a large part of current research consists of developing theories based on desirable properties under certain transformations, so it is useful to learn to think in this way.

A bit of terminology: as we have discussed before, a quantity is **invariant** under a transformation if its value in different reference frames is the same. The invariant interval associated with a space-time event is such a quantity. More frequently, we have a physical entity – the space-time vector describing an event, for example – whose coordinate representations in different frames are related by a special set of rules set by the transformation between the frames. Such a physical entity is said to be **covariant** under the transformation. It is not invariant, but there are rules that it does obey.

Notation

We will from this point begin to use the following notation for frames: frames and variables with a tilde refer to the "reference" or "fixed" system and frames and variables with no tildes or other accents refer to the "moving" system. The "moving" system will move with velocity $\vec{\beta}$ relative to the fixed system. In relativity, any two inertial systems are equally good, so this distinction is purely notational; it only serves to make clear the sign of the velocity. Note that Hand and Finch use primed coordinates to refer to the moving frame and unprimed for the fixed frame, which is different both from their notation for rotating systems and from our notation. We use tildes rather than primes because a primed, raised-index symbol like x'^{μ} can be hard to distinguish from the unprimed version x^{μ}.

We will also use four-vector notation to distinguish space-time vectors from spatial vectors, and use a notation analogous to what we developed for rotations to distinguish the coordinate-free, coordinate representation, and coordinate representation component versions of space-time vectors. The coordinate-free space-time position vector is denoted by $\vec{\vec{x}}$. The coordinate representation in a particular frame F is $\underline{\vec{x}}$ with components x^{μ} and in a frame $\widetilde{F}$ is $\underline{\widetilde{\vec{x}}}$ with components $\tilde{x}^{\mu}$. For the coordinate representations, $\mu = 0, 1, 2, 3$ is the index giving the four components of the space-time vector; 0 is the time component, $1, 2, 3$ are the spatial components.

Four-Vectors and the Lorentz Transformation Matrix

We first begin by defining the space-time analogy of a vector. A **Lorentz-covariant vector** is a physical entity whose coordinate representations in different frames are related by the Lorentz transformation. So far we have only discussed Lorentz transformations along the x-axis; but, obviously, we may generalize the transformation rules. Let us first write the

transformation matrix for a x-direction Lorentz transformation, converting the space-time event vector $\vec{\vec{x}}$ from its coordinate representation $\underline{\vec{\vec{x}}}$ with components x^μ in a frame F to its representation $\underline{\tilde{\vec{\vec{x}}}}$ with components $\tilde{x}^\mu$ in a frame $\widetilde{F}$ relative to which F is moving at velocity $\vec{\beta}$:

$$\underline{\vec{\vec{x}}} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad \underline{\tilde{\vec{\vec{x}}}} = \begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \qquad \tilde{x}^\mu = \Lambda^\mu{}_\nu \, x^\nu \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The meaning of "up" and "down" indices will be defined below. For now, space-time vectors may have only "up" indices and Lorentz transformation matrices have their first index "up" and their second index "down." The form of the Lorentz transformation matrix for an arbitrary velocity direction $\vec{\beta}$ is

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\,\beta_x & \gamma\,\beta_y & \gamma\,\beta_z \\ \gamma\,\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\,\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\,\beta_z}{\beta^2} \\ \gamma\,\beta_y & (\gamma-1)\frac{\beta_x\,\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\,\beta_z}{\beta^2} \\ \gamma\,\beta_z & (\gamma-1)\frac{\beta_x\,\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\,\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix}$$

which can be rewritten in a more memorable form as

$$\Lambda^0{}_0 = \gamma \qquad \Lambda^0{}_i = \Lambda^i{}_0 = \gamma\,\beta_i \qquad \Lambda^i{}_j = \delta_{ij} + (\gamma-1)\,\frac{\beta_i\,\beta_j}{\beta^2} \qquad i,j = 1,2,3$$

Metric Functions, the Metric Tensor, and Invariant Intervals

The standard dot-product clearly does not return the invariant interval, so we need to define a generalization of the vector dot product for Lorentz-covariant vectors. The standard dot-product is a special case of a class of functions that are generically called **metric functions**; their defining characteristics are that:

1. They map two vectors $\vec{u}$ and $\vec{v}$ to a single number; $g(\vec{u},\vec{v}) = a$ where a is just a real number.
2. The real number returned is independent of the reference frame in which the representations of $\vec{u}$ and $\vec{v}$ are specified. This is to some extent obvious – $\vec{u}$ and $\vec{v}$ refer to abstract vectors, not to particular coordinate representations.
3. They are symmetric under exchange of the two vectors: $g(\vec{u},\vec{v}) = g(\vec{v},\vec{u})$.
4. They are linear in both vectors: $g(\vec{u}, a\,\vec{v} + b\,\vec{w}) = a\,g(\vec{u},\vec{v}) + b\,g(\vec{u},\vec{w})$.

Note that we have so far made no mention of coordinate systems, reference frames, or transformation properties between frames. Metric functions are so far defined simply to act on coordinate-free vectors. The term "metric" is used because such functions define the norm of a vector via $|\vec{u}|^2 = g(\vec{u},\vec{u})$.

We shall prove here, though, that, when considering specific reference frames and coordinate systems, any metric function g can be written as a real, symmetric $N \times N$ matrix that transforms like a rank 2 tensor.

Since a metric function is a symmetric linear function, its action in a particular reference frame and coordinate system is determined entirely by its action on the unit vectors of that reference frame/coordinate system.[1] That is, we may define a symmetric $N \times N$ matrix

$$\underline{g}_{ij} = g\left({}_F\vec{e}_i, {}_F\vec{e}_j\right)$$

where ${}_F\vec{e}_i$ is the ith unit vector of the frame F with coordinate representation in frame F denoted by ${}_F\underline{\vec{e}}_i$. The components of the coordinate representation ${}_F\underline{\vec{e}}$ are ${}_F e_i^j = \delta_{ij}$.[2] If we have an arbitrary vector $\vec{a}$ whose coordinate representation components in F are given by a^i (we use "raised" indices because we have made the disinction between raised and lowered indices for four-vectors), then we have that $\vec{a}$ can be written as the linear combination

$$\vec{a} = a^i_F \vec{e}_i$$

Note that the unit vectors are written as vectors, not using their component representations. This decomposition is valid in any frame, but of course the coordinate representation of ${}_F\vec{e}_i$ will change with frame. Given the above, it then holds that we can calculate the metric function of two arbitrary vectors $\vec{a}$ and $\vec{b}$ using the coordinate representations in F and the matrix $\underline{g}_{ij}$:

$$g(\vec{a},\vec{b}) = g\left(a^i_F \vec{e}_i, b^j_F \vec{e}_j\right) = a^i b^j g\left({}_F\vec{e}_i, {}_F\vec{e}_j\right) = a^i b^j \underline{g}_{ij}$$

Now, there is nothing special about the frame F in which we first wrote down $\underline{g}_{ij} = g({}_F\vec{e}_i, {}_F\vec{e}_j)$. It seems sensible to expect that, for any other frame $\widetilde{F}$, the coordinate representation of g in that frame should be given by the same kind of formula using the unit vectors of $\widetilde{F}$. That is, we expect

$$\underline{\tilde{g}}_{ij} = g\left({}_{\widetilde{F}}\vec{e}_i, {}_{\widetilde{F}}\vec{e}_j\right)$$

where ${}_{\widetilde{F}}\vec{e}_i$ is the ith unit vector of the frame $\widetilde{F}$. The question we have to ask is whether this representation in $\widetilde{F}$ is compatible with the defining properties of metric functions. Properties 1, 3, and 4 are obviously verified because $\underline{\tilde{g}}_{ij}$ is, by its definition in terms of the function g and the unit vectors of the frame $\widetilde{F}$, a real, symmetric $N \times N$ matrix and properties 1, 3, and 4 hold for any real, symmetric $N \times N$ matrix. Property 2 is the one that is trickier to check. Let $\tilde{a}^i$ and $\tilde{b}^j$ be the coordinate representation components of the same vectors $\vec{a}$ and $\vec{b}$ in the frame $\widetilde{F}$. The action of $\underline{\tilde{g}}_{ij}$ on $\tilde{a}^i$ and $\tilde{b}^j$ is

$$\underline{\tilde{g}}_{ij}\tilde{a}^i\tilde{b}^j = \tilde{a}^i\tilde{b}^j g\left({}_{\widetilde{F}}\vec{e}_i, {}_{\widetilde{F}}\vec{e}_j\right) = g\left(\tilde{a}^i_F \vec{e}_i b^j_F \vec{e}_j\right) = g(\vec{a},\vec{b})$$

So, our expression for $\underline{\tilde{g}}_{ij}$ is compatible with property 2 – the value of the action of $\underline{\tilde{g}}_{ij}$ on the component representations $\tilde{a}^i$ and $\tilde{b}^j$ is the same as the action of $\underline{g}_{ij}$ on the component representations a^i and b^j.

[1] We distinguish reference frames and coordinate systems here: one can describe a single reference frame with multiple coordinate systems, and of course different reference frames require different coordinate systems (that nevertheless may be coincident at a particular instant in time). For brevity, we will use the term "frame" as shorthand for reference frame/coordinate system.

[2] Note on notation: in a different frame $\widetilde{F}$, the unit vectors ${}_F\vec{e}_i$ of the frame F will have different coordinate representations ${}_F\underline{\widetilde{\vec{e}}}$ whose components ${}_F\tilde{e}_i^j$ will in general not be so simple. The unit vectors of that other frame $\widetilde{F}$ are denoted by ${}_{\widetilde{F}}\vec{e}_i$ with coordinate representations ${}_{\widetilde{F}}\underline{\widetilde{\vec{e}}}$ in $\widetilde{F}$ whose components are ${}_{\widetilde{F}}\tilde{e}_i^j = \delta_{ij}$. Note the importance of the positioning of the tilde symbols!

Now that we have a means to calculate the components of the coordinate representations $\underline{g}_{ij}$ in any particular frame given the original metric function $g(\ ,\ )$, we can test whether g is rank 2 tensor. We have in fact already proven this, indirectly: since $\underline{g}_{ij}a^i b^j = \underline{\tilde{g}}_{ij}\tilde{a}^i\tilde{b}^j$, we see that the contraction of the coordinate representations of g with the coordinate representations of two arbitrary vectors $\vec{a}$ and $\vec{b}$ is a single number whose value is frame independent. That was exactly our definition of a second-rank tensor in Section 5.1.

Our discussion so far has been generic – any metric function g that satisfies the above four properties is always a rank 2 tensor under the transformations that act on the vectors that g acts on. If we specifically consider Lorentz transformations between different inertial reference frames, then the most reasonable metric tensor to consider is the one that returns the invariant interval. In any frame, its coordinate representation is $\underline{g} = \mathrm{diag}(1,-1,-1,-1)$. We are assured that this representation is correct in all frames because of the way we have defined invariant interval and our earlier proof that it is invariant under Lorentz transformations. We will from here on refer to the g defined in this way as *the* metric tensor and will not distinguish between g and its coordinate representation $\underline{g}$; we will in general just write $g_{\mu\nu}$.

Covariant and Contravariant Vectors and Indices

In general, for any metric tensor, we may now define "down" indices to be obtained by contraction of the metric tensor with a vector:

$$a_\mu = g_{\mu\nu}\, a^\nu$$

so then

$$|a^\mu|^2 = g_{\mu\nu}\, a^\mu\, a^\nu = a_\mu\, a^\mu$$

"up" indices are called **covariant** indices and "down" indices are called **contravariant** indices. We'll explain these names below. Now that we have a mechanism to "raise" indices, we need one to lower them. Obviously, we would like the operations of raising and lowering to be inverses of each other, so it is natural to define

$$g^{\mu\nu} = \left(g^{-1}\right)_{\mu\nu} \Longleftrightarrow g^{\mu\nu}\, g_{\nu\sigma} = g_{\mu\nu}\, g^{\nu\sigma} = \delta^\mu{}_\sigma = \delta_\mu{}^\sigma$$

where both versions of δ are the identity tensor. That is, $g^{\mu\nu}$ is just the matrix inverse of $g_{\mu\nu}$. The above statements are **definitions**; they need not be justified any further. Note that the above implies that

$$g_\mu{}^\nu = \delta_\mu{}^\nu \qquad g^\mu{}_\nu = \delta^\mu{}_\nu$$

That is, the metric tensor with one up and one down index is always the identity tensor. Given the above definition, it is consistent to **define** the operation of raising a lowered index as contraction with $g^{\mu\nu}$:

$$g^{\mu\nu}\, a_\nu = g^{\mu\nu}\, g_{\nu\sigma}\, a^\sigma = a^\mu$$

Finally, the name **covariant** comes from the fact that the original vectors are covariant under Lorentz transformations – they follow a certain set of rules for transforming under Lorentz transformations. The name **contravariant** indicates that contravariant vectors may transform under Lorentz transformations in a different way than covariant vectors.

Contravariant vectors are also known as **1-forms**. We will not make use of that nomenclature, but those familiar with differential geometry will recognize it.

We now explain the choice of index location on Lorentz transformation matrices: we want to contract Λ with a covariant vector and obtain another covariant vector, so Λ must be defined with one up and one down index. This holds not just for Lorentz transformations, but any kind of coordinate transformation that acts on covariant vector representations.

For our specific case of the special relativity metric tensor, $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, we see that $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the same. Also, with g in hand, we can see how contravariant vectors transform. The coordinates representation of a contravariant vector differs only from that of its corresponding covariant vector by a sign flip in all all the spatial coordinates. How this affects Lorentz transformation properties is seen as follows:

$$\tilde{x}_\mu = g_{\mu\nu}\,\tilde{x}^\nu = g_{\mu\nu}\,\Lambda^\nu{}_\sigma\,x^\sigma = g_{\mu\nu}\,\Lambda^\nu{}_\sigma\,g^{\sigma\lambda}\,x_\lambda = \Lambda_\mu{}^\lambda\,x_\lambda$$
$$\text{with}\quad \Lambda_\mu{}^\lambda = g_{\mu\nu}\,g^{\sigma\lambda}\,\Lambda^\nu{}_\sigma$$

(we have extended the raising/lower effect of g from vectors to the Λ matrices). If one works out component-by-component the last line, one sees that $\Lambda_\mu{}^\lambda$ is simply $\Lambda^\mu{}_\lambda$ with the sign of the velocity $\vec{\beta}$ reversed. Thus, contravariant vectors do indeed transform in a different way from covariant vectors for Lorentz transformations.

As a counterexample, consider rotations in three spatial dimensions. The metric tensor is the identity matrix (as is its inverse), so there is no distinction between covariant and contravariant vectors.

Finally, one can obviously define more general second-rank and higher-rank tensors under Lorentz transformations. For such objects, depending on how they are defined, some indices may transform like covariant vectors, others like contravariant vectors. These are therefore called **covariant indices** and **contravariant indices**. Just as we saw that rotation matrices in three spatial dimensions are second-rank tensors, we see that Lorentz transformation matrices are second-rank tensors under Lorentz transformations with one covariant and one contravariant index.

Lorentz Transformations, Rotations, and Group Structure

Let's return briefly to rotations in three dimensions. We decided that all rotations must preserve vector norms. We proved that all norm-preserving transformations are orthonormal. If we consider the larger set of norm-preserving transformation, we must also include reflections and products of reflections and rotations. The group of all orthonormal transformations in three dimensions, including both reflections and rotations, is known as $O(3)$; the O is for "orthonormal". The subgroup consisting only of rotations is known as $SO(3)$. The S is for "special," referring to the fact that these transformations have determinant $+1$, while elements of the $O(3)$ group may have determinant $+1$ or -1.

Taking rotations as inspiration, we may ask two questions: 1) how do we generically write the requirement that a transformation preserve the invariant interval; and 2) is there a larger set of transformations beyond Lorentz boosts that preserve the invariant interval. Question 1) first. Algebraically, we consider transformations Λ^μ_ν with $\tilde{x}^\mu = \Lambda^\mu{}_\nu\,x^\nu$ and require

$$g_{\mu\nu}\,\tilde{x}^\mu\,\tilde{x}^\nu = g_{\mu\nu}\,x^\mu\,x^\nu$$

The definition of invariant interval requires g to have the same coordinate representation in all frames, so there is no ambiguity there. Rewriting, we have

$$g_{\mu\nu}\,\Lambda^{\mu}{}_{\lambda}\,x^{\lambda}\,\Lambda^{\nu}{}_{\sigma}\,x^{\sigma} = g_{\mu\nu}\,x^{\mu}\,x^{\nu}$$

Since x^{μ} is arbitrary, we thus require

$$\Lambda^{\mu}{}_{\nu}\,\Lambda^{\sigma}{}_{\lambda} g_{\mu\sigma} = g_{\nu\lambda} \qquad \Longrightarrow \qquad \Lambda^{\mu}{}_{\nu}\,\Lambda_{\mu}{}^{\alpha} = \delta_{\nu}{}^{\alpha}$$

(For Lorentz boosts, we had earlier seen that $\Lambda_{\mu}{}^{\nu}$ is $\Lambda^{\mu}{}_{\nu}$ with the sign of the velocity reversed, so the two matrices are inverses. We see that same property here.) We define the set of all transformations that satisfy the above to be the **homogeneous Lorentz group**.[3]

This set of transformations turns out to be more general than just the boosts. Clearly, spatial rotations and reflections belong to the group because they preserve the invariant interval (by preserving the three-dimensional norm). It may include time and/or spatial inversions also. If you restrict to transformations satisfying $\Lambda^{0}{}_{0} \geq 1$ and $\det(\Lambda) = |\Lambda| = 1$, then one obtains the **proper homogeneous Lorentz group**.[4] This is the subset of the homogeneous Lorentz group that preserves time and space orientation (hence the name "proper") (and also preserves the sign of the volume element). Clearly, Lorentz boosts and pure spatial rotations are included therein. In fact, one can show that any member of this group can be decomposed purely in terms of boosts and spatial rotations (we won't show that here). Another interesting fact is that, when one combines two boosts along different directions, the product can in general be decomposed in terms of a single boost and a spatial rotation (nor will we show that here). Hence, boosts themselves do not form a closed subgroup. Boosts along one direction form a closed subgroup, but any set of boosts along more than one dimension must include rotations to form a closed group.[5]

The Lorentz Group as a Lie Group

Just as we were able to define generators for spatial rotations and obtain finite rotations as exponentials of the product of the generators and a rotation axis, we may similarly define generators for Lorentz transformations. The obvious set (based on the simple decomposition of the time-time and space-time components of the Lorentz transformation matrix) is

$$\mathbf{K}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K}_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K}_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Boosts may be written in terms of these generators in the form

$$\begin{aligned} \text{Infinitesimal}: \quad & \Lambda(\delta\vec{\beta}) = 1 + \delta\vec{\beta}\cdot\vec{\mathbf{K}} \\ \text{Finite}: \quad & \Lambda(\vec{\beta}) = \exp\left(\vec{\eta}\cdot\vec{\mathbf{K}}\right) \qquad \vec{\eta} = \hat{\beta}\,\left(\tanh^{-1}\beta\right) \end{aligned}$$

[3] One can show that the set of transformations satisfying the given conditions is indeed a group by simply showing that the product of two elements satisfies the same condition – the set is closed under the matrix multiplication operation. Associativity is ensured by the associativity of matrix multiplication. Existence of an identity is obvious and the existence of inverses is obvious from the second half of the condition given.

[4] One can show $|\Lambda^{0}{}_{0}| \geq 1$ always holds. $\Lambda^{0}{}_{0} \geq 1$ transformations are called **orthochronous** and $\Lambda^{0}{}_{0} \leq -1$ transformations are called **non-orthochronous**.

[5] We note that, if instead of requiring invariance of $x_{\mu}\,x^{\mu}$, one instead requires only that the invariant intervals for differences of coordinates – *e.g.*, $(a_{\mu} - b_{\mu})\,(a^{\mu} - b^{\mu})$ – be preserved, then one may include spatial translations also. This group is called the **inhomogeneous Lorentz group** or **Poincaré group**. Obviously, it is simply a larger group of transformations under which we expect physics to be covariant.

One can calculate commutation relations among the boost generators and between the boost generators and rotation generators:

$$\begin{aligned}
[\mathbf{M}_i, \mathbf{M}_j] &= \epsilon_{ijk}\, \mathbf{M}_k \\
[\mathbf{K}_i, \mathbf{K}_j] &= -\epsilon_{ijk}\, \mathbf{M}_k \\
[\mathbf{M}_i, \mathbf{K}_j] &= \epsilon_{ijk}\, \mathbf{K}_k
\end{aligned}$$

As already mentioned above, we see the intimate relationship between boosts and rotations. An arbitrary combination of boosts and rotations can be written as

$$\begin{aligned}
\text{Infinitesimal}: &\qquad \Lambda(\delta\vec{\beta}, \delta\vec{\theta}) = 1 + \delta\vec{\beta}\cdot\vec{\mathbf{K}} + \delta\vec{\theta}\cdot\vec{\mathbf{M}} \\
\text{Finite}: &\qquad \Lambda(\vec{\beta}, \vec{\theta}) = \exp\left(\vec{\eta}\cdot\vec{\mathbf{K}} + \vec{\theta}\cdot\vec{\mathbf{M}}\right)
\end{aligned}$$

The Lie algebra and group defined by the above commutation relations and exponentials is known as $SO(3,1)$ (with the S indicating proper transformations only). We state without further discussion that the above relationship between boosts and rotations is a hint of the intimate relationship between Lorentz covariance and spin in quantum field theory. You will also see an application of this intermingling when you study Thomas precession in E&M.

