

Physics 106a/196a – Problem Set 7 – Due Dec 1, 2006

This problem set focuses on both coupled oscillations and central force motion and scattering, Sections 3.2 and 4 of the lecture notes. Note that the set is 50% longer than usual because it covers 2 weeks' worth of material and you have (almost) 2 weeks to do it. Problems 1 and 2 are for 106a students only, Problems 3 through 6 are for 106a and 196a students, and Problems 7 and 8 for 196a students only. **More hints may be forthcoming next week, keep an eye on the web page.**

1. (106a) Hand and Finch 1-22, except this time do it in terms of normal modes – find the normal mode frequencies and normal mode vectors (appropriately normalized). You don't need to rederive the Lagrangian, you may use the Lagrangian provided in the solutions to Problem Set 3 (#3), but you will have to do the Taylor expansion to find the \mathbf{t} and \mathbf{v} matrices. Provide a physical interpretation of the two modes. Why does one mode have zero frequency?
2. (106a) Hand and Finch 9-9. The fact that the spring mass m_s is small is only intended to imply that the spring does not sag under its own weight – you need not make any other approximations using $m_s \ll m$.
3. (106a/196a) A satellite is to be launched from the surface of the earth. Assume the earth is a sphere of radius R and neglect friction with the atmosphere. The satellite is to be launched at an angle α with the vertical, with a velocity v_0 , so as to coast without power until its velocity is horizontal at an altitude h_1 above the earth's surface. A horizontal thrust is then applied by the last stage rocket so as to add an additional velocity Δv_1 to the velocity of the satellite. The final orbit is to be an ellipse with perigee h_1 (point of closest approach) and apogee h_2 (point farthest away) as measured from the earth's surface. Find the required initial velocity v_0 and additional velocity Δv_1 in terms of R , α , h_1 , h_2 , and g , the acceleration of gravity at the earth's surface. (Of course, you must not assume the acceleration of gravity is independent of distance from the earth's surface – g is only used to obviate G and M_{earth} in the result.) Hint: Use conservation of energy and conservation of angular momentum when you know they are conserved, and use the fact that you know exactly how they change when the thrust is applied.
4. (106a/196a) Hand and Finch 9-2, triple pendulum.
5. (106a/196a) It can be shown that the orbit given by the special theory of relativity for a particle of mass m moving under a potential energy $V(r)$ is the same as the orbit which the particle would follow according to Newtonian mechanics if the potential energy were

$$U(r) = V(r) - \frac{[E - V(r)]^2}{2m c^2}$$

where E is the total energy (kinetic plus potential) and c is the speed of light. Show by comparing the orbital angular velocity with the frequency of radial oscillation for nearly circular motion that the nearly circular orbits, when the relativistic correction is small, are

precessing ellipses, and calculate the angular velocity of precession. See the attached solution to Hand and Finch 4-24 for something of a hint (but you can't just apply it blindly!). You must be careful to carry your Taylor expansions to the appropriate number of terms to avoid a trivial result. This is a nontrivial problem.

6. (106a/196a) Show that in a scattering situation between particles of unit charge (in Gaussian units, $V(r) = q/r$, $q = 1$ in this case), if the incident particle is much heavier than the target particle ($m_1 \gg m_2$), the Rutherford scattering cross section in laboratory coordinates is approximately

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{1}{2m_2v_0^2} \right)^2 \frac{4\gamma^2}{\left[1 - \sqrt{1 - \gamma^2\theta^2}\right]^2 \sqrt{1 - \gamma^2\theta^2}}$$

if $\gamma\theta < 1$, where $\gamma = \frac{m_1}{m_2}$. Otherwise, $d\sigma/d\Omega = 0$.

7. (196a) We saw in the lecture notes (Section 3.2.2, Subsection **The Congruence Transformation...**) that the Hamiltonian for a set of coupled simple harmonic oscillators may be written in the form

$$H = \dot{\vec{\psi}}^T \dot{\vec{\psi}} + \vec{\psi}^T \Omega^2 \vec{\psi}$$

Let us rewrite this using standard symplectic notation. Since $m = 1$ in our usual way of writing the problem, it holds that $p_{\psi_i} = \dot{\psi}_i$. Define as usual $\vec{\xi}$ to be the symplectic coordinate, the $2M$ -dimensional combination of $\vec{\psi}$ and $\vec{p}_{\vec{\psi}} = \dot{\vec{\psi}}$. Do the following:

- (a) Show that the Hamiltonian may be written in the form

$$H = \frac{1}{2} \vec{\xi}^T \mathbf{S} \vec{\xi}$$

where \mathbf{S} is a square $2M \times 2M$ matrix.

