Physics 106a/196a – Problem Set 1 – Due Oct 6, 2006

v. 2: updated Oct 1, 2006

These problems cover the material on Newtonian mechanics and gravitation in Sections 1.1 and 1.2 of the lecture notes. Problems 1, 2 are for 106a students only, 3-6 for 106a and 196a students, and 7, 8 for 196a students only. Problem sets will typically only have 4 problems, but this is relatively easy review material.

However, 106a students ought to be able to do problems 7 and 8 – they're not too hard, I just didn't want to overload you. If you don't have an idea of how to do them, you should look at the solutions.

You are reminded that:

- The force exerted by a spring is $F = -k\,(x - x_0)$ where x is the length of the spring and x_0 is the rest length of the spring; the force is opposite the displacement from the rest length. In vector notation, $\vec{F} = -k\,(\vec{r} - \vec{r}_0)$.

- The potential energy function of a spring is $U(x) = \frac{1}{2}\,k\,(x - x_0)^2$, or $U(\vec{r}) = \frac{1}{2}\,k\,|\vec{r} - \vec{r}_0|^2$.

v. 2: Typo in Problem 5 fixed: was $F_z = a\,x\,z = b\,z$, should be $F_z = a\,x\,y + b\,y$.

1. (106a) Redo example 1.9 from the lecture notes (last item in Section 1.1.3) using forces – *i.e.*, find out if there is an equilibrium where the net force vanishes (including an equation expressing that equilibrium height in terms of b, d, m_1, and m_2). Also, given m_1, m_2, and d, find the condition on the length of the rope b for the equilibrium height to be a physically allowed value. You may be tempted to calculate $\ddot{z}_2$ in terms of $\ddot{z}_1$ – are you sure you need it?

2. (106a) A particle of mass m moves according to the equations

$$x = x_0 + a\,t^2$$
$$y = b\,t^3$$
$$z = c\,t$$

Find the angular momentum $\vec{L}$ at any time t. Find the force $\vec{F}$ and from it the torque $\vec{N}$ acting on the particle. Verify that the torque gives the rate of change of angular momentum.

3. (106a/196a) A moving boat of mass m stops its engine and consequently is slowed by a frictional force $F(v)$. Its velocity is observed to decrease according to the formula

$$v = C\,(t - t_1)^2$$

where C is a constant and t_1 is the time at which it stops. If the initial speed is v_0 at $t = 0$, rewrite the above velocity equation in terms of v_0. Find the force $F(v)$ in terms of m, v_0,

t_1 and v. Find the distance x that the boat travels between $t = 0$ and the time at which it stops.

Rewrite the force in the generic form $F = -k\, v^\alpha$, relating k to m, v_0, and t_1. What are t_1 and x in terms of k, m, and v_0? In the example from class with $F = -k\, v$ we saw that the characteristic timescale to stop depended only on k and m, here we find dependence on v_0 also. On dimensional grounds, why is this necessary?

4. (106a/196a) A particle of mass m moves under the influence of an isotropic spring, $\vec{F}(\vec{r}) = -k\,\vec{r}$, or, equivalently,

$$ F_x = -k\,x \qquad F_y = -k\,y \qquad F_z = -k\,z $$

Suppose the initial conditions are $x(t = 0) = x_0$, $y(t = 0) = y_0$, $z(t = 0) = 0$, $\dot{x}(t = 0) = \dot{x}_0$, $\dot{y}(t = 0) = \dot{y}_0$, and $\dot{z}(t = 0) = 0$. Calculate the angular momentum of the mass.

5. (106a/196a) We explained in class that a conservative force is one for which the work done in moving from one point to another is path-independent. It is equivalent to say that the work done around a closed path vanishes. Stokes' theorem relates the line integral of a vector around a closed path to the integral of the curl of the vector over the area enclosed by that path:

$$ \oint_C \vec{F} \cdot d\vec{r} = \int_{A_C} d\vec{a} \cdot \left(\vec{\nabla} \times \vec{F} \right) $$

Thus, any conservative force must have $\vec{\nabla} \times \vec{F} = 0$ everywhere to ensure the left side vanishes for an arbitrary loop C. Calculate the curl to determine which of the following force fields is conservative. For any that are conservative, find the potential energy $U(\vec{r})$.

(a) $F_x = a\,y\,z + b\,x + c$, $F_y = a\,x\,z + b\,z$, $F_z = a\,x\,y + b\,y$

(b) $F_x = -z\,e^{-x}$, $F_y = \log z$, $F_z = e^{-x} + \frac{y}{z}$

(c) $F(\vec{r}) = \vec{h} \times \vec{r}$.

where a, b, c, and $\vec{h}$ are constants. You may find some of the relations provided in Appendix A of the notes useful. This problem answers the question asked in the lecture notes, "Do there exist position-dependent forces for which the work is not path-independent?" You can construct them mathematically, but it's hard to think of physical examples.

6. (106a/196a) Calculate the gravitational potential at off-axis points due to a thin circular ring of radius a and mass M. Let R be the distance from the center of the ring to the field point, and let θ be the angle between the line connecting the center of the ring with the field point and the axis of the ring. Assume $R \gg a$ so that terms of order $(a/R)^3$ and higher may be neglected. This is an excellent example of the use of Taylor expansions to obtain approximate solutions, a techique we will use frequently. Note that you need to carry the expansion to higher than first order in a/R!

7. (196a) A mass m is connected to the origin with a spring of constant k, whose length when relaxed is l. The restoring force is very nearly proportional to the amount the spring has been stretched or compressed so long as it is not stretched or compressed very far. However, when the spring is compressed too far, the force increases very rapidly, so that it is impossible to compress the spring to less than half its relaxed length. When the spring is stretched more than about twice its relaxed length, it begins to weaken, and the restoring force becomes zero when it is stretched to very great lengths.

(a) Devise a force function $F(x)$ that represents this behavior.

(b) Find the potential energy as a function of position, $U(x)$. Describe the types of motion that may occur (bounded, unbounded, are there stable and/or unstable equilibria, etc.).

To make (b) easy, you should choose a force function in (a) that is as easy to integrate as possible.

8. (196a) Consider a massive body of arbitrary shape and a spherical surface completely outside the body. Show that the average value of the gravitational potential on the sphere due to the mass is equal to the value of the potential that would exist on the surface of the sphere if all the mass of the body were concentrated at the center of the sphere. Hint: 1) One way of doing this problem requires essentially no calculation, just a clever argument. 2) Another way of doing this problem involves a technique similar to that used for proving Newton's Iron Sphere Theorem.