6.1.4 Physical Implications

In this section, we extend to special relativity many of the quantities we are familiar with from classical mechanics – velocity, energy, momentum, angular momentum, and the Lagrangian and Hamiltonian functions.

Addition of Velocities

Let's get back to physics. We can determine the law for addition of velocities in special relativity by combining two boosts. That is, suppose frame F_1 is moving with speed $\beta_1\,\hat{x}$ with respect to frame F and frame F is moving at speed $\beta_2\,\hat{x}$ with respect to $\widetilde{F}$. We can obtain the speed of F_1 in $\widetilde{F}$ by asking at what speed the origin of F_1 moves in F. The origin of the F_1 frame has coordinate representation $x^\mu = (\tau, 0, 0, 0)$ for all time in F_1. We need to obtain the product of the two individual boost matrices to transform the F_1 origin's position four-vector to the frame $\widetilde{F}$:

$$\begin{aligned}
\Lambda(\beta_{12}) &= \Lambda(\beta_2)\,\Lambda(\beta_1) = \begin{pmatrix} \gamma_2 & \gamma_2\,\beta_2 & 0 & 0 \\ \gamma_2\,\beta_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_1\,\beta_1 & 0 & 0 \\ \gamma_1\,\beta_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma_2\,\gamma_1\,(1+\beta_2\,\beta_1) & \gamma_2\,\gamma_1\,(\beta_2+\beta_1) & 0 & 0 \\ \gamma_2\,\gamma_1\,(\beta_2+\beta_1) & \gamma_2\,\gamma_1\,(1+\beta_2\,\beta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

To obtain the coordinate representation of F_1's origin in frame $\widetilde{F}$, we apply the above product boost matrix to it, which yields $\tilde{x}^\mu = \gamma_2\,\gamma_1\,(1+\beta_1\,\beta_2, \beta_1+\beta_2, 0, 0)\,\tau$. The speed of the origin is thus

$$\beta_{12} = \frac{\tilde{x}^1}{\tilde{x}^0} = \frac{\beta_1+\beta_2}{1+\beta_1\,\beta_2}$$

One can obtain this through simpler, though less general, means by simply realizing that the Lorentz transformation rules apply equally well to differential displacements in space and time. That is, if a particle has velocity u_x in the F frame, then our vector in the F frame is $(dt, dx) = (1, u_x)\, dt$. Let's transform this to frame $\widetilde{F}$ as defined above. We Lorentz transform the differential displacements:

$$\begin{aligned} d\tilde{x} &= \gamma\, (dx + \beta\, dt) = \gamma\, (u_x + \beta)\, dt \\ d\tilde{t} &= \gamma\, (\beta\, u_x + 1)\, dt \end{aligned}$$

So then we have

$$\tilde{u}_x = \frac{d\tilde{x}}{d\tilde{t}} = \frac{u_x + \beta}{1 + u_x\, \beta}$$

as we found using the boost matrices. Either way one derives it, the formula has a reasonable form – it is limited above by 1 and it reduces to the Galilean limit if $\beta \ll 1$. Note that, if we work in terms of rapidity, the parallel velocity addition rule implies that rapidities add simply:

$$\beta_1 = \tanh \eta_1 \qquad \beta_2 = \tanh \eta_2 \qquad \beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1\, \beta_2} = \frac{\tanh \eta_1 + \tanh \eta_2}{1 + \tanh \eta_1\, \tanh \eta_2} = \tanh \eta_1 + \eta_2$$

Note that all our different formulae for the addition of parallel velocities imply that such boosts commute. This only holds for **parallel** boosts.

Now, let's consider the case of the first velocity being in an arbitrary direction. We'll keep the second velocity along the x axis for simplicity without loss of generality.

- Boost matrix method. Our first boost matrix must now be in an arbitrary direction. Rather than calculate the full matrix, let's realize that, above, we only applied the matrix to the four-vector $(\tau, 0, 0, 0)$. Application of a boost matrix to this four-vector returns the product of the first column of the boost matrix and τ; that first column is just $\gamma_1(1, \vec{\beta}_1)$. So then we only need apply the second boost matrix to this four-vector. That is:

$$\begin{aligned} \tilde{x}^\mu &= \Lambda(\beta_2 \hat{x})\, \Lambda(\vec{\beta}_1) \begin{pmatrix} \tau \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma_2 & \gamma_2\, \beta_2 & 0 & 0 \\ \gamma_2\, \beta_2 & \gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \beta_{1,x} \\ \beta_{1,y} \\ \beta_{1,z} \end{pmatrix} \gamma_1\, \tau \\ &= \begin{pmatrix} \gamma_2\, (1 + \beta_2\, \beta_{1,x}) \\ \gamma_2\, (\beta_2 + \beta_{1,x}) \\ \beta_{1,y} \\ \beta_{1,z} \end{pmatrix} \gamma_1\, \tau \end{aligned}$$

  So we obtain

$$\begin{aligned} \beta_{12,x} &= \frac{\beta_2 + \beta_{1,x}}{1 + \beta_2\, \beta_{1,x}} \\ \beta_{12,y} &= \frac{1}{\gamma_2} \frac{\beta_{1,y}}{1 + \beta_2\, \beta_{1,x}} \\ \beta_{12,z} &= = \frac{1}{\gamma_2} \frac{\beta_{1,z}}{1 + \beta_2\, \beta_{1,x}} \end{aligned}$$

- Simple method. If a particle has velocity $\vec{\mathbf{u}}$ in the F frame, then the vector in the F frame we will work with is $(dt, dx, dy, dz) = (1, u_x, u_y, u_z)\, dt$. The Lorentz transformation gives

$$\begin{aligned} d\tilde{t} &= \gamma\,(\beta\, u_x + 1)\, dt \\ d\tilde{x} &= \gamma\,(u_x + \beta)\, dt \\ d\tilde{y} &= u_y\, dt \\ d\tilde{z} &= u_z\, dt \end{aligned}$$

  Thus, we have

$$\begin{aligned} \tilde{u}_x &= \frac{d\tilde{x}}{d\tilde{t}} = \frac{\beta + u_x}{1 + \beta\, u_x} \\ \tilde{u}_y &= \frac{d\tilde{y}}{d\tilde{t}} = \frac{1}{\gamma}\frac{u_y}{1 + \beta\, u_x} \\ \tilde{u}_z &= \frac{d\tilde{y}}{d\tilde{t}} = \frac{1}{\gamma}\frac{u_z}{1 + \beta\, u_x} \end{aligned}$$

  as expected.

Essentially, the perpendicular velocities have no effect on the addition of the velocities along the direction of motion, while the perpendicular velocities are affected by both the velocities along the direction of motion and by time dilation. In the limit $u_x \to 0$, the transformation of perpendicular velocities becomes only a correction for time dilation. Time runs more slowly in F, so the particle with speed $u_\perp$ in F moves apparently more quickly than it does in $\widetilde{F}$. For the transverse velocity case, we again have the limitation that the transformed velocity can be no larger than the speed of light.

Four-Velocity

It is natural to ask whether it is possible to create a space-time vector for the velocity – an entity that transforms like the space-time position four-vector. This would provide a "unified" treatment of velocity and position. The trick, of course, is to find a set of four numbers that are covariant – that transform in the appropriate way under Lorentz transformations.

First, we must define the **proper time**. Consider a frame that is moving with the particle whose velocity we want to specify. In that frame, the particle's space-time position is always $x^\mu = (\tau, 0, 0, 0)$ because the particle is at the origin. τ specifies the time in the frame moving with the particle. Now, τ^2 is the invariant interval of the particle's position; this will be the same in all reference frames that share the same origin at their time origin. So, one reasonable definition of a **four-velocity** is (coordinate-free definition)

$$\vec{\vec{u}} = \frac{d\,\vec{\vec{x}}}{d\tau}$$

or, if we want the representation in a given frame

$$u^\mu = \frac{d\,x^\mu}{d\tau}$$

where x^μ is the position of the particle in the frame in which one wants to know the four-velocity (at some time t in that frame) and τ is the invariant interval associated with the

particle's position at that same time t. The notation is somewhat confusing because τ seems like a rest-frame quantity, so how can it be specified in a different frame? It is a rest-frame quantity but it is also an invariant quantity. Perhaps a more obvious, though more cumbersome, definition would be

$$u^\mu = \frac{d\,x^\mu}{d\sqrt{|x^\mu|^2}}$$

where $|x^\mu|^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ is the invariant interval associated with the position of the particle. The definition of four-velocity clearly transforms like the space-time position Lorentz vector because it is, essentially, the ratio of the space-time position Lorentz vector to a Lorentz scalar.

So, we have a good formal definition. What does it look like in terms of quantities we have easy access to – three-velocities? Consider a particle moving at velocity $\vec{\beta}_p$ (possibly a function of time) in some frame F. (We use the subscript $_p$ to distinguish the particle velocity from that of the frame F relative to another frame $\widetilde{F}$.) The particle's trajectory is $x^\mu(t)$ in this frame. We have the obvious differential relations

$$\begin{aligned} dx^\mu &= (1, \beta_{px}, \beta_{py}, \beta_{pz})\, dt \\ d\tau &= \gamma_p^{-1}\, dt \end{aligned}$$

where the second relation is simply time dilation. These relations are instantaneously true even if the particle is accelerating. Thus, the four-velocity in F is

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{(1, \beta_{px}, \beta_{py}, \beta_{pz})\ dt}{\gamma_p^{-1} dt} = \gamma_p\,(1, \beta_{px}, \beta_{py}, \beta_{pz}) = \gamma_p \left(1, \vec{\beta}_p\right)$$

Obviously, then the general expression is

$$u^\mu = \gamma_p \left(1, \vec{\beta}_p\right)$$

where $\gamma_p = \left(1 - |\vec{\beta}_p|^2\right)^{-1/2}$ is the γ factor associated with the particle velocity $\vec{\beta}_p$. Thus, given a particle trajectory, we can easily calculate the four-velocity in any frame.

Is the above definition consistent with the velocity addition rules found earlier? Yes, it is. The four-velocity calculated above is essentially the first step of our derivation of velocity addition: it is the transformation of the four-vector $(1, \vec{0})$ from the rest frame of the particle F_1 to the frame F in which its velocity is measured to be $\vec{\beta}_p$. That Lorentz transformation matrix's first column is $\gamma_p(1, \vec{\beta}_p)$, which, acting on $(1, \vec{0})$, returns the four-vector $\gamma_p(1, \vec{\beta}_p)$, which matches our definition of four-velocity. The second step would be to transform to the final frame $\widetilde{F}$ in which F moves at velocity $\vec{\beta}$. That would just be the Lorentz transformation of the result of the first step. But we would do the exact same thing in transforming our four-velocity form to the frame $\widetilde{F}$. So the definitions are consistent.

The Energy-Momentum Four-Vector

One can demonstrate using many different kinds of thought experiments that the classical definitions of momentum of kinetic energy can no longer hold true in special relativity. Perhaps this simplest way to see this is simple to realize that, if $T = |\vec{p}|^2/2m$ continues to

hold, and $\vec{p} = m\,\vec{u}$, then velocities exceeding the speed of light can be obtained by simply giving a particle enough kinetic energy. We need a generalization of energy and momentum.

The definition of four-velocity leads us to suggest the following generalization of momentum:

$$\begin{aligned} &\text{coordinate-free:} && \vec{\vec{p}} = m\vec{\vec{u}} = m\frac{d\vec{\vec{x}}}{d\tau} \\ &\text{coordinate-representation:} && p^\mu = m\,u^\mu = \gamma_p\, m\left(1, \vec{\beta}_p\right) \end{aligned}$$

We assume that the mass of a particle is a Lorentz scalar in order for the above expression to be a four-vector. The space terms of our definition are a clear extension of our classical definition of linear momentum to special relativity. But what about the time term? As you certainly know, it is the total particle energy, including the rest mass m. We may motivate that statement by returning to our original relation between kinetic energy and work done by a force. That is,

$$\begin{aligned} T_2 - T_1 &= \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 \frac{d\vec{p}}{dt}\cdot d\vec{r} \\ &= \int_1^2 \vec{\beta}_p\, dt \cdot \frac{d}{dt}\left(\gamma\, m\,\vec{\beta}_p\right) = m\int_0^{\beta_p} \beta'\, d(\gamma\beta') \\ &\overset{ibp}{=} \gamma\, m\,\beta_p^2 - m\int_0^{\beta_p} \frac{\beta'\, d\beta'}{\sqrt{1-(\beta')^2}} \\ &= \gamma_p\, m\,\beta_p^2 - m\left(\sqrt{1-\beta_p^2} - 1\right) \\ &= (\gamma_p - 1)\, m \quad \overset{\beta_p\to 0}{\longrightarrow} \quad \frac{1}{2}\, m\,\beta_p^2 \end{aligned}$$

where in the last step we have taken the classical limit to recover our usual low-energy kinetic energy as a check. So, if $(\gamma_p - 1)\, m$ gives the kinetic energy, then we have little choice but to consider $p^0 = \gamma_p m = T + m$ to be some kind of total energy, including the energy of the mass of the particle. Thus, our relation for the norm of p^μ gives

$$E^2 - |\vec{p}|^2 = m^2 \qquad \vec{p} = \gamma_p m\,\vec{\beta}_p$$

Note that $\vec{p}$ reduces to but, is not exactly, the classical momentum.

Note that our definition of the energy-momentum four-vector requires experimental verification where our definition of four-velocity did not. Here, we are implying a modified relationship between physically measurable quantities. Of course, the relation has been verified thoroughly by experiment over the last 100 years.

Kinematics of Scattering

Scattering of relativistic particles is such a ubiquitous application of conservation of four-momentum that it is worth going through the important parts. Recall that, in our original description of elastic collisions of particles (Section 1.3.3), we made use of the separate conservation of energy and momentum to find three equations relating post-collision to pre-collision quantities. We also transformed to the frame moving with the center of mass.

In our relativistic case, we must modify this somewhat. First, the equivalent of the center-of-mass system is the center-of-momentum system, where the total spatial component of the

four-momentum vanishes. This reduces to center-of-mass in the nonrelativistic limit. Second, to go into the center-of-momenutm system, we must perform a Lorentz transformation (or relativistic velocity addition) not a simple linear velocity addition.

We don't have time to go through the entire exercise; some of it will be assigned as a homework. One of the more interesting aspects is the phenomenon of **boosting** or **beaming**. In the nonrelativistic limit, for particles of equal mass, the outgoing particle trajectories in the lab frame always made a right angle. In relativistic scattering, this angle becomes smaller as γ increases. Even for nonidentical particles, the lab frame angle between outgoing trajectories will tend to be squeezed as γ increases.

The Four-Wavevector for Light and the Relativistic Doppler Shift

Light has no rest mass, so our definition of four-momentum does not immediately apply to light. However, we can think about what are the analogous quantities. One quickly sees that we might try to define a four-wavevector,

$$k^\mu = (\omega, \vec{k})$$

where $\omega = 2\,\pi\,\nu$ and $|\vec{k}| = 2\,\pi/\lambda$. The invariant length of k^μ is

$$k_\mu\,k^\mu = (2\,\pi)^2\left(\nu^2 - \frac{1}{\lambda^2}\right) = 0$$

Since light has no rest mass, this seems like a useful analogy to four-momentum for massive particles.

One way we can test the above definition is to derive the prediction for the Doppler effect and test it against other methods of deriving. Suppose we have two frames F and $\widetilde{F}$, both moving on the same axis as the direction of the light wave, with frame F moving with velocity $\vec{\beta}$ relative to $\widetilde{F}$ in the same direction as the light's wavevector $\vec{k}$. Let the origins of the two frames coincide at $t = \tilde{t} = 0$.

Suppose an observer at the origin of the frame $\widetilde{F}$ makes a measurement of the period of the light wave by measuring the time interval between two crests of the wave passing his location; call this time $\tilde{T}$. Suppose the same observer also measures the separation between two crests at no time separation, giving the wavelength $\tilde{\lambda}$. These are time and length measurements, so we can do the analysis by the same kind of technique as we used to discuss time dilation and length contraction. Note that basic electromagnetism tells us $\tilde{\nu} = \tilde{\lambda}^{-1}$, so $\tilde{T} = \tilde{\lambda}$.

So, suppose an observer at the origin of the F frame makes the same kind of measurements. Let's determine where those events are in the $\widetilde{F}$ frame. For the period measurement, we can assume that the first event occurs at the space-time origin of both frames (since those origins coincide at the time origin of both frames). The second event occurs when the next crest of the light wave has caught up with the moving observer. This happens when the trajectories of the crest and the observer intersect. At $t = \tilde{t} = 0$, the second crest is at position $-\tilde{\lambda}$, and the crest moves at speed 1, so it covers a distance $\tilde{t}$ in the time $\tilde{t}$. The origin of the F frame coincides with the origin of the $\widetilde{F}$ frame at $t = \tilde{t} = 0$ and has moved a distance $\beta\,\tilde{t}$ in the $\widetilde{F}$ frame in the time $\tilde{t}$. Requiring that the second crest coincide with the F origin at $\tilde{t}$ gives

$$-\tilde{\lambda} + \tilde{t} = \beta\,\tilde{t} \quad \Longrightarrow \quad \tilde{t} = \frac{\tilde{\lambda}}{1-\beta}$$

These two events are separated in space in $\widetilde{F}$ by the distance the moving observer travels in that time, $\beta\,\tilde{t}$. So our space-time interval in $\widetilde{F}$ is $\left(\frac{1}{1-\beta}, \frac{\beta}{1-\beta}\right)\tilde{\lambda}$. The moving observer sees this as the interval $(T, 0)$ where T is the apparent period measured by the moving observer. The Lorentz transformation gives us

$$T = -\gamma\left(-\beta\,\frac{\beta}{1-\beta} + \frac{1}{1-\beta}\right)\tilde{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}}\,\tilde{\lambda} \qquad 0 = \gamma\left(\frac{\beta}{1-\beta} - \beta\,\frac{1}{1-\beta}\right)\tilde{\lambda}$$

The spatial separation vanishes as expected and we obtain the relation

$$\frac{\nu}{\tilde{\nu}} = \sqrt{\frac{1-\beta}{1+\beta}}$$

where F and $\widetilde{F}$ are both moving in the same direction as the wave and β is the velocity of F relative to $\widetilde{F}$. This result is as we expect, because, if the F observer is moving faster than $\widetilde{F}$ in the same direction as the wave, F will tend to "catch" up with the crests and see them as separated by a longer time than $\widetilde{F}$ will. F will see a lower frequency, as indicated by the above.

Note that we could have applied similar logic to a wavelength measurement by the F observer and would have obtained the same relationship between quantities in the F and $\widetilde{F}$ frames.

Now, let's try the calculation by applying the Lorentz transformation to the four-wavevector. We have

$$k = \gamma\left(\tilde{k} - \beta\,\tilde{\omega}\right) \qquad \omega = -\gamma\left(\beta\,\tilde{k} - \tilde{\omega}\right)$$

Now, use $k = \frac{2\,\pi}{\lambda} = 2\,\pi\,\nu = \omega$ and $\tilde{k} = \tilde{\omega}$ to obtain

$$\omega = \gamma\left(\tilde{\omega} - \beta\,\tilde{\omega}\right) \qquad \Longrightarrow \qquad \frac{\nu}{\tilde{\nu}} = \sqrt{\frac{1-\beta}{1+\beta}}$$

We obtain the same result as by the argument that did not rely on the definition of the four-wavevector, so the four-wavevector definition is validated.

6.1.5 Lagrangian and Hamiltonian Dynamics in Relativity

It is interesting to ask whether it is possible to define Lagrangian and Hamiltonian functions in special relativity. The first requirement we must make is that there is no potential energy function; such a function would pick a particular frame of reference to be special, so it would be impossible to establish any kind of Lorentz-covariant formulation of the Lagrangian. So we will consider only free particles. In E&M, you will consider the Lagrangian for a particle moving in an electromagnetic field, which can be written in a Lorentz-covariant way due to the inherent Lorentz covariance of the electromagnetic field.