- (b) Assuming all modes are nondegenerate, let \mathbf{M} be a square $2M \times 2M$ -dimensional matrix formed by the components of the set of normal mode vectors in the $2M$ -dimensional $\vec{\xi}$ space (like $\mathbf{\Phi}$ is the corresponding matrix in M -dimensional configuration space). Show that the conventional normalization of the normal mode vectors makes \mathbf{M} the Jacobian matrix of a canonical transformation.
- (c) Show that the canonical transformation so found transforms the effective Hamiltonian to the form

$$H = \sum_j i \omega_j q_j p_j \tag{1}$$

What are the equations of motion in this set of canonical coordinates?

- (d) Finally, show that

$$F_2 = \sum_j \left[q_j P_j + \frac{i}{2} \frac{P_j^2}{\omega_j} - \frac{i}{4} \omega_j q_j^2 \right] \tag{2}$$

leads to a canonical transformation that decomposes H into the Hamiltonians for a set of uncoupled linear harmonic oscillators that oscillate in the normal modes.

8. (196a) A mass m of gass and debris surrounds a star of mass M . The radius of the star is negligible in comparison with the distances to the particles of gas and debris. The material surrounding the star has initially a total angular momentum \vec{L} and a total kinetic and potential energy E . Assume that $m \ll M$, so that the gravitational fields due to the mass m are negligible in comparison with that of the star. Due to internal friction, the surrounding material continually loses mechanical energy. Show that there is a maximum energy ΔE which can be lost in this way, and that when this energy has been lost, the material must all lie on a circular ring around the star (but not necessarily uniformly distributed). Find ΔE and the radius of the ring. You will need to use Lagrange multipliers. This problem qualitatively explains why planetary systems and galaxies collapse to a disk of gas and dust rather than retaining the spherical halo of such material that they likely begin with.

Hand and Finch 4-24 Solution

The force is

$$F(r) = \frac{dU(r)}{dr} = -\frac{k}{r^2} + \frac{2\beta}{r^3} \quad (3)$$

so the equation of orbit is

$$\frac{d^2u}{d\varphi^2} + u = -\frac{\mu}{l^2 u^2} F\left(\frac{1}{u}\right) \implies \frac{d^2u}{d\varphi^2} + \left(1 + \frac{2\beta\mu}{l^2}\right)u = \frac{\mu k}{l^2} \quad (4)$$

We will get a similar solution as in the pure r^{-2} force case, except that, we have $\omega^2 = 1 + \frac{2\beta\mu}{l^2} \equiv \alpha^2$, instead of unity. The solution is

$$u = A \cos \alpha\varphi + \frac{\mu k}{l^2 + 2\beta\mu} = A \cos \alpha\varphi + \frac{G\mu^2 M}{l^2 + 2\beta\mu} \quad (5)$$

and the polar equation of the orbit is now

$$\frac{p}{r} = 1 + \epsilon \cos \alpha\varphi \quad (6)$$

Thus, we got precession because r repeats with period $\frac{2\pi}{\alpha} = \frac{2\pi}{\sqrt{1 + \frac{2\beta\mu}{l^2}}}$ in φ , and α is not unity here.

r_{\min} occurs at $\alpha\varphi = 2n\pi \implies \varphi = \frac{2n\pi}{\alpha}$, so the apsides precess an angle $\frac{2\pi}{\alpha} - 2\pi$ per period, and the precession frequency is

$$\dot{\Omega} = \frac{2\pi(\frac{1}{\alpha} - 1)}{\tau} \quad (7)$$

When the additional potential (β term) is small, we have

$$\frac{1}{\alpha} = \frac{1}{\sqrt{1 + \frac{2\beta\mu}{l^2}}} \simeq \frac{1}{1 + \frac{\beta\mu}{l^2}} = 1 - \frac{\beta\mu}{l^2} \quad (8)$$

so

$$\dot{\Omega} = -\frac{2\pi\beta\mu}{l^2\tau} \quad (9)$$

Since the question asks to calculate only the speed of precession, we will not require that the sign have been obtained correctly. Regardless, we may discuss its significance.

Notice the minus sign in the expression. When $\beta < 0$, i.e. the additional potential gives an attraction force, the orbit precesses forward, and *vice versa*. This is a reasonable result, physically. Compare the two orbits with the same apogee and l without and with the β term. If the force is attractive, the particle will have a smaller perigee than in the $\beta = 0$ case, and in general its orbit will be “smaller” than the $\beta = 0$ orbit. Since angular momentum is conserved, the particle must have larger $\dot{\phi}$ in the $\beta < 0$ orbit. If $\dot{\phi}$ is larger, the $\beta < 0$ orbit progresses through 2π more quickly, so by the time the particle has returned to the apogee radius, it has passed 2π , indicating forward precession of the orbit. Similarly, if you analyse the case where $\beta > 0$, i.e. the additional potential gives a repelling force, you will ended up with a backward precession conclusion.