The Action as a Lorentz Scalar and the Lorentz-Covariant Lagrangian

The action is an obvious candidate for promotion to a Lorentz scalar. For a free particle, there are only two Lorentz scalars available to work with, the particle mass and its proper time (time in its rest frame). The action for a free particle must be

$$\mathcal{S} = \lambda \int_A^B m\, d\tau = \lambda \int_{t_A}^{t_B} \frac{m}{\gamma_p}\, dt$$

We have begun by using the proper time and then we have rewritten for an arbitrary frame using time dilation. We can figure out the appropriate constant by requiring that we obtain the correct nonrelativistic limit for L, $L \to \frac{1}{2}\, m\, v^2$. We have

$$L = \lambda\, m \sqrt{1 - \beta_p^2} \approx \lambda\, m - \lambda\, m\, \frac{1}{2}\, \beta_p^2$$

If we take $\lambda = -1$, we obtain the correct limit. So our Lorentz covariant Lagrangian is

$$L = -\frac{m}{\gamma_p}$$

Note that L is not Lorentz invariant; rather, it is the **action** that is Lorentz invariant!

Relativistic Canonical Momentum and the Hamiltonian

Performing the usual Euler-Lagrange procedure obtains the conserved canonical momenta

$$\vec{p} = \frac{m\, \vec{\beta}_p}{\sqrt{1 - \beta_p^2}} = \gamma_p\, m\, \vec{\beta}_p$$

It is reassuring that the canonical momentum thus obtained is the space components of the four-momentum we defined using only the principles of Lorentz covariance and recovery of the nonrelativistic limit. The Hamiltonian is obtained in the obvious way

$$H = \vec{p} \cdot \vec{\beta}_p - L = \frac{m}{\sqrt{1 - \beta_p^2}} = m\, \gamma_p$$

Similarly, we recover the time component of the four-momentum. Because there is no time dependence, $H = E$ is conserved **in any given reference frame**. It of course is a component of the four-momentum when one considers transformations between frames.

Appendix A

Mathematical Appendix

This appendix is a reference for various mathematical definitions, relations, and theorems.

A.1 Notational Conventions for Mathematical Symbols

Typefaces:

Category	Typeface	Example
scalar	lower case, normal weight, italic	s
vector	lower case, normal weight, italic, with arrow	$\vec{a}$
vector coordinate representation	lower case, boldface, with arrow	$\vec{\mathbf{a}}$
vector coordinate representation component	lower case, roman, subscript	a_i
unit vector	normal weight, italic, with hat	$\hat{a}$
tensor	upper case, math calligrahic	$\mathcal{A}$
tensor coordinate representation or matrix	upper case, boldface	$\mathbf{A}$
tensor coordinate representation component	upper case, roman, with subscripts	$\mathrm{A}_{i_1 \cdots i_n}$

We will violate the above convention is some special cases where using the standard notation is more important; *e.g.*, total angular momentum will be referred to as $\vec{L}$, position and momentum of center of mass as $\vec{R}$ and $\vec{P}$.

The following special symbols are defined:

δ_{ij} Kronecker delta
$= 1$ when $i = j$
$= 0$ otherwise

ϵ_{ijk} Levi-Civita density or tensor
$= 1$ when i, j, k are an even permutation of $1, 2, 3$
$= -1$ when odd permutation
$= 0$ otherwise (two or more of i, j, k are equal)

When written as matrices, we will write the Kronecker delta as the identity matrix, $\mathbf{1}$, and the Levi-Civita symbol as $\underline{\vec{\epsilon}}$ (the underline implies two of the indices, the vector implies the third).

Einstein summation convention: When any two indices are repeated, they are summed over; *e.g.*, $a_i\, b_i \equiv \sum_i a_i\, b_i$

A.2 Coordinate Systems

The rectangular coordinate system (with unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$) is assumed to be obvious and taken as our starting point. These unit vectors are independent of position, so they serve as a natural basis in which to define the unit vectors of curvilinear systems.

Cylindrical

The three coordinates are

$$\rho = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right) \qquad z = z \tag{A.1}$$

The inverse relations are

$$x = \rho\cos\phi \qquad y = \rho\sin\phi \qquad z = z \tag{A.2}$$

The area elements are

$$\rho\, d\rho\, d\phi \qquad d\rho\, dz \qquad \rho\, d\phi\, dz \tag{A.3}$$

The volume element is

$$\rho\, d\rho\, d\phi\, dz \tag{A.4}$$

The line element is

$$ds^2 = d\rho^2 + \rho^2\, d\phi^2 + dz^2 \tag{A.5}$$

The unit vectors are

$$\hat{\rho} = \hat{x}\,\cos\phi + \hat{y}\,\sin\phi \qquad \hat{\phi} = -\hat{x}\,\sin\phi + \hat{y}\,\cos\phi \qquad \hat{z} = \hat{z} \tag{A.6}$$

Spherical

The three coordinates are

$$r = \sqrt{x^2 + y^2 + y^2}\phi \qquad = \tan^{-1}\left(\frac{y}{x}\right)\theta = \cos^{-1}\left(\frac{z}{r}\right) \tag{A.7}$$

The inverse relations are

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta \tag{A.8}$$

The area elements are

$$r\,\sin\theta\, dr\, d\phi \qquad r\, dr\, d\theta \qquad r^2\,\sin\theta\, d\theta\, d\phi \tag{A.9}$$

The volume element is

$$r^2\,\sin\theta\, dr\, d\theta\, d\phi \tag{A.10}$$

The line element is

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2\, \sin^2\theta\, d\phi^2 \tag{A.11}$$

The unit vectors are

$$\begin{aligned} \hat{r} &= \hat{x}\, \sin\theta\, \cos\phi + \hat{y}\, \sin\theta\, \sin\phi + \hat{z}\, \cos\theta \\ \hat{\phi} &= -\hat{x}\, \sin\phi + \hat{y}\, \cos\phi \\ \hat{\theta} &= \hat{x}\, \cos\theta\, \cos\phi + \hat{y}\, \cos\theta\, \sin\phi - \hat{z}\, \sin\theta \end{aligned} \tag{A.12}$$

A.3 Vector and Tensor Definitions and Algebraic Identities

Definitions

Dot and Cross Products

$$\vec{a}\cdot\vec{b} = a\, b\, \cos(\vec{a},\vec{b}) = \delta_{ij}\, a_i\, b_j = A_i\, B_j \tag{A.13}$$

$$\vec{a}\times\vec{b} = \hat{n}\, A\, B\, \sin(\vec{a},\vec{b}) = \epsilon_{ijk}\, \vec{e}_i\, A_j\, B_k \tag{A.14}$$

where $(\vec{a},\vec{b})$ is the angle from $\vec{a}$ to $\vec{b}$ and $\hat{n}$ is the unit vector normal to the plane defined by $\vec{a}$, $\vec{b}$, and the right-hand rule, and the secondary expression using vector components hold in rectangular coordinates.

Matrix Representation of Cross Products

Define the three matrices

$$(\mathbf{M}_i)_{jk} = -\epsilon_{ijk} \tag{A.15}$$

or

$$\mathbf{M}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{M}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathbf{M}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.16}$$

Then any cross product operation can be rewritten using these matrices:

$$\vec{a}\times\vec{b} = (\vec{a}\cdot\hat{x}\,\mathbf{M}_x + \vec{a}\cdot\hat{y}\,\mathbf{M}_y + \vec{a}\cdot\hat{z}\,\mathbf{M}_z)\,\vec{b} = \left(\vec{a}\cdot\vec{\mathbf{M}}\right)\vec{b} \tag{A.17}$$

The above matrices satisfy the following relations

$$\mathbf{M}_x^2 = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{M}_y^2 = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{M}_z^2 = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{A.18}$$

and

$$\mathbf{M}_i\mathbf{M}_j - \mathbf{M}_j\mathbf{M}_i \equiv [\mathbf{M}_i,\mathbf{M}_j] = \epsilon_{ijk}\,\mathbf{M}_k \tag{A.19}$$

Algebraic Identities

Most of the following identities rely on a few facts about products of ϵ_{ijk} symbols:

$$\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl} \tag{A.20}$$
$$\epsilon_{ijk}\,\delta_{ij} = 0 \tag{A.21}$$
$$\epsilon_{ijk}\,\epsilon_{ljk} = 2\,\delta_{il} \tag{A.22}$$
$$\epsilon_{ijk}\,\epsilon_{ijk} = 6 \tag{A.23}$$

Using the above, one can easily prove

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) \equiv \vec{a}\,\vec{b}\,\vec{c} \tag{A.24}$$
$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{a}\cdot\vec{b})\,\vec{c} \tag{A.25}$$
$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{a}\cdot\Big[\vec{b}\times(\vec{c}\times\vec{d})\Big] \tag{A.26}$$
$$= \vec{a}\cdot\Big[(\vec{b}\cdot\vec{d})\,\vec{c} - (\vec{b}\cdot\vec{c})\,\vec{d}\Big] \tag{A.27}$$
$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}) \tag{A.28}$$
$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \Big[(\vec{a}\times\vec{b})\cdot\vec{d}\Big]\;\vec{c} - \Big[(\vec{a}\times\vec{b})\cdot\vec{c}\Big]\;\vec{d} \tag{A.29}$$
$$= \Big(\vec{a}\,\vec{b}\,\vec{d}\Big)\,\vec{c} - \Big(\vec{a}\,\vec{b}\,\vec{c}\Big)\,\vec{d} \tag{A.30}$$
$$= \Big(\vec{a}\,\vec{c}\,\vec{d}\Big)\,\vec{b} - \Big(\vec{b}\,\vec{c}\,\vec{d}\Big)\,\vec{a} \tag{A.31}$$

Note that, if any of the vectors are the gradient vector $\vec{\nabla}$, care must be taken in how the above expressions are written out to ensure $\vec{\nabla}$ acts on the appropriate vectors. Any two quantities that commute in the above should be commuted as necessary to get reasonable behavior of $\vec{\nabla}$. But in some cases, even that may not be sufficient and you will have to keep track of which vector should be acted on by $\vec{\nabla}$. A good example is the second line when $\vec{b} = \vec{\nabla}$. In the simple case of $\vec{a}$ being constant, one simply needs to move the $\vec{b}$ in the first term:

$$\vec{a}\times\Big(\vec{\nabla}\times\vec{c}\Big) = \vec{\nabla}\,(\vec{a}\cdot\vec{c}) - \Big(\vec{a}\cdot\vec{\nabla}\Big)\,\vec{c}$$

But if $\vec{a}$ depends on position and does not give zero when acted on by $\vec{\nabla}$, then the above must be read with care. One has to somehow remember that $\vec{\nabla}$ should not be allowed to act on $\vec{a}$ since it does not act on $\vec{a}$ in the original expression. Since the above expression does not correctly convey that meaning, it is better to abandon the vector notation. The completely unambiguous way to write it, using index notation, is

$$\Big[\vec{a}\times(\vec{\nabla}\times\vec{c})\Big]_i = \sum_j a_j\nabla_i c_j - \sum_j a_j\nabla_j c_i$$

The key point is that in the first term, $\vec{a}$ is in a dot product with $\vec{c}$, but $\vec{\nabla}$ must be allowed to act on $\vec{c}$ first, and not as $\vec{\nabla}\cdot\vec{c}$.

Rotations

Infinitesimal Rotations

Infinitesimal rotation of a vector $\vec{r}$ by an infinitesimal angle $\delta\theta$ around the axis $\delta\vec{\theta}/|\delta\vec{\theta}|$ is given by

$$\vec{r}\,' = \vec{r} + \delta\vec{\theta}\times\vec{r}$$

This lets us define angular velocity implicitly via

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

A matrix operator version of this is given by

$$\begin{aligned} \vec{r}' &= \mathbf{R}_{\delta\vec{\theta}}\,\vec{r} \\ \mathbf{R}_{\delta\vec{\theta}} &= \mathbf{1} + \left(\delta\vec{\theta}\cdot\hat{x}\,\mathbf{M}_x + \delta\vec{\theta}\cdot\hat{y}\,\mathbf{M}_y + \delta\vec{\theta}\cdot\hat{z}\,\mathbf{M}_z\right) \equiv \mathbf{1} + \delta\vec{\theta}\cdot\vec{\mathbf{M}} \\ (\mathbf{M}_i)_{jk} &= -\epsilon_{ijk} \end{aligned}$$

Infinitesimal rotations add linearly and commutatively like vectors.

Finite Rotations

Finite rotations may be obtained from infinitesimal rotations, giving

$$\mathbf{R}_{\vec{\theta}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\vec{\theta}\cdot\vec{\mathbf{M}}\right)^n \equiv \exp\left(\vec{\theta}\cdot\vec{\mathbf{M}}\right)$$

For θ along, for example, the z axis, this form reduces to

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Finite rotations do not commute unless about the same axis.

Algebraic Properties of Finite Rotations

Rotation matrices must preserve the norms and dot products of vectors since they only rotate coordinate axes. Therefore, they satisfy a number of equivalent algebraic relations, collectively known as orthonormality:

$$\delta_{ij} = \sum_k R_{ki}R_{kj} = \sum_k R_{ik}R_{jk}$$
$$\mathbf{1} = \mathbf{R}\,\mathbf{R}^T = \mathbf{R}^T\,\mathbf{R}$$
$$\mathbf{R}^T = \mathbf{R}^{-1}$$

Interpretation of Rotations

- **Active transformation**

  In an active transformation, we think of the transformation as actively rotating the particle whose position is given by the vector $\vec{r}(t)$ relative to the coordinate axes. The rotations of the form $\vec{\omega}\times\vec{r}$ that we began with are really of that form. The coordinate system that the rotation is relative to is inertial, the coordinate system in which the rotating vector is fixed is noninertial. This point of view is most appropriate for studying motion of a rigid body as viewed from the "space", or inertial, frame.

- **Passive transformation**

  We think of a passive transformation as simply a relabeling of points in space according to a new coordinate system. In this picture, the coordinate system in which the particle is at rest, F, is inertial, while the F' system coordinate axes rotate and so F' is not an inertial system. When discussing dynamics of particles *as viewed from a rotating system*, this will be the appropriate point of view to take.

- **Why there can be confusion**

  The difficulty arises because the transformation relating the two systems is mathematically identical, but the physical interpretation is very different. We use the mathematical equivalence to relate one type of transformation to another to allow us to write down the rules. But, the definition of which system is inertial differs between the cases, so we must be very careful. In our discussion of rotating physical systems, it is usually quite clear that the coordinate fixed to the rotating object is noninertial, in which case the **active** viewpoint is the appropriate one.

Representation of Rotations by Euler Angles

Any rotation matrix can be represented in terms of a product of three rotations by the Euler angles ϕ, θ, and ψ. If $\mathbf{R}$ is the matrix that rotates from F to F', then we have

$$\begin{aligned}
\mathbf{R}(\phi,\theta,\psi) &= \mathbf{R}_1(\phi)\,\mathbf{R}_2(\theta)\,\mathbf{R}_3(\psi) && \text{(A.32)}\\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} && \text{(A.33)}\\
&= \begin{pmatrix} c_\psi\,c_\phi - c_\theta\,s_\phi\,s_\psi & -s_\psi\,c_\phi - c_\theta\,s_\phi\,c_\psi & s_\theta\,s_\phi \\ c_\psi\,s_\phi + c_\theta\,c_\phi\,s_\psi & -s_\phi\,s_\psi + c_\theta\,c_\phi\,c_\psi & -s_\theta\,c_\phi \\ s_\theta\,s_\psi & s_\theta\,c_\psi & c_\theta \end{pmatrix} && \text{(A.34)}\\
\vec{\mathbf{r}}' &= \mathbf{R}(\phi,\theta,\psi)\,\vec{\mathbf{r}} && \text{(A.35)}
\end{aligned}$$

where $c_\psi = \cos\psi$, $s_\psi = \sin\psi$, etc. is a convenient shorthand. $\mathbf{R}_3(\psi)$ is a rotation by angle ψ about the z axis of the F system. $\mathbf{R}_2(\theta)$ is a rotation by an angle θ about the x axis of the F_1 system that is obtained from the F system by $\mathbf{R}_1(\psi)$. $\mathbf{R}_3(\phi)$ is a rotation about the z axis of the F_2 system that is obtained from the F_1 system by $\mathbf{R}_2(\theta)$. The Euler angles are convenient because ψ naturally parameterizes the spin of a symmetric top about its symmetry axis, θ gives the polar tip angle of the spin axis from the precession axis, and ϕ is the precession angle. One can show that any rotation is a rotation about a particular vector (by using the orthonormality of rotation matrices). The rotation angle about that vector, Φ, can be related to the Euler angles by

$$\cos\left(\frac{\Phi}{2}\right) = \cos\left(\frac{\phi+\psi}{2}\right)\cos\left(\frac{\theta}{2}\right) \qquad \text{(A.36)}$$

In terms of Euler angles, the angular velocity vector has representations in the space (inertial) frame F' and the body (rotating, fixed to the rigid body) frame F:

$$\mathrm{w}' = \begin{pmatrix} \dot\theta\,c_\phi + \dot\psi\,s_\phi\,s_\theta \\ \dot\theta\,s_\phi - \dot\psi\,c_\phi\,s_\theta \\ \dot\phi + \dot\psi\,c_\theta \end{pmatrix} \qquad \mathrm{w} = \begin{pmatrix} \dot\theta\,c_\psi + \dot\phi\,s_\psi\,s_\theta \\ -\dot\theta\,s_\psi + \dot\phi\,c_\psi\,s_\theta \\ \dot\psi + \dot\phi\,c_\theta \end{pmatrix}$$

Tensors

A rank n tensor $\mathcal{T}$ is an object that has coordinate representations $\underline{\mathcal{T}}$ each consisting of N^n components $T_{i_1\cdots i_n}$ (where N is the dimensionality of the physical space, $N = 3$ for what we consider in this course) with transformation properties

$$T'_{i_1\cdots i_n} = R_{i_1 j_1} \cdots R_{i_n j_n} T_{j_1\cdots j_n}$$

under rotations of the coordinate system by the rotation matrix $\mathbf{R}$ (which rotates from the unprimed to the primed frame, $\underline{\vec{r}}' = \mathbf{R}\,\underline{\vec{r}}$). A vector is a rank 1 tensor, a scalar is a rank 0 tensor. An equivalent definition is to require that for n arbitrary vectors $\{\vec{a}_i\}$ with coordinate representation $\{a_{i,j_i}\}$, the quantity

$$\underline{s} = T_{i_1\cdots i_n}\, a_{1,i_1} \cdots a_{n,i_n}$$

is a scalar (invariant under choice of rotational orientation of coordinate axes). An isotropic tensor is one whose coordinate representation is the same in all coordinate systems. $\mathbf{1}$ is an isotropic rank 2 tensor. $\vec{\mathbf{M}} = \mathcal{M}$, with components $(\mathbf{M}_i)_{jk} = -\epsilon_{ijk}$, is an isotropic rank 3 tensor.
The coordinate transformation of rank 2 tensors can be represented as simple matrix multiplication.

$$T'_{ij} = R_{ik}\, R_{jl}\, T_{kl} = R_{ik}\, T_{kl}\, R^T_{lj}$$
$$\underline{\mathcal{T}}' = \mathbf{R}\,\underline{\mathcal{T}}\,\mathbf{R}^T = \mathbf{R}\,\underline{\mathcal{T}}\,\mathbf{R}^{-1}$$

A particularly interesting second-rank tensor is the contraction of any vector with $\vec{\mathbf{M}}$. The action of this entity is the same as that of a vector cross product:

$$\left(\vec{a}\cdot\vec{\mathbf{M}}\right)\vec{b} = \vec{a}\times\vec{b}$$

Because it is a second-rank tensor, and because $\vec{\mathbf{M}}$ is isotropic, the transformation properties of $\vec{a}\cdot\vec{\mathbf{M}}$ are simply

$$\underline{\vec{a}}'\cdot\vec{\mathbf{M}} = \mathbf{R}\left(\underline{\vec{a}}\cdot\vec{\mathbf{M}}\right)\mathbf{R}^T \tag{A.37}$$

This implies that rotation of a cross product is equivalent to the cross product of the rotations of the individual vectors:

$$\mathbf{R}\left(\underline{\vec{a}}\times\underline{\vec{b}}\right) = \mathbf{R}\left(\underline{\vec{a}}\cdot\vec{\mathbf{M}}\right)\underline{\vec{b}} = \mathbf{R}\left(\underline{\vec{a}}\cdot\vec{\mathbf{M}}\right)\mathbf{R}^T\,\mathbf{R}\,\underline{\vec{b}} = \left(\underline{\vec{a}}'\cdot\vec{\mathbf{M}}\right)\underline{\vec{b}}' = \underline{\vec{a}}'\times\underline{\vec{b}}'$$

Extension to Relativistic Systems

Lorentz transformations (boosts) are generated by

$$\mathbf{K}_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{K}_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathbf{K}_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Boosts may be written in terms of these generators in the form

$$\text{Infinitesimal}: \quad \Lambda(\delta\vec{\beta}) = 1 + \delta\vec{\beta}\cdot\vec{\mathbf{K}}$$
$$\text{Finite}: \quad \Lambda(\vec{\beta}) = \exp\left(\vec{\eta}\cdot\vec{\mathbf{K}}\right) \qquad \vec{\eta} = \hat{\beta}\,\left(\tanh^{-1}\beta\right)$$

The set of all spatial rotations, reflections, and Lorentz transformations form the **homogeneous Lorentz group**. The subgroup containing transformations with ${\Lambda^0}_0 \geq 1$ and $\det(\Lambda) = 1$ is called the **proper homogenous Lorentz group**. Boosts alone do not form a closed subgroup unless one restricts to boosts along one direction only. Boosts along a single direction always form a closed group. The transformations in the proper homogeneous Lorentz group can be generated using the rotation generators $\mathbf{M}_x$, $\mathbf{M}_y$, and $\mathbf{M}_z$ and the boost generators. One can calculate commutation relations among the boost generators and between the boost generators and rotation generators:

$$\begin{aligned} [\mathbf{M}_i, \mathbf{M}_j] &= \epsilon_{ijk}\, \mathbf{M}_k \\ [\mathbf{K}_i, \mathbf{K}_j] &= -\epsilon_{ijk}\, \mathbf{M}_k \\ [\mathbf{M}_i, \mathbf{K}_j] &= \epsilon_{ijk}\, \mathbf{K}_k \end{aligned}$$

An arbitrary combination of boosts and rotations can be written as

$$\begin{aligned} \text{Infinitesimal}: &\quad \Lambda(\delta\vec{\beta}, \delta\vec{\theta}) = 1 + \delta\vec{\beta}\cdot\vec{\mathbf{K}} + \delta\vec{\theta}\cdot\vec{\mathbf{M}} \\ \text{Finite}: &\quad \Lambda(\vec{\beta}, \vec{\theta}) = \exp\left(\vec{\eta}\cdot\vec{\mathbf{K}} + \vec{\theta}\cdot\vec{\mathbf{M}}\right) \end{aligned}$$

A.4 Vector Calculus

Differentiation

Given a function g of multiple coordinate variables $\{q_k\}$ and time t, the **total time derivative** of g is

$$\frac{d}{dt}\, g(\{q_k\}, t) = \frac{\partial g}{\partial t} + \sum_k \frac{\partial g}{\partial q_k} \tag{A.38}$$

Vector Differential Operators in Different Coordinate Systems

Rectangular Coordinates

$$\vec{\nabla} U = \vec{e}_i\, \frac{\partial U}{\partial x_i} \tag{A.39}$$

$$\vec{\nabla}\cdot\vec{a} = \frac{\partial A_i}{\partial x_i} \tag{A.40}$$

$$\nabla^2 U = \vec{\nabla}\cdot\vec{\nabla} U = \frac{\partial^2 U}{\partial x_i^2} \tag{A.41}$$

$$\vec{\nabla}\times\vec{a} = \epsilon_{ijk}\, \vec{e}_i\, \frac{\partial A_k}{\partial x_j} \tag{A.42}$$

Cylindrical Coordinates

$$\vec{\nabla}U = \hat{\rho}\,\frac{\partial U}{\partial \rho} + \frac{\hat{\phi}}{\rho}\,\frac{\partial U}{\partial \phi} + \hat{z}\,\frac{\partial U}{\partial z} \tag{A.43}$$

$$\vec{\nabla}\cdot\vec{a} = \frac{1}{\rho}\,\frac{\partial}{\partial \rho}\left(\rho\, a_\rho\right) + \frac{1}{\rho}\,\frac{\partial a_\phi}{\partial \phi} + \frac{\partial a_z}{\partial z} \tag{A.44}$$

$$\nabla^2 U = \vec{\nabla}\cdot\vec{\nabla}U = \frac{1}{\rho}\,\frac{\partial}{\partial \rho}\left(\rho\,\frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \tag{A.45}$$

$$\vec{\nabla}\times\vec{a} = \left[\frac{1}{\rho}\,\frac{\partial a_z}{\partial \phi} - \frac{\partial a_\phi}{\partial z}\right]\hat{\rho} + \left[\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right]\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\, a_\phi\right) - \frac{\partial a_r}{\partial \phi}\right]\hat{z} \tag{A.46}$$

Spherical Coordinates

$$\vec{\nabla}U = \hat{r}\,\frac{\partial U}{\partial r} + \frac{\hat{\theta}}{r}\,\frac{\partial U}{\partial \theta} + \frac{\hat{\phi}}{r\,\sin\theta}\,\frac{\partial U}{\partial \phi} \tag{A.47}$$

$$\vec{\nabla}\cdot\vec{a} = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2 a_r\right) + \frac{1}{r\,\sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\, a_\theta\right) + \frac{1}{r\,\sin\theta}\,\frac{\partial a_\phi}{\partial \phi} \tag{A.48}$$

$$\nabla^2 U = \vec{\nabla}\cdot\vec{\nabla}U = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 U}{\partial \phi^2} \tag{A.49}$$

$$\vec{\nabla}\times\vec{a} = \frac{1}{r\,\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\, a_\phi\right) - \frac{\partial a_\theta}{\partial \phi}\right]\hat{r} \tag{A.50}$$

$$+\frac{1}{r}\left[\frac{1}{\sin\theta}\,\frac{\partial a_r}{\partial \phi} - \frac{\partial}{\partial r}\left(r\, a_\phi\right)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\, a_\theta\right) - \frac{\partial a_r}{\partial \theta}\right]\hat{\phi} \tag{A.51}$$

Relations between Differential and Integral Forms

$$\vec{a} = -\vec{\nabla}U \quad\Longleftrightarrow\quad U(\vec{r}_2) - U(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{a}(\vec{r})\cdot d\vec{r} \tag{A.52}$$

Vector Theorems

Gauss' Theorem

Let S be a closed surface with outward pointing normal $\hat{n}(\vec{r})$ enclosing a volume V. Let $\vec{f}(\vec{r})$ be a vector field. Gauss' divergence theorem is

$$\int_S d^2 r\,\hat{n}(\vec{r})\cdot\vec{f}(\vec{r}) = \int_V d^3 r\,\vec{\nabla}\cdot\vec{f}(\vec{r}) \tag{A.53}$$

Stokes' Theorem

Let C be a closed loop with line element $d\vec{r}$. Let S be any surface bounded by C, with surface normal $\hat{n}(\vec{r})$, with sign defined with respect to $d\vec{r}$ by the right-hand rule. Let $\vec{f}(\vec{r})$ be vector field. Stokes' Thorem is

$$\int_C d\vec{r}\cdot\vec{f}(\vec{r}) = \int_S d^2 r\,\hat{n}(\vec{r})\cdot\vec{\nabla}\times\vec{f}(\vec{r}) \tag{A.54}$$

A.5 Taylor Expansion

The Taylor expansion is one of the most useful techniques in physics to find limiting behaviors or dynamics for small deviations from an equilibrium point. Generically, the Taylor expansion of a function of one variable is

$$f(x_0+\delta x) = f(x_0) + \sum_{n=1}^{\infty} \frac{1}{n} \frac{d^n f}{dx^n}\bigg|_{x=x_0} (\delta x)^n \tag{A.55}$$

$$= f(x_0) + \frac{df}{dx}\bigg|_{x=x_0} \delta x + \frac{1}{2}\frac{d^2 f}{dx^2}\bigg|_{x=x_0} (\delta x)^2 + \cdots \tag{A.56}$$

For multiple dimensions, one has

$$f(\vec{x}_0+\delta\vec{x}) = f(\vec{x}_0) + \sum_{n=1}^{\infty} \frac{1}{n} \left(\delta\vec{x}\cdot\vec{\nabla}\right)^n f\Big|_{\vec{x}=\vec{x}_0} (\delta x)^n \tag{A.57}$$

$$= f(\vec{x}_0) + \left(\delta\vec{x}\cdot\vec{\nabla}\right) f\Big|_{\vec{x}=\vec{x}_0} + \frac{1}{2}\left(\delta\vec{x}\cdot\vec{\nabla}\right)^2\Big|_{\vec{x}=\vec{x}_0} + \cdots \tag{A.58}$$

When expanding a potential function about an equilibrium point $\vec{x}_0$, the first-order term will vanish, leaving a function that is quadratic in the displacement from the equilibrium – a simple harmonic oscillator. A frequently used Taylor expansion is

$$\frac{1}{1\pm x} = 1 \mp x - x^2 + \cdots$$

Another frequent use is to expand denominators of the form $|\vec{R}-\vec{r}|^n$ for the case $|\vec{r}| \ll |\vec{R}|$:

$$\begin{aligned}
\frac{1}{\left|\vec{R}-\vec{r}\right|^n} &= \frac{1}{\left(R^2+r^2-2\,\vec{R}\cdot\vec{r}\right)^{n/2}} = \frac{1}{R}\frac{1}{\left(1-2\left(\hat{R}\cdot\hat{r}\right)\frac{r}{R}+\frac{r^2}{R^2}\right)^{n/2}} \\
&= \frac{1}{R}\frac{1}{1+\frac{n}{2}\left[-2\left(\hat{R}\cdot\hat{r}\right)\frac{r}{R}+\frac{r^2}{R^2}\right]+\frac{1}{2}\frac{n}{2}\frac{n-2}{2}\left[-2\left(\hat{R}\cdot\hat{r}\right)\frac{r}{R}+\frac{r^2}{R^2}\right]^2+\cdots} \\
&= \frac{1}{R}\frac{1}{1+\frac{n}{2}\left[-2\left(\hat{R}\cdot\hat{r}\right)\frac{r}{R}+\frac{r^2}{R^2}\right]+\frac{n(n-2)}{8}\left[4\left(\hat{R}\cdot\hat{r}\right)^2\frac{r^2}{R^2}\right]+\cdots} \\
&= \frac{1}{R}\frac{1}{1-n\left(\hat{R}\cdot\hat{r}\right)\frac{r}{R}+\frac{n}{2}\frac{r^2}{R^2}\left[1+(n-2)\left(\hat{R}\cdot\hat{r}\right)^2\right]+\cdots} \\
&= \frac{1}{R}\left\{1+n\frac{r}{R}\left(\hat{R}\cdot\hat{r}\right)-\frac{n}{2}\frac{r^2}{R^2}\left[1+(n-2)\left(\hat{R}\cdot\hat{r}\right)^2\right]-n^2\left(\hat{R}\cdot\hat{r}\right)^2\frac{r^2}{R^2}+\cdots\right\} \\
&= \frac{1}{R}\left\{1+n\frac{r}{R}\left(\hat{R}\cdot\hat{r}\right)+\frac{n}{2}\frac{r^2}{R^2}\left[-1-[(n-2)-2]\left(\hat{R}\cdot\hat{r}\right)^2\right]+\cdots\right\} \\
&= \frac{1}{R}\left\{1+n\frac{r}{R}\left(\hat{R}\cdot\hat{r}\right)+\frac{n}{2}\frac{r^2}{R^2}\left[-1-(n-4)\left(\hat{R}\cdot\hat{r}\right)^2\right]+\cdots\right\}
\end{aligned}$$

Note that we could have obtained the above somewhat more quickly by just directly Taylor expanding the function $\left|1-\frac{\vec{R}\cdot\vec{r}}{R^2}\right|^{-n}$ rather than first expanding $\left|1-\frac{\vec{R}\cdot\vec{r}}{R^2}\right|^{n}$ in the denominator and

then using our $(1 \pm x)^{-1}$ expansion. But the above is instructive in terms of showing how to keep terms to a self-consistent order of approximation – notice how we had to carry each expansion to the second term because there were pieces in both the first and second terms that were of second order in $\frac{r}{R}$.
For the specific case usually under consideration, $n = 1$, we find

$$\frac{1}{\left|\vec{R} - \vec{r}\right|} = \frac{1}{R}\left\{1 + \frac{r}{R}\left(\hat{R}\cdot\hat{r}\right) + \frac{1}{2}\,\frac{r^2}{R^2}\left[3\left(\hat{R}\cdot\hat{r}\right)^2 - 1\right] + \cdots\right\}$$

This is known as the **multipole expansion** – the first term is the dipole, the second term the quadrupole, etc. You will see this explicitly in electromagnetism.

A.6 Calculus of Variations

Given a functional I of a the function y of the form

$$I[y] \equiv \int_{x_0}^{x_1} dx\, F\left(y, \frac{dy}{dx}, x\right) \tag{A.59}$$

Its variation δI for a variation δy is

$$\begin{aligned} \delta I &= \int_{x_0}^{x_1} dx \left[\frac{\partial F}{\partial y}\,\delta y + \frac{\partial F}{\partial \frac{dy}{dx}}\,\delta\frac{dy}{dx}\right] &\text{(A.60)}\\ &= \int_{x_0}^{x_1} dx \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right)\right]\delta y &\text{(A.61)} \end{aligned}$$

The extremum condition $\delta I = 0$ for arbitrary δy away from the extremizing $y(x)$ generates the Euler equation for the function $y(x)$:

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial \frac{dy}{dx}}\right) = 0 \tag{A.62}$$

A.7 Legendre Transformations

The **Legendre transformation** is a generic, fully invertible technique for generating a new pair of independent variables (x, z) from an initial pair (x, y). A Legendre transformation applied twice is the identity transformation. It is defined as follows.
Consider a function $A(x, y)$ of two variables x and y with the quantity

$$z \equiv \left.\frac{\partial A}{\partial y}\right|_x \tag{A.63}$$

being an invertible function of y. Then the function $B(x, z)$ defined by

$$B(x, z) \equiv y\, z - A(x, y) \tag{A.64}$$

has partial derivatives

$$\left.\frac{\partial B}{\partial x}\right|_z = -\left.\frac{\partial A}{\partial x}\right|_y \qquad \left.\frac{\partial B}{\partial z}\right|_x = y \tag{A.65}$$

B is computed explicitly by inverting the definition of z to find $y(x,z)$ and then substituting for y in the definition of B. B is then indeed a function of only x and z. The variable x is called the **passive** variable and y the **active** variable because of their different roles in the transform.

Geometric interpretation: Fix the x variable momentarily. The new variable z is the slope of the tangent to $A(x,y)$ when considered as a function of y only. Since the tangent line must have value $A(x,y)$ at ordinate value y, the tangent line equation is $yz + b = A(x,y)$ where b is the intersection of the tangent line with the vertical axis. But, by our Legendre transformation, $b = -B(x,z)$. That is, the Legendre transform is a mapping, for each x, from ordinate values y to tangent slope values z, and from the function $A(x,y)$ to the vertical-axis intercept of the line tangent to $A(x,y)$ in the y direction, which is $B(x,z)$.

Appendix B

Summary of Physical Results

This appendix contains relations that have been assumed or proven and that arise from physical considerations.

B.1 Elementary Mechanics

Newtonian Mechanics

Newton's Second Law

In an inertial reference frame, the rate of change of the momentum of a particle is determined by the total force on the particle:

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{B.1}$$

Newton's Third Law

weak form
The forces exerted by two particles a and b on each other are equal in magnitude and opposite in direction. That is, if $\vec{f}_{ab}$ is the force exerted on particle a by particle b, then

$$\vec{f}_{ab} \;=\; -\vec{f}_{ba} \tag{B.2}$$

strong form
In addition to the above, the force between the two particles a and b is a function of only the difference in the two particles' positions and is directed along the vector between them:

$$\vec{f}_{ab} \;=\; f_{ab}(r_{ab})\,\hat{r}_{ab} \tag{B.3}$$

where $\hat{r}_{ab} = \vec{r}_{ab}/|\vec{r}_{ab}|$ and $\vec{r}_{ab} = \vec{r}_a - \vec{r}_b$. That is, the force is a scalar function of the magnitude of the position difference and is directed along $\hat{r}_{ab}$. The mathematical form may seem like a stronger statement than the verbal form. But such a dependence implies the force must be directed along $\hat{r}_{ab}$. The remaining dependence on $\vec{r}_{ab}$ must therefore be a scalar, and the only nonzero scalar that can be formed from $\vec{r}_{ab}$ is r_{ab}, so the scalar function f_{ab} must only be a function of r_{ab}.

Gravitation

The relative position vector is

$$\vec{r}_{21} \quad = \quad \vec{r}_2 - \vec{r}_1 \tag{B.4}$$

The length is denoted by r_{21} and the unit vector is $\hat{r}_{21} = \vec{r}_{21}/r_{21}$
The generic form for the gravitational force on mass distribution 2 exerted by mass distribution 1 is

$$\vec{F}_{21} = -G \int_{V_2} \int_{V_1} d^3 r_2 \, d^3 r_1 \, \frac{\rho_1(\vec{r}_1) \, \rho_2(\vec{r}_2)}{r_{21}^2} \, \hat{r}_{21} \tag{B.5}$$

Point masses for 1 or 2 are special cases of the above with $\rho(\vec{r}) = M \, \delta(\vec{r} - \vec{r}_M)$ where $\vec{r}_M$ is the location of the mass.
The gravitational field of a mass distribution at position $\vec{r}_2$ is

$$\vec{g}(\vec{r}_2) \quad = \quad -G \int_{V_1} d^3 r_1 \, \frac{\rho(\vec{r}_1)}{r_{21}^2} \, \hat{r}_{21}$$

The generic form for the gravitational potential energy of two mass distributions is

$$U = -G \int_{V_2} \int_{V_1} d^3 r_2 \, d^3 r_1 \, \frac{\rho_1(\vec{r}_1) \, \rho_2(\vec{r}_2)}{r_{21}} \tag{B.6}$$

The gravitational potential of a mass distribution at position $\vec{r}_2$ is

$$\Psi(\vec{r}_2) = -G \int_{V_1} d^3 r_1 \, \frac{\rho_1(\vec{r}_1)}{r_{21}} \tag{B.7}$$

Newton's iron sphere theorem says that the gravitational potential of a spherically symmetric mass distribution at a point outside the distribution at radius R from the center of the distribution is is the same as the potential of a point mass equal to the total mass enclosed by the radius R, and that the gravitational field at a radius R depends only on mass enclosed by the radius R. Explicitly, the potential of the mass distribution (with density function $\rho(r)$ and inner and outer radii r_i and r_o) is

$$\Psi(R) = \begin{cases} -G \displaystyle\int_{r_i}^{r_o} 4 \pi \, r \, \rho(r) \, dr & R < r_i \\ -\dfrac{G}{R} \, M(R) - G \displaystyle\int_{R}^{r_o} 4 \pi \, r \, \rho(r) \, dr & r_i < R < r_o \\ -\dfrac{G}{R} \, M(r_o) & R > r_o \end{cases} \tag{B.8}$$

where $M(R)$ is the mass contained by radius R,

$$M(R) = \int_{r_i}^{R} 4 \pi \, r^2 \, \rho(r) \, dr \tag{B.9}$$

The gravitational field is

$$\vec{g}(R) = \begin{cases} 0 & R < r_i \\ -\dfrac{G}{R^2} \, M(R) \, \hat{R} & r_i < R < r_o \\ -\dfrac{G}{R^2} \, M(r_o) \, \hat{R} & R > r_o \end{cases} \tag{B.10}$$

Poisson's equation for gravity is

$$-\vec{\nabla}\cdot\vec{g} = \nabla^2\Psi = 4\,\pi\,G\,\rho(\vec{r}) \tag{B.11}$$

It holds because gravity is a $1/r^2$ force.

Dynamics of Systems of Particles

Newtonian Mechanical Concepts for Systems of Particles

A system of particles has constant masses $\{m_a\}$ and variable positions $\{\vec{r}_a\}$. The total mass, center-of-mass coordinate, and momentum of the system are

$$M = \sum_a m_a \qquad \vec{R} = \frac{1}{M}\sum_a m_a\vec{r}_a \qquad \vec{P} = \sum_a m_a\,\dot{\vec{r}}_a = M\,\dot{\vec{R}} \tag{B.12}$$

$$= \int d^3r\,\rho(\vec{r}) \qquad = \frac{1}{M}\int d^3r\,\rho(\vec{r})\,\vec{r} \qquad = \int d^3r\,\rho(\vec{r})\,\dot{\vec{r}} \tag{B.13}$$

where $M = \sum_a m_a$ is the total mass. Newton's third law assures us that the center-of-mass coordinate satisfies a version of Newton's second law:

$$M\,\ddot{\vec{R}} = \dot{\vec{P}} = \vec{F}^{(e)} = \sum_a \vec{F}_a^{(e)} \tag{B.14}$$

where $\vec{F}^{(e)}$ is the sum of all external forces acting on the system and $\vec{F}_a^{(e)}$ is the total external force acting on particle a. The total angular momentum of the system is

$$\vec{L} = \vec{R}\times\vec{P} + \sum_a \vec{s}_a \times m_a\,\dot{\vec{s}}_a \tag{B.15}$$

where $\vec{s}_a = \vec{r}_a - \vec{R}$ is the coordinate relative to the center of mass. Newton's second law for angular coordinates is

$$\dot{\vec{L}} = \vec{N}^{(e)} = \sum_a \vec{N}_a^{(e)} \tag{B.16}$$

where $\vec{N}^{(e)}$ is the total external torque on the system and $\vec{N}_a^{(e)}$ is the external torque on particle a. The total kinetic energy is

$$T = \frac{1}{2}\,M\,\dot{R}^2 + \sum_a \frac{1}{2}\,m_a\,\dot{s}_a^2 \tag{B.17}$$

The total potential energy is the sum of two terms, one due to external potentials and one due to internal potentials:

$$U^{(e)} = \sum_a U_a(\vec{r}_a) \qquad U^{(i)} = \frac{1}{2}\sum_{a,b,b\neq a} U_{ab}^{(i)}(\vec{r}_{ab}) \tag{B.18}$$

For a rigid body, the distances r_{ab} are constant. Combined with the third law, this implies that $U^{(i)}$ is therefore also fixed.

Rocket motion is one application of dynamics of systems of particles. Given a rocket that exhausts mass at a rate $\dot{m} < 0$ at speed u as measured in the rocket frame, then the equation we start with is conservation of momentum,

$$m\,v = p(t) = p(t+dt) = (m + \dot{m}\,dt)\,(v + dv) + (-\dot{m}\,dt)\,(v - u) \tag{B.19}$$

which reduces to the differential equation

$$m\,dv = -\dot{m}\,u\,dt = -u\,dm \qquad \Longleftrightarrow \qquad m\,\frac{dv}{dt} = u\,|\dot{m}| \tag{B.20}$$

The second equation is an effective Newton's second law.

Elastic Collisions: Kinematics of Scattering

Given particles of mass m_1 and m_2, with m_1 arriving with velocity $\vec{u}_1$ and m_2 at rest, one can calculate various features of the collision result. The velocity of the center of mass, and the velocities of the particles in the center-of-mass frame, are

$$\vec{V} = \frac{m_1}{m_1 + m_2}\,\vec{u}_1 \qquad \vec{u}_1' = \vec{u}_1 - \vec{V} = \frac{m_2}{m_1 + m_2}\,\vec{u}_1 \qquad \vec{u}_2' = -\vec{V} \tag{B.21}$$

Because the center-of-mass is stationary in the center-of-mass frame, the outgoing angle between the two particles is always π and the individual speeds and kinetic energies are preserved,

$$v_1' = u_1' \qquad v_1' = u_1' \tag{B.22}$$

The cm-frame final velocity vectors are

$$\vec{v}_1' = \frac{m_2\,u_1}{m_1 + m_2}\,(\hat{x}\,\cos\theta + \hat{y}\,\sin\theta) \qquad \vec{v}_2' = -\frac{m_1\,u_1}{m_1 + m_2}\,(\hat{x}\,\cos\theta + \hat{y}\,\sin\theta) \tag{B.23}$$

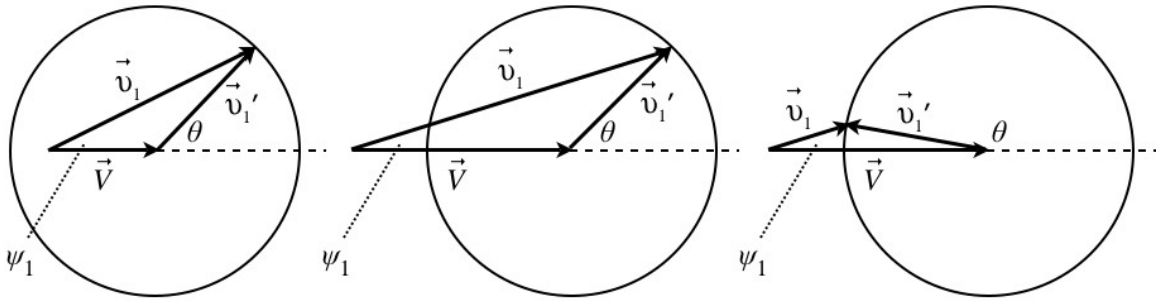
The lab frame final velocities are

$$\begin{aligned} \vec{v}_1 &= \hat{x}\left[\frac{m_2\,u_1}{m_1 + m_2}\,\cos\theta + \frac{m_1\,u_1}{m_1 + m_2}\right] + \hat{y}\,\frac{m_2\,u_1}{m_1 + m_2}\,\sin\theta \\ \vec{v}_2 &= \hat{x}\left[-\frac{m_1\,u_1}{m_1 + m_2}\,\cos\theta + \frac{m_1\,u_1}{m_1 + m_2}\right] - \hat{y}\,\frac{m_1\,u_1}{m_1 + m_2}\,\sin\theta \end{aligned}$$

The angles ψ_1 and ψ_2 between $\vec{v}_1'$ and the x-axis and $\vec{v}_2'$ and the x-axis are

$$\tan\psi_1 = \frac{\sin\theta}{\frac{m_1}{m_2} + \cos\theta} \qquad \tan\psi_2 = \frac{\sin\theta}{1 - \cos\theta} \tag{B.24}$$

Basic rules about outgoing velocities based on ψ_1 and ψ_2 formulae and the vector diagrams for $\vec{v}_1$:



$m_1 < m_2$ $\qquad$ $m_1 > m_2$

- $m_1 = m_2$: The two particles emerge in the lab frame at right angles.
- $m_1 < m_2$: Particle 1 may forward or backward scatter, there is a one-to-one relationship between outgoing cm-frame angle θ and outgoing particle 1 angle ψ_1.
- $m_1 > m_2$: Particle 1 may only forward scatter, there are two solutions in the forward direction because there is a two-to-one relationship between cm-frame angle θ and outgoing particle 1 angle ψ_1

The maximum scattering angle is given by

$$\sin\psi_{1,max} = \frac{m_2}{m_1} \tag{B.25}$$

The kinetic energies are:

$$\text{lab frame initial total KE} \qquad T_0 = \frac{1}{2}\, m_1\, u_1^2 \tag{B.26}$$

$$\text{cm frame initial total KE} \qquad \frac{T_0'}{T_0} = \frac{m_2}{m_1+m_2} < 1 \tag{B.27}$$

$$\text{cm frame initial/final KE 1} \qquad \frac{T_1'}{T_0} = \left(\frac{m_2}{m_1+m_2}\right)^2 \tag{B.28}$$

$$\text{cm frame initial/final KE 2} \qquad \frac{T_2'}{T_0} = \frac{m_1 m_2}{(m_1+m_2)^2} \tag{B.29}$$

$$\text{lab frame final KE 1} \qquad \frac{T_1}{T_0} = 1 - \frac{2\, m_1 m_2}{(m_1+m_2)^2}\,(1-\cos\theta) \tag{B.30}$$

$$= \frac{m_1^2}{(m_1+m_2)^2}\left[\cos\psi_1 \pm \sqrt{\frac{m_2^2}{m_1^2} - \sin^2\psi_1}\right] \tag{B.31}$$

$$\text{lab frame final KE 2} \qquad \frac{T_2}{T_0} = \frac{2\, m_1 m_2}{(m_1+m_2)^2}\,(1-\cos\theta) \tag{B.32}$$

$$= \frac{4\, m_1 m_2}{(m_1+m_2)^2}\cos^2\psi_2 \tag{B.33}$$

In the expression for T_1, one should use only the positive sign when when $m_1 < m_2$ and both signs otherwise (see the diagram).

Inelastic Scattering

The only general rule is that, in addition to mechanical kinetic energy, there is thermal kinetic energy Q that may be liberated ($Q < 0$) or absorbed ($Q > 0$) and that conservation of energy is

$$Q + T_1 + T_2 = T_1' + T_2' \tag{B.34}$$

B.2 Lagrangian and Hamiltonian Dynamics

Lagrangian Mechanics

Holonomic Constraints

Holonomic constraints are constraints that can be written in the form

$$f(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_M, t) \;=\; 0 \tag{B.35}$$

i.e., there is some sort of condition on the coordinates and possibly time. The condition may not involve the coordinate velocities. Holonomic constraints are also termed **integrable** because a differential version of the constraint can be integrated to yield the full constraint. The constraint is **rheonomic** if time appears explicitly, **scleronomic** if not.

Generalized Coordinates

Given a set of holonomic constraints, one can use the constraints to implicitly define a set of independent **generalized coordinates**. The transformation relations between the generalized coordinates and the original physical coordinates are of the form

$$\vec{r}_i = \vec{r}_j(q_1, q_2, \ldots, q_{3M-j}, t) \tag{B.36}$$

where we assume there are j constraint equations. It is assumed that all the constraints are used so that the generalized coordinates truly form an independent set.

Independence of q_k and $\dot{q}_k$ and Nothing Else

Because Newton's second law is a second-order differential equation, it establishes a relationship between $\ddot{\vec{r}}$ and the force. In general, the force can depend on $\vec{r}$ and $\dot{\vec{r}}$. Thus, Newton's second law establishes a relationship between the second (and higher-order) derivatives of $\vec{r}$ and $\vec{r}$ and $\dot{\vec{r}}$. Thus, at most $\vec{r}$ and $\dot{\vec{r}}$ are independent. Because Newton's second law is a differential equation, it holds that two initial conditions are required to specify the motion, $\vec{r}(t=0)$ and $\dot{\vec{r}}(t=0)$. Since there is complete freedom in these two quantities, $\vec{r}$ and $\dot{\vec{r}}$ can be treated as independent.

Dot Cancellation

For holonomic constraints, it holds that

$$\frac{\partial \vec{r}_i}{\partial q_k} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} \tag{B.37}$$

This was proven in Equation 2.3.

Generalized Force

The **generalized force** along the generalized coordinate q_k is defined to be

$$\mathcal{F}_k = \sum_i \vec{F}_i^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} \tag{B.38}$$

where $\vec{F}_i^{(nc)}$ is the sum of all non-constraint forces acting on particle i.

Virtual Displacement and Work

A **virtual displacement** is a set of coordinate displacements $\{\delta\vec{r}_i\}$ that obey the following conditions:

1. The displacement satisfies the constraint equations.
2. Time is held fixed during the displacement.

3. The generalized velocities $\{\dot{q}_k\}$ are held fixed during the displacement.

Any displacement of a truly independent set of generalized coordinates $\{q_k\}$ satisfies (1) automatically.

Virtual work is the work done during a *virtual* displacement. It is defined as

$$\delta W = \sum_{ij} \vec{F}_{ij} \cdot \delta \vec{r}_i \tag{B.39}$$

Because the displacement is virtual, we *make the assumption* that the contribution of constraint forces to the virtual work vanishes, leaving

$$\delta W = \sum_{ij} \vec{F}_{ij}^{(nc)} \cdot \delta \vec{r}_i \tag{B.40}$$

where $^{(nc)}$ refers to non-constraint forces only.

d'Alembert's Principle

d'Alembert's principle is

$$\sum_i \left[\vec{F}_i^{(nc)} - \dot{\vec{p}}_i \right] \cdot \delta \vec{r}_i \;\; = \;\; 0 \tag{B.41}$$

where $\delta \vec{r}_i$ is a virtual displacement that is differential and satisfies the constraints. d'Alembert's principle may be rewritten in terms of generalized coordinates and forces:

$$\mathcal{F}_k \;\; = \;\; \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \tag{B.42}$$

Generalized Equation of Motion

d'Alembert's principle can be used to prove the **generalized equation of motion**:

$$\mathcal{F}_k \;\; = \;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \tag{B.43}$$

where $T = T(\{q_k\}, \{\dot{q}_k\}, t)$ is the kinetic energy written as a function of the generalized coordinates.

Euler-Lagrange Equation

When the non-constraint forces are conservative, they can be written as gradients of a time-independent potential, $\vec{F}_i = -\vec{\nabla}_i U(\{\vec{r}_j\})$. From this we can prove that the generalized forces can also be written as gradients:

$$\mathcal{F}_k \;\; = \;\; -\frac{\partial}{\partial q_k} U(\{q_l\}, t) \tag{B.44}$$

If we then define the Lagrangian as

$$L \;\; = \;\; T - U \tag{B.45}$$

then we can prove the **Euler-Lagrange equation**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{B.46}$$

In some cases, it is possible to write nonconservative forces using a potential function. If the nonconservative force can be written in terms of a function $U(\{q_k\}, \{\dot{q}_k\})$ in the following manner:

$$\mathcal{F}_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \tag{B.47}$$

then it one may include this function U as a potential energy in the Lagrangian and apply the Euler-Lagrange equation.
If one has nonconservative forces that cannot be written in the above form, one can still write down a generalized Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \mathcal{F}_k^{no-L} \tag{B.48}$$

where $\mathcal{F}^{no-L}$ encompasses all forces that cannot be included in the Lagrangian.

Symmetry Transformations and Noether's Theorem

Given a coordinate transformation

$$q' = q'(q, \dot{q}, t) \tag{B.49}$$
$$\dot{q}' = \dot{q}'(q, \dot{q}, t) \tag{B.50}$$

Then the Lagrangian in the new coordinate system is found by inverting the transformation and writing

$$L'(q', \dot{q}', t) = L(q(q', \dot{q}', t), \dot{q}(q', \dot{q}', t), t) \tag{B.51}$$

The coordinate transformation is a **symmetry transformation** of the Lagrangian if

$$L'(q', \dot{q}', t) = L(q', \dot{q}', t) \tag{B.52}$$

Noether's theorem tells us that if a Lagrangian has a set of continuous symmetry transformations $\{Q_k(\{s_j\})\}$, with $\{Q_k(s_j = 0)\} = \{q_k\}$ being the untransformed coordinates, then there is a set of conserved quantities:

$$I_j(\{q_k\}, \{\dot{q}_k\}, t) \equiv \sum_{k=1}^{N} p_k \left.\frac{dQ_k}{ds_j}\right|_{\{s_j=0\}} \tag{B.53}$$

where $\{p_k\}$ are the canonical momenta conjugate to $\{q_k\}$,

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \tag{B.54}$$

When a coordinate does not appear in the Lagrangian, it is called **cyclic** and the Euler-Lagrange equations imply the conjugate canonical momentum is conserved. This is a particular case of Noether's theorem with the transformation being translation in the cyclic coordinate.

Variational Calculus and Dynamics

Principle of Least Action

The Euler-Lagrange equations can be generated via the calculus of variations, requiring that the action functional $S[q]$ of the coordinate path $q(t)$

$$S[q] \quad = \quad \int_{t_0}^{t_1} dt\, L\left(q(t), \dot{q}(t), t\right) \tag{B.55}$$

satisfy $\delta S = 0$ for the true physical path $q(t)$.

Incorporating Constraints via Lagrange Multipliers

If it is not convenient to include constraints via appropriate definition of generalized coordinates, constraint functions may be included via **Lagrange multipliers**. Given a set of constraints $\{G_p(\{y_k\}, x) - C_p = 0\}$, one may incorporate them by introducing additional dynamical variables $\{\lambda_p\}$ called Lagrange multipliers and obtain the following modification of the Euler-Lagrange equations:

$$\frac{\partial L}{\partial y_k} - \frac{d}{dx}\left(\frac{\partial L}{\partial \frac{dy_k}{dx}}\right) + \sum_p \lambda_p(x)\,\frac{\partial G_p}{\partial y_k} \quad = \quad 0 \tag{B.56}$$

$$G_p(\{y_k\}, x) - C_p \quad = \quad 0 \tag{B.57}$$

The Lagrange multipliers provide the generalized constraint forces:

$$\mathcal{N}_k = \sum_p \lambda_p(t) \frac{\partial G_p}{\partial q_k} \tag{B.58}$$

Incorporating Nonholonomic Constraints via the Principle of Least Action

If a nonholonomic constraint may be written as a differential relation among coordinates, then it may be included by applying it during minimization of the action functional: such constraints provide relations among the variations in the coordinates δq and velocities $\delta \dot{q}$. Requiring extremization of the action results in generalized Euler-Lagrange equations incorporating the constraints.

Hamiltonian Dynamics

The Hamiltonian and Hamilton's Equations

The **Hamiltonian function** is derived from the Lagrangian function via the Legendre transformation

$$H = \sum_k p_k\, \dot{q}_k - L \tag{B.59}$$

$$H = H(\{q_k\}, \{p_k\}, t) \tag{B.60}$$

Hamilton's Equations of Motion are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k} \qquad \frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{B.61}$$

Liouville's Theorem

From Hamilton's equations of motion, we can derive **Liouville's Theorem**. If $\rho(\{q_k\},\{p_k\},t)$ is the density of an ensemble of systems in phase space, defined by

$$N = \int d^M q\, d^M p\, \rho(\{q_k\},\{p_k\},t) \tag{B.62}$$

then ρ satisfies

$$\frac{d\rho}{dt}(\{q_k\},\{p_k\},t) = 0 \tag{B.63}$$

This can be rewritten using the symplectic vector $\vec{\xi}$ as

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \dot{\vec{\xi}}\cdot\vec{\nabla}_\xi\rho \tag{B.64}$$

Theoretical Mechanics

Canonical Transformations

A **contact transformation** is a transformation of the form

$$Q = Q(q(t),p(t),t) \qquad P = P(q(t),p(t),t) \tag{B.65}$$

A **canonical transformation** $(\vec{q},\vec{p}) \to (\vec{Q},\vec{P})$ is one that preserves Hamilton's equations. A transformation is canonical if the determinant of the Jacobian of the transformation is 1, which is equivalent to requiring that the Poisson brackets of the new variables satisfy

$$[Q_k(\vec{q},\vec{p}), P_l(\vec{q},\vec{p})]_{\vec{q},\vec{p}} = \delta_{kl} \tag{B.66}$$

There are four types of **generating functions** that can generate canonical transformations, though canonical transformations need not come from a generating function to be canonical. In general, a generating function is a function of two of the four variables involved (q, p, Q, and P), and the other two variables are obtained via partial derivatives of the generating function. The partial derivative relations are used to find the transformation $Q = Q(q,p)$ and $P = P(q,p)$. The four forms and their partial derivative relations are

$$\begin{aligned}
F_1: &\quad F_1(q,Q) \qquad P = -\frac{\partial F_1}{\partial Q} \qquad p = \frac{\partial F_1}{\partial q}\\
F_2: &\quad F_2(q,P) \qquad p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P}\\
F_3: &\quad F_3(p,Q) \qquad P = -\frac{\partial F_3}{\partial Q} \qquad q = -\frac{\partial F_3}{\partial p}\\
F_4: &\quad F_4(p,P) \qquad q = -\frac{\partial F_4}{\partial p} \qquad Q = \frac{\partial F_4}{\partial P}
\end{aligned} \tag{B.67}$$

A necessary and sufficient condition for a generating function to be valid is $\frac{\partial^2 F}{\partial q\,\partial Q} \neq 0$, which ensures invertibility of the canonical (Legendre) transformation of the Hamiltonian. The transformation rule for the Hamiltonian under a canonical transformation is

$$\widetilde{H}(Q,P,t) = H(q(Q,P,t),p(Q,P,t),t) + \frac{\partial}{\partial t}F_1(q(Q,P,t),Q,t) \tag{B.68}$$

where F_1 is the generating function that generates the transformation. If one uses a different type of generating function, the result still holds, one simply has different arguments for the generating function.

Symplectic Notation

Symplectic notation is a simple way to combine the position and momentum coordinates in the Hamiltonian formalism and write things in a unified fashion. The symplectic coordinate is defined to be $\vec{\xi}$ with $(k = 1, \ldots, 2M)$

$$\xi_k = \begin{cases} q_{(k+1)/2} & k = \text{even} \\ p_{k/2} & k = \text{odd} \end{cases} \tag{B.69}$$

We define the matrix $\mathbf{\Gamma}$ by

$$\mathbf{\Gamma}_{ij} = (j - i)\, \delta_{|i-j|,1} \qquad \text{for } i \text{ odd and } j \text{ even} \tag{B.70}$$

With $\mathbf{\Gamma}$, Hamilton's equations are written

$$\frac{d\vec{\xi}}{dt} = \mathbf{\Gamma} \vec{\nabla}_\xi H \tag{B.71}$$

where $\vec{\nabla}_\xi$ is simple the vector of partial derivatives with respect to the components of $\vec{\xi}$.
A contact transformation generates a new set of phase space coordinates $\vec{\Xi}$. The Jacobian of transformation is $\mathbf{J}$

$$J_{ij} = \frac{\partial \Xi_i}{\partial \xi_j} \tag{B.72}$$

$\mathbf{J}$ gives the transformation of differential line and volume elements:

$$d\vec{\Xi} = \mathbf{J}\, d\vec{\xi} \tag{B.73}$$

$$\prod_k d\Xi_k = \prod_k \left(\sum_l J_{kl}\, d\xi_l \right) = [\det \mathbf{J}] \prod_l d\xi_l \tag{B.74}$$

The Jacobian transforms gradients:

$$\vec{\nabla}_\xi H = \mathbf{J}^T \vec{\nabla}_\Xi H \tag{B.75}$$

A contact transformation is canonical if its Jacobian is **symplectic**; *i.e.*, satisfies

$$\mathbf{J}\, \mathbf{\Gamma}\, \mathbf{J}^T = \mathbf{\Gamma} \tag{B.76}$$

This is equivalent to the requirement $\det \mathbf{J} = 1$.

Poisson Brackets

The **Poisson Bracket** of two functions of F and G of the coordinates and canonical momenta q and p is defined to be

$$[F, G]_{\vec{q},\vec{p}} = \sum_k \left[\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right] \tag{B.77}$$

In symplectic notation, the Poisson bracket is written

$$[F, G]_{\vec{\xi}} = \left[\vec{\nabla}_\xi F \right]^T \mathbf{\Gamma} \vec{\nabla}_\xi G \tag{B.78}$$

Poisson brackets are invariant under canonical transformations. As noted earlier, the Poisson bracket can be used to test whether a contact transformation is canonical (old-coordinate Poisson bracket of new coordinates must be 1). Poisson brackets also provide the time evolution of functions of the coordinates:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H]_{\vec{\xi}} \tag{B.79}$$

Action-Angle Variables and Adiabatic Invariance

Given a 1-dimensional system whose Hamiltonian is conserved and that undergoes periodic motion, it is possible to define via canonical transformation **action-angle variables** (ψ, I). The action variable is given by

$$I = \frac{1}{2\pi} \oint p(q, E)\, dq \tag{B.80}$$

In practice, to get the function $p(q, E)$ in order to do the integral, one obtains from $H(p, q)$ the function $p(q, H)$ and then uses the fact that energy is conserved for periodic systems to replace H by its constant value E. I will depend on E.
The canonical transformation, combined with the fact that H is conserved, allows the Hamiltonian to be rewritten in terms of I. ψ evolves linearly with time according to

$$\psi(t) = \omega\, t + \psi_0 \qquad \text{with} \qquad \omega \equiv \frac{\partial H(I)}{\partial I} \tag{B.81}$$

ω is constant in time but may depend on I. ω gives the period of the motion,

$$T = \frac{2\,\pi}{\omega} \tag{B.82}$$

ψ advances by $2\,\pi$ when the motion goes through one period. **Adiabatic invariance** says that if the Hamiltonian has slow time dependence via a parameter α, then the action variable I is to first order independent of changes in H that are slow compared to the period of the periodic motion. This property can be used determine features of the evolution of such systems.

The Hamilton-Jacobi Equation

The **Hamilton-Jacobi equation** is a partial differential equation for **Hamilton's Principal Function** $S(\vec{q}, \vec{P})$:

$$H\left(\vec{q}, \frac{\partial S(\vec{q}, \vec{\alpha})}{\partial \vec{q}}, t\right) + \frac{\partial S(\vec{q}, \vec{\alpha})}{\partial t} = 0 \tag{B.83}$$

$\vec{\alpha}$ are the constant values for the momenta $\vec{P}$ resulting from the canonical transformation generated by S. Once the function S is found by solving the equation, the solutions to Hamilton's equations of motions are provided by

$$\vec{p} = \frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial \vec{q}} \tag{B.84}$$

$$\vec{\beta} = \frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial \vec{\alpha}} \tag{B.85}$$

The constants $\vec{\alpha}$ and $\vec{\beta}$ are found by applying the above equations at $t = 0$:

$$\vec{p}(t=0) = \left.\frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial \vec{q}}\right|_{t=0, \vec{q}(t=0), \vec{\alpha}} \tag{B.86}$$

$$\vec{\beta} = \left.\frac{\partial S(\vec{q}, \vec{\alpha}, t)}{\partial \vec{\alpha}}\right|_{t=0, \vec{q}(t=0), \vec{\alpha}} \tag{B.87}$$

When the Hamiltonian is time-independent, we may write an alternate version in terms of **Hamilton's Characteristic Function** $W(\vec{q},\vec{P})$:

$$H\left(\vec{q},\frac{\partial W(\vec{q},\vec{P})}{\partial \vec{q}}\right)=E \tag{B.88}$$

where E is the conserved value of H. W is related to S and E by

$$\begin{aligned} W &= S+E\,t=\int dt\;(L+H) \\ &= \int dt\sum_k p_k\,\dot{q}_k=\int d\vec{q}\cdot\vec{p} \end{aligned} \tag{B.89}$$

Once the differential equation for W has been solved, we may obtain the solution to Hamilton's equation of motion via

$$p_k = \frac{\partial W(\vec{q},E,\alpha_2,\ldots,\alpha_M)}{\partial q_k} \tag{B.90}$$

$$Q_k = \frac{\partial W(\vec{q},E,\alpha_2,\ldots,\alpha_M)}{\partial \alpha_k}=\beta_k \qquad k>1 \tag{B.91}$$

$$Q_1 = \frac{\partial W(\vec{q},E,\alpha_2,\ldots,\alpha_M)}{\partial E}+t=\beta_1'+t \tag{B.92}$$

and incorporate initial conditions via the equations

$$p_k(t=0) = \left.\frac{\partial W(\vec{q},E,\alpha_2,\ldots,\alpha_M)}{\partial q_k}\right|_{q(t=0),E,\alpha_2,\ldots,\alpha_M} \tag{B.93}$$

$$\beta_k = \left.\frac{\partial W(\vec{q},E,\alpha_2,\ldots,\alpha_M)}{\partial \alpha_k}\right|_{q(t=0),E,\alpha_2,\ldots,\alpha_M} \tag{B.94}$$

$$\beta_1' = \left.\frac{\partial W(\vec{q},E,\alpha_2,\ldots,\alpha_M)}{\partial E}\right|_{q(t=0),E,\alpha_2,\ldots,\alpha_M} \tag{B.95}$$

B.3 Oscillations

Simple Harmonic Oscillator

Setting up and Solving the SHO

Given an arbitrary Lagrangian with an equilibrium point q_0, $\dot{q}_0$, the equation of motion around the equilibrium point is given by

$$\ddot{q}-\frac{D}{F}\,q = 0 \tag{B.96}$$

with

$$D = \left.\frac{1}{2}\frac{\partial^2 L}{\partial q^2}\right|_{(q_0,\dot{q}_0)} \tag{B.97}$$

$$F = \left.\frac{1}{2}\frac{\partial^2 L}{\partial \dot{q}^2}\right|_{(q_0,\dot{q}_0)} \tag{B.98}$$

The system's stability and oscillation frequency are given by

$$\omega^2 \equiv -\frac{D}{F} \tag{B.99}$$

If ω^2 is positive then the system undergoes stable oscillation. If we rescale the time coordinate using $\beta = \omega^{-1} = \sqrt{\left|\frac{F}{D}\right|}$ and τ by $t = \beta\,\tau = \omega^{-1}\tau$ (invert by replacing all occurrences of t by $\omega\,t$ and all occurrences of modified frequencies ω' by ω'/ω), the equation of motion becomes

$$\ddot{q} \pm q \;=\; 0 \tag{B.100}$$

and the Lagrangian and Hamiltonian are

$$L \;=\; \frac{1}{2}\left(\dot{q}^2 \mp q^2\right) \tag{B.101}$$

$$H \;=\; \frac{1}{2}\left(\dot{q}^2 \pm q^2\right) \tag{B.102}$$

The solutions to the equation of motion are of the form

$$a(t) = A\,\sin(t+\phi) = A'\cos t + B'\sin t \tag{B.103}$$

where the two sets of coefficients are related to each other and the initial conditions by

$$A\,\sin\phi = A' = q(0) \qquad\qquad A\,\cos\phi = B' = \dot{q}(0) \tag{B.104}$$

In complex notation, we have

$$q_c(t) = \mathcal{A}_c\,e^{it} = A_c\,e^{i\phi}\,e^{it} \tag{B.105}$$

where $\mathcal{A}_c = A_c\,e^{i\phi}$ (and thus $A_c = |\mathcal{A}_c|$). The initial conditions are

$$q(0) \;=\; \mathcal{R}[q_c(0)] = A_c\,\cos\phi = \mathcal{R}[\mathcal{A}_c] \tag{B.106}$$

$$\dot{q}(0) \;=\; \mathcal{R}[\dot{q}_c(0)] = -A_c\,\sin\phi = -\mathcal{I}[\mathcal{A}_c] \tag{B.107}$$

or, equivalently,

$$\mathcal{A}_c = q(0) - i\,\dot{q}(0) \tag{B.108}$$

The Undriven, Damped SHO

The equation of motion for the damped SHO is

$$\ddot{q} + \frac{\dot{q}}{Q} + q = 0 \tag{B.109}$$

where the damping force is

$$F_{damp} = -\frac{\dot{q}}{Q} \tag{B.110}$$

Q can also be shown to be the product of the natural frequency and the damping time

$$Q = \omega\,\tau_{damp} \tag{B.111}$$

where τ_{damp} is the exponential decay time arising from the damping term. The version of the above with physical units reinserted is

$$F_{restore} = -k\,x$$
$$F_{damp} = -b\,\dot{x} = \frac{m\,\omega}{Q}\,\dot{x} = \frac{\sqrt{k\,m}}{Q}\,\dot{x}$$
$$\ddot{x} + \frac{\omega}{Q}\,\dot{x} + \omega^2\,x = 0$$
$$Q = \omega\,\tau_{damp} = \frac{\omega}{b/m}$$

Depending on the size of Q, there are three types of solutions:

- **Underdamped**: $Q > \frac{1}{2}$. In this case, the discriminant is positive and we obtain oscillatory motion from the second term. The complex solution is given by

$$q_c(t) = \mathcal{A}_c\,\exp\left(-\frac{t}{\tau_d}\right)\,\exp\left(\pm i\,\omega'\,t\right) \qquad \tau_d \equiv 2\,Q \qquad \omega' \equiv \sqrt{1 - \frac{1}{4\,Q^2}} \tag{B.112}$$

- **Overdamped**: $Q < \frac{1}{2}$. In this case, the discriminant becomes negative and the second term also is decaying. There are actually two possible decay times due to the sign freedom for the radical. The general solution is

$$q(t) = A\,\exp\left(-\frac{t}{\tau_{d,+}}\right) + B\,\exp\left(-\frac{t}{\tau_{d,-}}\right) \qquad \tau_{d,\mp}^{-1} = \frac{1}{2\,Q} \pm \sqrt{\frac{1}{4\,Q^2} - 1} \tag{B.113}$$

  The inversion of the sign in subscripting the solution is so that the + subscript goes with the larger time constant. We refer to the decay constants as times (unlike Hand and Finch) because it's more intuitive.

- **Critically damped**: $Q = \frac{1}{2}$. The discriminant vanishes and the two solutions become degenerate. The degenerate time constant is $\tau_d = 2\,Q = 1$ (*i.e.*, the damping time becomes equal to half the undamped oscillation period). The generic solution is

$$q(t) = A\,\exp\left(-t\right) + B\,t\,\exp\left(-t\right) \tag{B.114}$$

The energy satisfies the differential equation

$$\frac{dE}{dt} = -\frac{E}{Q} \tag{B.115}$$

Since the time has been rescaled so the angular frequency of oscillation is 1 and the period is 2π. The energy decay time is therefore $\frac{Q}{2\,\pi}$ periods of the oscillator (assuming $Q \gg \frac{1}{2}$ so we may ignore the frequency shift).

The Driven, Damped SHO – Transient Response and Green's Function

The full solutions are provided by **Green's functions**: If $G(t, t')$ satisfies the differential equation

$$\ddot{G} + \frac{\dot{G}}{Q} + G = \delta(t - t') \tag{B.116}$$

where the dots indicate derivatives with respect to t and t' should be regarded as a parameter of the driving function, then the full solution can be calculated by

$$q(t) = \int_{-\infty}^{\infty} dt'\, F(t')\, G(t,t') \tag{B.117}$$

The Green's functions for the three cases are:

- **Underdamped**:

$$G(t-t') \;=\; \frac{1}{\omega'}\, \exp\left(-\frac{t-t'}{2Q}\right)\, \sin\left[\omega'\,(t-t')\right]\, \theta(t-t') \tag{B.118}$$

  Note the use of the Heaviside function to impose causality. The generic solution for an arbitrary forcing function is therefore

$$q(t) = \int_{-\infty}^{\infty} dt'\, F(t')\, G(t-t') = \int_{-\infty}^{t} dt' F(t') \frac{1}{\omega'}\, \exp\left(-\frac{t-t'}{2Q}\right)\, \sin\left[\omega'\,(t-t')\right] \tag{B.119}$$

- **Overdamped**:

$$G(t-t') \;=\; \frac{\tau_{d,+}\,\tau_{d,-}}{\tau_{d,+}-\tau_{d,-}}\, \left[\exp\left(-\frac{t-t'}{\tau_{d,+}}\right) - \exp\left(-\frac{t-t'}{\tau_{d,-}}\right)\right] \theta(t-t') \tag{B.120}$$

  The generic solution for an arbitrary forcing function is

$$q(t) = \int_{-\infty}^{t} dt' F(t')\, \frac{\tau_{d,+}\,\tau_{d,-}}{\tau_{d,+}-\tau_{d,-}}\, \left[\exp\left(-\frac{t-t'}{\tau_{d,+}}\right) - \exp\left(-\frac{t-t'}{\tau_{d,-}}\right)\right] \tag{B.121}$$

- **Critically damped**:

$$G(t-t') \;=\; (t-t')\, \exp\left(-(t-t')\right)\, \theta(t-t') \tag{B.122}$$

  and the generic solution is

$$q(t) = \int_{-\infty}^{t} dt' F(t')\, (t-t')\, \exp\left(-(t-t')\right) \tag{B.123}$$

The Driven, Damped SHO – Steady-State Response and Resonance Phenomena

If the driving function is sinusoidal and steady state, then any transient response dies out and the system responds at the drive frequency. The complex amplitude response is

$$\mathcal{A}_c = \frac{F_0}{1-\omega_d^2 + \frac{i\,\omega_d}{Q}} \tag{B.124}$$

where F_0 is the drive amplitude. The square of the amplitude response gives the stored potential energy. It is

$$E = \frac{F_0^2}{2}\, \frac{1}{(1-\omega_d^2)^2 + \frac{\omega_d^2}{Q^2}} \tag{B.125}$$

The stored energy is maximized at the **resonant frequency**, ω_r, which is related to the characteristic and damped characteristic frequencies by

$$\text{undamped characteristic}: \quad \omega_0 = \sqrt{\frac{k}{m}} \tag{B.126}$$

$$\text{damped characteristic}: \quad \omega' = \omega_0 \sqrt{1 - \frac{1}{4\,Q^2}} < \omega_0 \tag{B.127}$$

$$\text{damped resonant}: \quad \omega_r = \omega_0 \sqrt{1 - \frac{1}{2\,Q^2}} < \omega' < \omega_0 \tag{B.128}$$

The stored kinetic energy is

$$T = \frac{F_0^2}{2} \frac{\omega_d^2}{(1-\omega_d^2)^2 + \frac{\omega_d^2}{Q^2}} \tag{B.129}$$

The maximum in the stored kinetic energy occurs at the undamped characteristic frequency because the velocity and driving force are in phase, providing the maximum power $P = F\,v$.
For high Q systems, the stored potential energy response can be simplified:

$$E \approx \frac{F_0^2}{8} \frac{1}{(\omega_d - 1)^2 + \frac{1}{4\,Q^2}} \tag{B.130}$$

In this limit, the potential and kinetic energy resonances occur at the same frequency. The full-width at half-maximum of the resonance curve is

$$\Delta\omega \quad = \quad \frac{1}{Q} \tag{B.131}$$

The stored energy on resonance is

$$E \approx \frac{F_0^2}{8}\, 4\, Q^2 = \frac{1}{2}\, F_0^2\, Q^2 \tag{B.132}$$

The phase response is given by

$$\tan\phi = -\frac{\frac{\omega_d}{Q}}{1-\omega_d^2} \tag{B.133}$$

For high Q, the phase response traces out a circle in the complex plane of radius $\frac{Q}{2}$ and whose origin is at $-i\,\frac{Q}{2}$.

Coupled Oscillations

Given an arbitrary multidimensional system with coordinates $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_M)$ with a stable equilibrium point, we can expand the kinetic and potential energies about the equilibrium point

and rewrite in matrix form:

$$t_{ij} = \frac{1}{2}\frac{\partial^2 T}{\partial\,\dot{\phi}_i\,\partial\,\dot{\phi}_j}\bigg|_{\dot{\phi}_i=0,\dot{\phi}_j=0} \tag{B.134}$$

$$T = \sum_{i,j} t_{ij}\,\dot{\phi}_i\,\dot{\phi}_j = \dot{\vec{\phi}}^T\mathbf{t}\,\dot{\vec{\phi}} \tag{B.135}$$

$$v_{ij} = \frac{1}{2}\frac{\partial^2 V}{\partial\,\phi_i\,\partial\,\phi_j}\bigg|_{\phi_i=0,\phi_j=0} \tag{B.136}$$

$$V = \sum_{i,j} v_{ij}\,\phi_i\,\phi_j = \vec{\phi}^T\mathbf{v}\,\vec{\phi} \tag{B.137}$$

where we shift the origin so the equilibrium point is at the origin. The equation of motion becomes

$$\mathbf{t}\,\ddot{\vec{\phi}} + \mathbf{v}\,\vec{\phi} = 0 \tag{B.138}$$

Assuming harmonic solutions of the form

$$\vec{\phi}(t) = \vec{\Phi}\,e^{i\omega\,t} \tag{B.139}$$

we can find the **normal mode frequencies** $\{\omega_i\}$ by requiring the solutions be nontrivial:

$$\left|-\omega^2\,\mathbf{t} + \mathbf{v}\right| = 0 \tag{B.140}$$

Once the normal mode frequencies have been found, the **normal mode vectors** $\vec{\Phi}_i$ are the cofactors of any row of the matrix $-\omega_i^2\,\mathbf{t} + \mathbf{v}$. The normal mode vectors obey the **orthonormalization relation**

$$\vec{\Phi}_i^T\mathbf{t}\,\vec{\Phi}_j = \delta_{ij} \tag{B.141}$$

which should be use to determine the normalization of each normal mode vector. When degenerate normal modes are obtained, normal mode vectors for the degenerate modes may be chosen somewhat arbitrarily but ensuring that they satisfy the orthonormalization condition. The generic solution has the form

$$\vec{\phi}_r(t) = \mathcal{R}\left[\vec{\phi}(t)\right] = \mathcal{R}\left[\sum_{i=1}^{M} A_i\,\vec{\Phi}_i\,e^{i\omega_i t}\right] \tag{B.142}$$

with the mode coefficients given by

$$\mathcal{R}\left[A_i\right] = \vec{\Phi}_i^T\mathbf{t}\,\vec{\phi}_r(t=0) \tag{B.143}$$

$$\mathcal{I}\left[A_i\right] = \frac{1}{\omega_i}\,\vec{\Phi}_i^T\mathbf{t}\,\dot{\vec{\phi}}_r(t=0) \tag{B.144}$$

The Hamiltonian can be rewritten in diagonalized form via a **congruence transformation**. Define the matrix

$$\mathbf{\Phi}_{ji} = \left[\vec{\Phi}_i\right]_j \tag{B.145}$$

Then the following hold:

$$\mathbf{\Phi}^T\mathbf{t}\,\mathbf{\Phi} = \mathbf{\Phi}\,\mathbf{\Phi}^T\mathbf{t} = \mathbf{t}\,\mathbf{\Phi}\,\mathbf{\Phi}^T = \mathbf{I} \tag{B.146}$$

and

$$\mathbf{\Phi}^T \mathbf{v}\, \mathbf{\Phi} = \Omega^2 \tag{B.147}$$

where Ω is a matrix with

$$\Omega_{ij} = \omega_i\, \delta_{ij} \tag{B.148}$$

The the Lagrangian and Hamiltonian can be rewritten

$$L = \dot{\vec{\psi}}^T \mathbf{I}\, \dot{\vec{\psi}} - \vec{\psi}^T \Omega^2\, \vec{\psi} \tag{B.149}$$

$$H = \dot{\vec{\psi}}^T \mathbf{I}\, \dot{\vec{\psi}} + \vec{\psi}^T \Omega^2\, \vec{\psi} \tag{B.150}$$

where

$$\vec{\psi}(t) = \mathbf{\Phi}^T \mathbf{t}\, \vec{\phi}(t) \tag{B.151}$$

and conversely:

$$\vec{\phi}(t) = \mathbf{\Phi}\, \vec{\psi}(t) \tag{B.152}$$

The new coordinates $\vec{\psi}(t)$ satisfy the equation of motion

$$\ddot{\vec{\psi}} + \Omega^2 \vec{\psi} = 0 \tag{B.153}$$

The time evolution is

$$\vec{\psi}(t) = e^{i\Omega t} \vec{\psi}(t=0) \tag{B.154}$$

The initial conditions are applied via

$$\mathcal{R}\left[\vec{\psi}(t=0)\right] = \mathbf{\Phi}^T \mathbf{t}\, \vec{\phi}_r(t=0) \tag{B.155}$$

$$\mathcal{I}\left[\vec{\psi}(t=0)\right] = \Omega^{-1}\, \mathbf{\Phi}^T \mathbf{t}\, \dot{\vec{\phi}}_r(t=0) \tag{B.156}$$

and the original coordinates are recovered via

$$\vec{\phi}_r(t) = \mathcal{R}\left[\vec{\phi}(t)\right] = \mathcal{R}\left[\Phi\, \vec{\psi}(t)\right] \tag{B.157}$$

Waves

Loaded String

We began with the loaded string, with M masses of mass m separated by spacing d and held together by a massless string with tension τ. The normal modes giving the motion of the pth mass in the nth mode are

$$\omega_n = 2\sqrt{\frac{\tau}{m\, d}}\left|\sin\left(\frac{1}{2}\frac{n\,\pi}{M+1}\right)\right| \tag{B.158}$$

$$y_{n,p}(t) = \sqrt{\frac{2}{M+1}}\, \sin\left(\frac{n\, p\, \pi}{M+1}\right) \cos\left(\omega_n t\right) \tag{B.159}$$

$$\equiv \Phi_{n,p} \cos\left(\omega_n t\right) \tag{B.160}$$

The generic solution is

$$\vec{y}_r(t) = \mathcal{R}\left[\vec{y}(t)\right] = \mathcal{R}\left[\sum_{n=1}^{M} A_n\,\vec{\Phi}_n\,e^{i\omega_n t}\right] \tag{B.161}$$

with initial conditions applied via

$$\mathcal{R}\left[A_n\right] = \sqrt{\frac{2}{M+1}}\sum_{p=1}^{M}\sin\left(\frac{n\,p\,\pi}{M+1}\right)\,y_{r,p}(t=0) \tag{B.162}$$

$$\mathcal{I}\left[A_n\right] = \frac{1}{\omega_n}\sqrt{\frac{2}{M+1}}\sum_{p=1}^{M}\sin\left(\frac{n\,p\,\pi}{M+1}\right)\,\dot{y}_{r,p}(t=0) \tag{B.163}$$

Continuous String

The continuous string was taken as the $d \to 0, M \to \infty$ limit of the loaded string, having linear mass density Λ and length $L = (M+1)\,d$. The normal modes are

$$\omega_n = \frac{n\,\pi}{L}\sqrt{\frac{\tau}{\Lambda}} \tag{B.164}$$

$$y_n(x,t) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,\cos\left(\omega_n t\right) \tag{B.165}$$

$$\equiv \Phi_n(x)\,\cos\left(\omega_n t\right) \tag{B.166}$$

The generic solution is

$$y_r(x,t) = \mathcal{R}\left[\sum_{n=1}^{\infty} A_n\,\Phi_n(x)\,e^{i\omega_n t}\right] \tag{B.167}$$

Initial conditions are applied via

$$\mathcal{R}\left[A_n\right] = \sqrt{\frac{2}{L}}\int_0^L dx\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,y_r(x,t=0) \tag{B.168}$$

$$\mathcal{I}\left[A_n\right] = \frac{1}{\omega_n}\sqrt{\frac{2}{L}}\int_0^L dx\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,\dot{y}_r(x,t=0)$$

The correspondence to normal mode notation is

$$\begin{aligned}
p &\longleftrightarrow x \\
\phi_p(t) &\longleftrightarrow y(x,t) \\
\mathbf{t} &\longleftrightarrow \frac{\Lambda}{2} \\
\mathbf{\Phi}_{pn} = \Phi_{n,p} &\longleftrightarrow \sqrt{\frac{2}{\Lambda}}\,\Phi_n(x) = \sqrt{\frac{2}{\Lambda}}\sqrt{\frac{2}{L}}\,\sin\left(\frac{n\,\pi\,x}{L}\right) \\
\psi_n(t=0) &\longleftrightarrow \sqrt{\frac{\Lambda}{2}}\,A_n \\
\psi_n(t) &\longleftrightarrow \sqrt{\frac{\Lambda}{2}}\,\mathcal{R}\left[A_n\,e^{i\omega_n t}\right] \\
\dot{\psi}_n(t) &\longleftrightarrow \sqrt{\frac{\Lambda}{2}}\,\mathcal{R}\left[i\,\omega_n\,A_n\,e^{i\omega_n t}\right]
\end{aligned}$$

The diagonalized Hamiltonian can be written in the form

$$H = \frac{\Lambda}{2} \sum_{n=1}^{\infty} \omega_n^2 \, |A_n|^2 \tag{B.169}$$

which makes clear the contribution of each mode to the total energy.

Wave Equation and Wave Solutions

We derived on general grounds for the continuous string the **wave equations**

$$\frac{\partial^2}{\partial t^2} \, y(x,t) - \frac{\tau}{\Lambda} \, \frac{\partial^2}{\partial x^2} \, y\,(x,t) = 0 \tag{B.170}$$

A generic solution, including all frequencies ω_r allowed by the boundary conditions, has the form

$$\begin{aligned} y(x,t) &= \sum_r \left[a_{r<} \, e^{i(k_r x + \omega_r t)} + a_{r>} \, e^{i(k_r x - \omega_r t)} + a^*_{r>} \, e^{-i(k_r x - \omega_r t)} + a^*_{r<} \, e^{-i(k_r x + \omega_r t)} \right] \\ &= 2\,\mathcal{R} \left\{ \sum_r \left[a_{r<} \, e^{i(k_r x + \omega_r t)} + a_{r>} \, e^{i(k_r x - \omega_r t)} \right] \right\} \end{aligned}$$

where the propagation speed is $v^2 = \frac{\tau}{\Lambda}$ and the wavevector is $k = \frac{\omega}{v}$. The wavelength of the wave is $\lambda = \frac{2\,\pi}{k}$. To further reduce the freedom in the solution, boundary conditions and initial conditions must be applied. The velocity v is the **phase velocity**, the speed at which a point of constant wave phase moves. If the phase velocity in the medium is dependent on the frequency ω, then the phase velocity is not the speed at which a wave packet with finite spatial extent propagates. Rather a wave packet propagates at speed

$$v_g = \left. \frac{d\omega}{dk} \right|_{k=k_0} = \frac{v_0}{1 - \left. \frac{\omega_0}{v_0} \frac{dv}{d\omega} \right|_0} \tag{B.171}$$

where v_g is called the **group velocity**. A wave packet has the form

$$y(x,t) = \int_{-\infty}^{\infty} dk \, \alpha(k) \, e^{i(\omega(k)t - kx)} \tag{B.172}$$

$$= e^{i(\omega_0 t - k_0 x)} \int_{-\Delta k}^{\Delta k} d\tilde{k} \, \alpha(k_0 + \tilde{k}) \, e^{i\tilde{k}(v_g t - x)} + c.c. \tag{B.173}$$

where $\alpha(k)$ is nonzero for only a finite range of k, $k_0 - \Delta k$ to $k_0 + \Delta k$. We have written ω as a function of k because k is now effectively the mode index. $\alpha(k)$ is known as the **spectral distribution** of the wave solution $y(x,t)$. Initial conditions can be applied in the standard manner.

B.4 Central Forces and Dynamics of Scattering

Generic Central Forces

The problem of two particles interacting via a strong-form third law central force can be reduced to translational motion of the center-of-mass system combined with one particle in a central force.

If the two particles have masses m_a and m_b and position $\vec{r}_a$ and $\vec{r}_b$, then the relative coordinate and reduced mass are

$$\vec{r}_{ab} = \vec{r}_a - \vec{r}_b \qquad \mu \equiv \frac{m_a\, m_b}{m_a + m_b} \tag{B.174}$$

The original coordinates can be rewritten as

$$\vec{r}_a = \frac{\mu}{m_a}\, \vec{r}_{ab} \qquad \vec{r}_b = -\frac{\mu}{m_b}\, \vec{r}_{ab} \tag{B.175}$$

The Lagrangian can be rewritten in the following ways:

$$L = \frac{1}{2}\, \mu\, \dot{\vec{r}}_{ab}^{\,2} - U(r_{ab}) = \frac{1}{2}\, \mu \left(\dot{r}_{ab}^2 + r_{ab}^2\, \dot{\theta}_{ab}^2 + r_{ab}^2 \sin^2\theta_{ab}\, \dot{\phi}_{ab}^2 \right) - U(r_{ab}) \tag{B.176}$$

We obtain equations of motion (after eliminating the constant ϕ coordinate)

$$\mu\, \ddot{r} = -\frac{dU}{dr} + \frac{l_\theta^2}{\mu\, r^3} \qquad \mu\, r^2 \dot{\theta} = l_\theta = constant$$

We define the effective potential

$$U_{eff}(r) = U(r) + \frac{l_\theta^2}{2\, \mu\, r^2} \tag{B.177}$$

The Lagrangian and energy can be rewritten in a one-dimensional form

$$L_{1D} = \frac{1}{2}\, \mu\, \dot{r}^2 - \frac{l_\theta^2}{2\, \mu\, r^2} - U(r) = \frac{1}{2}\, \mu\, \dot{r}^2 - U_{eff}(r) \tag{B.178}$$

$$E = \frac{1}{2}\, \mu\, \dot{r}^2 + \frac{l_\theta^2}{2\, \mu\, r^2} + U(r) = \frac{1}{2}\, \mu\, \dot{r}^2 + U_{eff}(r) \tag{B.179}$$

where $\frac{l_\theta^2}{2\, \mu\, r^2}$ is the repulsive "centrifugal potential." Note that L_{1D} is not just L with l_θ substituted in – there is a sign flip in the centrifugal term needed to generate the correct equation of motion (see the original text for details). The qualitative behavior of the system can be obtained by examining the shape of the effective potential – where it is repulsive, attractive, where its slope vanishes, etc. The constancy of l_θ gives us Kepler's second law

$$\frac{dA}{dt} = \frac{1}{2}\, \frac{l_\theta}{\mu} = constant \tag{B.180}$$

The generic quadrature solution to the central force problem is given by

$$t = \pm \int_{r(0)}^{r} dr' \left[\frac{2}{\mu} \left(E - U(r) \right) - \frac{l_\theta^2}{\mu^2\, r'^2} \right]^{-1/2} \tag{B.181}$$

$$\theta - \theta(0) = \frac{l_\theta}{\mu} \int_0^t \frac{dt'}{\left[r(t') \right]^2} \tag{B.182}$$

Elimination of t from the original differential relations also lets us obtain the quadrature solution for θ in terms of r:

$$\theta(r) - \theta(0) = \pm l_\theta \int_{r(0)}^{r} \frac{dr'}{\left[r(t) \right]^2} \left[2\, \mu \left(E - U(r) \right) - \frac{l_\theta^2}{r'^2} \right]^{-1/2} \tag{B.183}$$

We may obtain a generic differential equation relating r and θ

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=-\frac{\mu\, r^2}{l_\theta^2}\,F(r) \tag{B.184}$$

Defining $u=\frac{1}{r}$, this may be rewritten

$$\frac{d^2u}{d\theta^2}+u=-\frac{\mu}{l_\theta^2\,u^2}\,F\left(\frac{1}{u}\right) \tag{B.185}$$

The total energy of the system is constant and can be written as:

$$E=\frac{l_\theta^2}{2\,\mu}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)+\frac{l_\theta^2}{2\,\mu\,r^2}+U(r)=\frac{l_\theta^2}{2\,\mu}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]+U\left(\frac{1}{u}\right) \tag{B.186}$$

The Kepler Problem

When we specialize to $U(r)\propto\frac{1}{r}$, we may obtain more specific results. The differential equation relating r and θ or u and θ is

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{G\,\mu^2\,M}{l_\theta^2}\qquad\frac{d^2u}{d\theta^2}+u=\frac{G\,\mu^2M}{l_\theta^2} \tag{B.187}$$

The generic solution may be written

$$u(\theta)=A\cos(\theta-\theta_0)+\frac{G\,\mu^2M}{l_\theta^2}\qquad p=r+\epsilon\,r\,\cos\theta\qquad p=\frac{l_\theta^2}{G\,\mu^2M}\qquad\epsilon=p\,A \tag{B.188}$$

We specify two initial conditions (neglecting θ_0): l_θ and the total energy E. The constants A and ϵ are related to the total energy by

$$E=\frac{l_\theta^2}{2\,\mu}\left[A^2-\left(\frac{G\,\mu^2M}{l_\theta^2}\right)^2\right]=\frac{G\,\mu\,M}{2\,p}\left(\epsilon^2-1\right)\equiv E_{scale}\left(\epsilon^2-1\right) \tag{B.189}$$

The energy may also be written as

$$E=-\frac{G\,\mu\,M}{2\,a} \tag{B.190}$$

The assorted orbital parameters are summarized in Table B.1:
Kepler's third law, which tells us the period of elliptical orbits, is obtained from Kepler's second law and the area of an elliptical orbit, giving

$$\tau=2\,\pi\sqrt{\frac{a^3}{G\,M}} \tag{B.191}$$

The full time dependence of elliptical orbits can be obtained through use of the eccentric anomaly, ε, which is defined implicitly in terms of the true anomaly, θ. One begins with

$$r(\theta)\cos\theta=x=x_c+a\,\cos\varepsilon \tag{B.192}$$

quantity	symbol	formula(e)	sign	significance
angular momentum	l_θ	$\mu\, r^2 \dot{\theta}$	≥ 0 $= 0$ gives trivial orbit	centrifugal potential (brings in effect of θ motion)
scale energy	E_{scale}	$\frac{1}{2}\frac{G\,\mu\,M}{p}$	> 0	scale energy $= \lvert E_{min}\rvert$ for attractive pot.
scale radius	p	$\frac{l_\theta^2}{G\,\mu^2\,M}$	> 0 for attr. pot. < 0 for repul. pot.	sets scale of orbit
eccentricity	ϵ	$\sqrt{1+\frac{E}{E_{scale}}}$ $-\sqrt{1+\frac{E}{E_{scale}}}$	≥ 0 attr. pot. < -1 repul. pot.	sets shape of conic section, related to ratio of energy to scale energy
orbit center	x_c	$-\frac{\epsilon\, p}{1-\epsilon^2}$ $= -\epsilon\, a$	$= 0$ circular < 0 elliptical > 0 hyperbolic	
semimajor axis	a	$\frac{p}{1-\epsilon^2}$	> 0 circ./ellip. < 0 hyperbolic attr. > 0 hyperbolic repul.	distance from x_c to vertices along major axis
semiminor axis	b	$\frac{p}{\sqrt{\pm(1-\epsilon^2)}}$	> 0 attr. pot. < 0 repul. pot.	distance from x_c to vertices along minor axis (circ./ellip.) helps set asymptotic slope of trajectory (hyperbol.)
turning points	x_1 x_2	$\frac{p}{1+\epsilon}$ $-\frac{p}{1-\epsilon}$	> 0 < 0 circ./ellip. > 0 hyperbolic	turning points of motion relative to CM = focus 1 apsides for circ./ellip. orbits
For hyperbolic orbits, x_1 is the turning point for attractive potentials, x_2 the turning point for repulsive potentials.				

Table B.1: Parameters of Kepler orbits

and one obtains

$$r(\varepsilon) = a\,(1 - \epsilon\cos\varepsilon) \qquad t(\varepsilon) = \sqrt{G\,M\,a^3}\,(\varepsilon - \epsilon\sin\varepsilon) \tag{B.193}$$

$$x(\varepsilon) = a\,(\cos\varepsilon - \epsilon) \qquad y(\varepsilon) = a\sqrt{1-\epsilon^2}\,\sin\varepsilon \tag{B.194}$$

An analogous parameterization can be done for parabolic orbits, though the geometrical interpretation of the eccentric anomaly is no longer valid. Begin with

$$r(\theta)\cos\theta = x_c \mp a\cosh\varepsilon \tag{B.195}$$

and one obtains

$$r(\varepsilon) = a\,(\epsilon\cosh\varepsilon \mp 1) \qquad t(\varepsilon) = \sqrt{G\,M\,a^3}\,(\epsilon\sinh\varepsilon \mp \varepsilon) \tag{B.196}$$

$$x(\varepsilon) = a\,(\epsilon \mp \cosh\varepsilon) \qquad y(\varepsilon) = a\sqrt{\epsilon^2-1}\,\sinh\varepsilon = b\,\sinh\varepsilon \tag{B.197}$$

where the upper signs are for an attractive orbit and the lower signs for a repulsive one.

Dynamics of Scattering

For central-force scattering problems, we generically consider unbound central-force orbits. Rather than parameterizing in terms of energy E and angular momentum l_θ, we use the velocity at infinity v_∞ and the impact parameter b (distance of closest approach to scattering center). These parameters are related by

$$E = \frac{1}{2}\,\mu\,v_\infty^2 \qquad l_\theta = \mu\,v_\infty\,b \tag{B.198}$$

We usually phrase scattering in terms of an incoming beam of particles of number flux

$$F = nv_\infty \tag{B.199}$$

The differential scattering cross section gives us the area of the beam that will be scattered into a solid angle $d\Omega$:

$$\frac{d\sigma}{d\Omega} \;=\; \frac{1}{F}\,\frac{dN}{d\Omega} \tag{B.200}$$

The differential cross section can be found if the relationship between the input impact parameter and the scattering angle θ_* (the angle between the incoming and outgoing particle trajectories) is known:

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta_*}\left|\frac{db}{d\theta_*}\right| \tag{B.201}$$

The total scattering cross section, which gives us the effective area of the scattering center, is

$$\sigma = \int d\Omega\frac{d\sigma}{d\Omega} = 2\,\pi\int_0^\pi d\theta_*\sin\theta_*\frac{d\sigma}{d\Omega} = 2\,\pi\int_0^\infty db\,b \tag{B.202}$$

The scattering angle is calculated from the potential function via

$$\theta_* = \mp\left(\pi - 2\left|\int_{r_{min}}^\infty \frac{b\,dr'}{r'^2}\left[1 - \frac{U(r)}{E} - \frac{b^2}{r'^2}\right]^{-1/2}\right|\right) \tag{B.203}$$

For a potential $U(r) = \pm k/r$, the relationship between impact parameter and scattering angle and the differential cross section are given by

$$|b| = \frac{k}{2E}\left|\cot\frac{\theta_*}{2}\right| \qquad \frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta_*}{2}} \tag{B.204}$$

The relative sign between b and θ_* is chosen based on whether the potential is attractive or repulsive ($E > 0$ always for unbound orbits).

B.5 Rotating Systems

Dynamics in Rotating Coordinate Systems

Transforming Between Inertial and Rotating Systems

If F is the rotating coordinate system and F' is the fixed, inertial system, then coordinate representations of the position vector in the two are related by

$$\underline{\vec{r}}\,'(t) = \mathbf{R}(t)\,\underline{\vec{r}}(t) \tag{B.205}$$

and the non-rotating frame coordinate representation of the velocity measured in the non-rotating frame is related to the rotating frame coordinate representation of the velocity measured in the rotating frame by

$$\underline{\vec{v}}'_{space}(t) = \dot{\mathbf{R}}(t)\left[\mathbf{R}(t)\right]^T \underline{\vec{r}}\,'(t) + \mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) \tag{B.206}$$

$$= \left(\underline{\vec{\omega}}\,'(t)\cdot\vec{\mathbf{M}}\right)\underline{\vec{r}}\,'(t) + \mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) \tag{B.207}$$

$$= \underline{\vec{\omega}}\,'(t)\times\underline{\vec{r}}\,'(t) + \mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) \tag{B.208}$$

Remember that $\mathbf{R}^T\underline{\vec{v}}'_{space}(t) \neq \underline{\vec{v}}_{body}$: $\vec{v}_{space}$ and $\vec{v}_{body}$ are different vectors.

Acceleration and Total Apparent Force in Rotating Frame

Taking a derivative of the above equation for $\underline{\vec{v}}'_{space}(t)$, we obtain

$$\frac{d}{dt}\underline{\vec{v}}'_{space}(t) = \underline{\vec{\omega}}\,'(t)\times\left[\underline{\vec{\omega}}\,'(t)\times\underline{\vec{r}}\,'(t)\right] + 2\,\underline{\vec{\omega}}\,'(t)\times\mathbf{R}(t)\,\underline{\vec{v}}_{body}(t) + \dot{\underline{\vec{\omega}}}\,'(t)\times\underline{\vec{r}}\,'(t) + \mathbf{R}(t)\,\underline{\vec{a}}_{body}(t) \tag{B.209}$$

We can rearrange the equation and make use of multiple applications of the generic fact

$$\mathbf{R}^T\left(\underline{\vec{a}}\,'\times\underline{\vec{b}}\,'\right) = \underline{\vec{a}}\times\underline{\vec{b}} \tag{B.210}$$

we obtain Newton's second law in the rotating frame:

$$\begin{aligned}\underline{\vec{F}}_{app} = {} & \underline{\vec{F}}_{true} \\ & - m\left[\mathbf{R}^T\underline{\vec{\omega}}\,'(t)\right](t)\times\left(\left[\mathbf{R}^T\underline{\vec{\omega}}\,'(t)\right]\times\underline{\vec{r}}(t)\right) - 2\,m\left[\mathbf{R}^T\underline{\vec{\omega}}\,'(t)\right]\times\underline{\vec{v}}_{body}(t) \\ & - m\left[\mathbf{R}^T\dot{\underline{\vec{\omega}}}\,'(t)\right]\times\underline{\vec{r}}(t)\end{aligned} \tag{B.211}$$

where $\underline{\vec{F}}_{app}$ is the rotating frame coordinate representation of the apparent force and $\underline{\vec{F}}_{true}$ is the rotating frame representation of the true force, $\underline{\vec{F}}_{true} = \mathbf{R}^T\underline{\vec{F}}'_{true}$. The equation tells us to add to the true force (written in rotating frame coordinate representation) a number of fictitious forces (centrifugal, Coriolis, and Euler, respectively) to obtain the apparent force in the rotating frame. The apparent force $\vec{F}_{app}$ then satisfies $\underline{\vec{F}}_{app} = m\,\underline{\vec{a}}_{body}$ in the rotating frame.

Lagrangian and Hamiltonian Dynamics in Rotating Frames

The only general result to keep in mind here is to write the Lagrangian first in the inertial frame, write the inertial frame coordinates and velocities in terms of rotating frame coordinates and velocities, and finally substitute. The Hamiltonian should be obtained from the Lagrangian in the rotating frame, not from the inertial frame Hamiltonian, to ensure that canonical momenta are defined properly. The rotating frame Hamiltonian will not in general equal the total energy.

Dynamics of Rigid Bodies

Basic Kinematics and Moment of Inertia Tensor

If $\vec{s}_a$ denotes the coordinates of a particle belonging to a rigid body relative to the body's center of mass, then it is true that all motion of such a particle relative to the center of mass is rotational; *i.e.*, it holds that

$$\dot{\vec{s}}_a = \vec{\omega} \times \vec{s}_a \tag{B.212}$$

with a common angular velocity vector $\vec{\omega}$ for all particles a.
The kinetic energy and angular momentum of the rigid body may be written

$$T_{rot} = \sum_a \frac{1}{2} m_a \left(\omega^2 \, s_a^2 - (\vec{\omega} \cdot \vec{s}_a)^2 \right) \tag{B.213}$$

$$\vec{L} = \sum_a m_a \left(s_a^2 \, \vec{\omega} - (\vec{\omega} \cdot \vec{s}_a) \, \vec{s}_a \right) \tag{B.214}$$

which motivate the definition of the moment of inertia tensor $\mathcal{I}$ with coordinate representation

$$I_{ij} = \sum_a m_a \left(s_a^2 \delta_{ij} - s_{a,i} \, s_{a,j} \right) \tag{B.215}$$

and coordinate-free form

$$\mathcal{I} = \sum_a m_a \left(\vec{s}_a \cdot \vec{s}_a \, \mathbf{1} - \vec{s}_a \, \vec{s}_a^{\,T} \right) \tag{B.216}$$

which allow us to write

$$T_{rot} = \frac{1}{2} \, \vec{\omega}^{\,T} \mathcal{I} \, \vec{\omega} = \frac{1}{2} \, \omega_i \, I_{ij} \, \omega_j \tag{B.217}$$

$$\vec{L} = \mathcal{I} \, \vec{\omega} \iff L_i = I_{ij} \, \omega_j \tag{B.218}$$

For continuous systems, the moment of inertial tensor has representation

$$I_{ij} = \int d^3 r \rho(\vec{s}) \left(s^2 \delta_{ij} - s_i \, s_j \right) \tag{B.219}$$

Since $\mathcal{I}$ is a symmetric matrix, there exists a rotation matrix that diagonalizes it. The unit vectors in the frame in which it is diagonalized are called the principal axes and the eigenvalues (diagonal components of $\mathcal{I}$ in the diagonal frame) are called the principal components. The principal axes are the natural rotation axes of the body; in particular, angular velocity and angular momentum are aligned if rotation is purely about one principal axis. The frame in which the inertia tensor is diagonalized is usually referred to as the body frame.

The displaced axis theorem lets us calculate the moment of inertia tensor of a rigid body about one axis if we know the moment of inertia tensor about some other axis. If the new axis is displaced by the vector $\vec{a}$ from the original axis, then the new moment of inertial tensor has coordinate representation

$$I_{ij}^{displaced} = I_{ij} + M\left(a^2\delta_{ij} - a_i\,a_j\right) \tag{B.220}$$

Note that the sign of $\vec{a}$ is not important.

Euler's Equations

Since the body frame is the frame in which the inertia tensor is diagonalized, it can be a convenient frame in which to calculate the equations of motion. From our generic results about relating derivatives of vectors in rotating and non-rotating frames, the rate of change of the angular momentum relative to the space frame is related to the rate of change of the angular momentum relative to the body frame by

$$\left.\frac{d}{dt}\,\underline{\vec{L}}\,'\right|_{space} = \underline{\vec{\omega}}\,' \times \underline{\vec{L}}\,' + \mathbf{R}\left.\frac{d}{dt}\,\underline{\vec{L}}\right|_{body} \tag{B.221}$$

Remember, $\left.\frac{d}{dt}\,\underline{\vec{L}}\,'\right|_{space}$ and $\left.\frac{d}{dt}\,\underline{\vec{L}}\right|_{body}$ are representations in different frames of the different vectors $\left.\frac{d}{dt}\,\vec{L}\right|_{space}$ and $\left.\frac{d}{dt}\,\vec{L}\right|_{body}$. The symbols $\underline{\vec{\omega}}\,'$ and $\underline{\vec{L}}\,'$ are non-rotating frame representations of the vectors $\vec{\omega}$ and $\vec{L}$, which are always measured relative to the non-rotating frame (and hence no $_{space}$ or $_{body}$ modifier is needed). We may rewrite the above as

$$\left.\frac{d}{dt}\,\underline{\vec{L}}\right|_{body} = \mathbf{R}^T\left(-\,\underline{\vec{\omega}}\,' \times \underline{\vec{L}}\,' + \underline{\vec{\tau}}\,'\right) = -\left[\mathbf{R}^T\underline{\vec{\omega}}\,'\right] \times \left[\mathbf{R}^T\underline{\vec{L}}\,'\right] + \mathbf{R}^T\underline{\vec{\tau}}\,' \tag{B.222}$$

where $\underline{\vec{\tau}}\,'$ is the coordinate representation in the non-rotating frame of the torque and $\mathbf{R}^T\underline{\vec{\tau}}\,'$ is its representation in the rotating frame. Making use of the confusing but less cumbersome notation $\omega_i = \left[\mathbf{R}^T\underline{\vec{\omega}}\,'\right]_i$ to denote the rotating frame coordinate representation components of the angular velocity (and similarly for $\vec{L}$ and $\vec{\tau}$), using the fact $\underline{\vec{L}} = \mathbf{R}^T\underline{\vec{L}}\,' = \mathbf{R}^T\underline{\mathcal{I}}\,'\underline{\vec{\omega}}\,' = \mathbf{R}^T\underline{\mathcal{I}}\,'\mathbf{R}\,\mathbf{R}^T\underline{\vec{\omega}}\,' = \underline{\mathcal{I}}\,\underline{\vec{\omega}}$, and also realizing that $\underline{\mathcal{I}}$ is diagonal because it is the body frame representation, we may simplify to find

$$I_1\,\frac{d}{dt}\,\omega_1 = \omega_2\,\omega_3\,(I_2 - I_3) + \tau_1 \tag{B.223}$$

$$I_2\,\frac{d}{dt}\,\omega_2 = \omega_1\,\omega_3\,(I_3 - I_1) + \tau_2 \tag{B.224}$$

$$I_3\,\frac{d}{dt}\,\omega_e = \omega_1\,\omega_2\,(I_1 - I_2) + \tau_3 \tag{B.225}$$

Once one has integrated the above equations to determine $\omega_i(t)$ in the body frame representation, one can obtain the representation in the space frame using $\omega_i(t) = \left[\mathbf{R}^T\,\underline{\vec{\omega}}\,'(t)\right]_i$.

Kinetic Energy of a Symmetric Top

The kinetic energy of a symmetric top has a particularly clean form when written in terms of Euler angles:

$$T = \frac{1}{2}\,I_1\left(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\theta\right) + \frac{1}{2}\,I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 \tag{B.226}$$

Dynamics of Torque-Free Symmetric Tops

For torque-free symmetric tops, we have $I_1 = I_2$, which thereby ensures that ω_3 is constant. We obtained the solution

$$\omega_3 = \omega_3 = \text{constant} \qquad \Omega_P = \omega_3 \left(\frac{I_3}{I_1} - 1 \right) \tag{B.227}$$

$$\omega_1 = A \, \sin\left(\Omega_P \, t + \phi\right) \qquad \omega_2 = A \, \cos\left(\Omega_P \, t + \phi\right) \tag{B.228}$$

with

$$|A| = \sqrt{\frac{I_3}{I_1 - I_3} \left(\frac{L^2}{I_1 \, I_3} - \frac{2\,T}{I_1} \right)} \tag{B.229}$$

$$\omega_3 = \sqrt{\frac{I_1}{I_3 - I_1} \left(\frac{L^2}{I_1 \, I_3} - \frac{2\,T}{I_3} \right)} \tag{B.230}$$

$$\Omega_p = \frac{I_3 - I_1}{|I_3 - I_1|} \sqrt{\frac{I_3 - I_1}{I_1} \left(\frac{L^2}{I_1 \, I_3} - \frac{2\,T}{I_3} \right)} \tag{B.231}$$

More insight was obtained by solving the problem in the space frame, which gave us the solution:

$$\underline{\vec{L}}' = \text{constant} = L \, \hat{z}' \qquad T = \text{constant} \tag{B.232}$$

$$\underline{\vec{\omega}}'(t) = \omega_P \, \hat{z}' + \Omega \, \hat{z} = \omega_P \, \hat{z}' + \Omega \, (\hat{z}' \, \cos\theta - \hat{x}' \, \sin\theta \, \sin \, \omega_P \, t \, + \hat{y}' \, \sin\theta \, \cos \, \omega_P \, t \,) \tag{B.233}$$

with

$$\omega_P = \frac{L}{I_1} \tag{B.234}$$

$$\cos\theta = \frac{I_3}{L} \sqrt{\frac{I_1}{I_3 - I_1} \left(\frac{L^2}{I_1 \, I_3} - \frac{2\,T}{I_3} \right)} \tag{B.235}$$

$$\tag{B.236}$$

$$\tan\gamma = \frac{I_3}{I_1} \, \tan\theta \tag{B.237}$$

$$\Omega = L \left(\frac{1}{I_3} - \frac{1}{I_1} \right) \cos\theta \tag{B.238}$$

where θ is the angle between the spin axis and the angular momentum vector, γ is the angle between the spin axis and the angular velocity vector, and $\theta - \gamma$ is the angle between the angular momentum and angular velocity vectors. The body and space frame solutions are related by

$$A = -\omega_P \, \sin\theta \qquad I_3 \, \omega_3 = I_1 \, \omega_P \, \cos\theta = L \, \cos\theta \qquad \Omega = -\Omega_p \qquad \frac{|A|}{\omega_3} = \tan\gamma \tag{B.239}$$

Finally, if we used Euler angles and the Lagrangian point of view, we would have obtained the results

$$\dot{\phi} \iff \omega_P \tag{B.240}$$

$$\text{constant} = p_\phi = I_1 \, \dot{\phi} \iff L = |\underline{\vec{L}}'| \tag{B.241}$$

$$\text{constant} = p_\psi \iff I_3 \, \omega_3 = L \, \cos\theta \tag{B.242}$$

$$\dot{\psi} \iff \Omega \tag{B.243}$$

Dynamics of Symmetric Tops with Torque

When torque is included, Euler angles and a Lagrangian approach are easiest. In general, the torque yields a potential energy function in the Euler angle θ, $U(\theta)$. In that case, the Lagrangian is

$$L = T - U = \frac{1}{2} I_{1d} \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi} \cos\theta \right)^2 - U(\theta) \tag{B.244}$$

There are two conserved canonical momenta

$$p_\psi = I_3 \left(\dot{\psi} + \dot{\phi} \cos\theta \right) \tag{B.245}$$

$$p_\phi = I_{1d} \dot{\phi} \sin^2 \theta + p_\psi \cos\theta \tag{B.246}$$

The equation of motion for θ is

$$I_{1d} \ddot{\theta} = I_{1d} \dot{\phi}^2 \sin\theta \cos\theta - I_3 \left(\dot{\psi} + \dot{\phi} \cos\theta \right) \dot{\phi} \sin\theta - \frac{\partial U(\theta)}{\partial \theta} \tag{B.247}$$

These may be used to rewrite the problem as one-dimensional using an effective potential energy function:

$$V_{eff} = \frac{1}{2 I_{1d}} \frac{(p_\phi - p_\psi \cos\theta)^2}{\sin^2 \theta} + U(\theta) \tag{B.248}$$

The equation of motion with V_{eff} is

$$I_1 \ddot{\theta} = -\frac{\partial}{\partial \theta} \left(\frac{1}{2 I_{1d}} \frac{(p_\phi - p_\psi \cos\theta)^2}{\sin^2 \theta} + U(\theta) \right) \tag{B.249}$$

The one-dimensional Lagrangian that would generate this equation of motion is

$$L_{1D} = \frac{I_{1d}}{2} \dot{\theta}^2 + \frac{1}{2 I_3} p_\psi^2 - V_{eff}(\theta) \tag{B.250}$$

Note that L_{1D} is **not** the original L with p_ϕ and p_ψ substituted back in; there is a sign flip on the kinetic terms that generate the effective potential. This kind of thing was encountered in defining the effective potential in the central force problem, too (see the text for details). With that correction, one can obtain a one-dimensional equation of motion in θ. Once the dynamics in θ have been obtained, the dynamics in ϕ and ψ can be found by using $\theta(t)$ and the conservation of p_ϕ and p_ψ. The total energy is

$$E = \frac{I_{1d}}{2} \dot{\theta}^2 + \frac{1}{2 I_3} p_\psi^2 + V_{eff}(\theta) \tag{B.251}$$

which is the same as what one would obtain if one just substituted for p_ϕ and p_ψ in the original expression for the energy – the total energy does not suffer the sign flip confusion. This equation is a nonlinear differential equation in $\dot{\theta}$ that can be studied qualitatively to understand the possible types of solutions.

B.6 Special Relativity

The fundamental postulates of special relativity are

1. Physics is the same in all inertial frames.

2. The speed of light is the same in all inertial frames.

These postulates can be used to derive the rules for the **Lorentz transformation** of space-time coordinates. If frame F moves at speed $\beta = v/c$ along the $+x$ direction relative to frame $\widetilde{F}$, and their space and time origins coincide at $t = \tilde{t} = 0$, then the coordinates of a space-time event in one frame are related to those in the other frame by

$$\tilde{x} = \gamma\,(x + \beta\,t) \qquad \tilde{t} = \gamma\,(\beta\,x + t) \qquad \tilde{y} = y \qquad \tilde{z} = z \tag{B.252}$$

$$x = \gamma\left(\tilde{x} - \beta\,\tilde{t}\right) \qquad t = -\gamma\left(\beta\tilde{x} - \tilde{t}\right) \qquad y = \tilde{y} \qquad z = \tilde{z} \tag{B.253}$$

The space-time coordinate $x^\mu = (t, \vec{x})$ is the archetype of a **four-vector** or **Lorentz covariant vector**, which is any set of four numbers that obey the Lorentz transformation. The Lorentz transformation can be used to derive time dilation, length contraction, and the relativity of simultaneity. The **invariant interval** is

$$s^2 = t^2 - (x^2 + y^2 + z^2) \tag{B.254}$$

and is independent of Lorentz frame. A similar invariant length can be formed for any four-vector. The invariant interval defines whether a space-time interval is **time-like** ($s^2 > 0$), **space-like** ($s^2 < 0$), or **light-like** ($s^2 = 0$). Only events with time-like or light-like separations can causally affect one another. This causal property is invariant under Lorentz transformations.
The **rapidity** η (also known as the **boost angle** or **boost parameter** is defined by

$$\beta = \tanh\eta \qquad \gamma = \cosh\eta \qquad \beta\,\gamma = \sinh\eta \tag{B.255}$$

The Lorentz transformation may thus be written

$$\tilde{x} = x\,\cosh\eta + t\,\sinh\eta \qquad \tilde{t} = x\,\sinh\eta + t\,\cosh\eta \tag{B.256}$$

$$x = \tilde{x}\cosh\eta - \tilde{t}\sinh\eta \qquad t = -\tilde{x}\sinh\eta + \tilde{t}\cosh\eta \tag{B.257}$$

The **velocity addition formula** tells us how to transform a velocity measured in frame F' to the frame F; essentially, it performs summing of relativistic velocities:

$$\tilde{u}_{||} = \frac{\beta + u_{||}}{1 + \beta\,u_{||}} \qquad \tilde{u}_\perp = \frac{1}{\gamma}\,\frac{u_\perp}{1 + \beta\,u_{||}} \tag{B.258}$$

where $u_{||}$ and $u_\perp$ are the components of the velocity measured in the F frame along and perpendicular to the velocity $\vec{\beta}$ of the F frame relative to $\widetilde{F}$ and $\tilde{u}_{||}$ and $\tilde{u}_\perp$ are the components in $\widetilde{F}$. The γ symbol here is $\gamma = \left(1 - \beta^2\right)^{-1}$ as usual. Alternately, we can use the Lorentz transformation law on the **four-velocity**, which is defined to be

$$u^\mu = \frac{d\,x^\mu}{d\tau} = \frac{d\,x^\mu}{d\sqrt{|x^\mu|^2}} = \gamma_p\left(1, \vec{\beta}_p\right) \tag{B.259}$$

Here, $\vec{\beta}_p$ is the three-velocity of the particle (not the frame) in the frame in which we want the representation of u^μ and $\gamma_p = \left(1-\beta_p^2\right)^{-1}$ again. The definition of the four-velocity leads to the **four-momentum** or the **energy-momentum four-vector**,

$$p^\mu = m\,u^\mu = \gamma_p\,m\left(1,\vec{\beta}_p\right) \tag{B.260}$$

which is also Lorentz covariant because m is a Lorentz scalar. For light, the analogous quantity is the **four-wavevector**

$$k^\mu = (\omega,\vec{k}) \tag{B.261}$$

The four-momentum of light is $p^\mu = \hbar\,k^\mu$. The Lorentz transformation of the four-wavevector can be used to derive the formulae for the **relativistic Doppler shift**:

$$\frac{\nu'}{\nu} = \sqrt{\frac{1-\beta}{1+\beta}} \qquad\qquad \frac{\lambda'}{\lambda} = \sqrt{\frac{1+\beta}{1-\beta}} \tag{B.262}$$

where ν' is measured in the frame F' which moves at speed β relative to F. ν is measured in F. And similarly for λ' and λ. The action and Lagrangian for a free particle is

$$S = -\int_A^B m\,d\tau = -\int_A^B \frac{m}{\gamma_p}\,dt \tag{B.263}$$

The action is explicitly Lorentz invariant. The Lagrangian depends on the frame and is not Lorentz invariant. Standard conservative potentials cannot be added to the action because they are manifestly not Lorentz covariant. The canonical momentum and the Hamiltonian derived from L are

$$\vec{p} = \gamma_p\,m\,\vec{\beta}_p \qquad\qquad H = \gamma_p\,m \tag{B.264}$$

Neither one of these alone is Lorentz invariant or covariant, but of course together they make the Lorentz covariant energy-momentum four-vector.
Lorentz transformations can be written in vector/matrix form (for $\vec{\beta}$ along the $+x$ axis)

$$\mathrm{x}^\mu = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \qquad \tilde{\mathrm{x}}^\mu = \begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} \qquad \tilde{\mathrm{x}}^\mu = \Lambda^\mu{}_\nu\,\mathrm{x}^\nu \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where x^μ and $\tilde{\mathrm{x}}^\mu$ are the representations of the four-vector x^μ in the frames F and $\widetilde{F}$. Such a vector is said to transform **Lorentz-covariantly**; it is a **Lorentz-covariant** vector. For an arbitrary direction, the Lorentz transformation matrix is

$$\Lambda^0{}_0 = \gamma \qquad \Lambda^0{}_i = \Lambda^i{}_0 = \gamma\,\beta_i \qquad \Lambda^i{}_j = \delta_{ij} + (\gamma-1)\,\frac{\beta_i\,\beta_j}{\beta^2} \qquad i,j = 1,2,3$$

or, in its fully glory,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\,\beta_x & \gamma\,\beta_y & \gamma\,\beta_z \\ \gamma\,\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\,\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\,\beta_z}{\beta^2} \\ \gamma\,\beta_y & (\gamma-1)\frac{\beta_x\,\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\,\beta_z}{\beta^2} \\ \gamma\,\beta_z & (\gamma-1)\frac{\beta_x\,\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\,\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix}$$

One can demonstrate addition of velocities, and the Lorentz covariance of four-velocity, four-momentum, and four-wavevector using boost matrices. The **metric tensor** calculates the invariant interval associated with a four-vector (Einstein summation convention):

$$|x^\mu|^2 = g_{\mu\nu}x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{B.265}$$

The metric tensor $g_{\mu\nu}$ has representation $\mathrm{diag}(1,-1,-1,-1)$ in all Lorentz frames. The metric tensor can be used to raise and lower indices. It holds that

$$g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \qquad g^\mu{}_\nu = g_\mu{}^\nu = \mathrm{diag}(1,1,1,1) \tag{B.266}$$

The **Lorentz-contravariant** four-vector x_μ transforms as follows

$$\tilde{\mathrm{x}}_\mu = \Lambda_\mu{}^\nu \mathrm{x}_\nu \qquad \Lambda_\mu{}^\nu = g_{\mu\lambda}\, g^{\nu\sigma}\Lambda^\lambda{}_\sigma \tag{B.267}$$