Physics 125a – Problem Set 1 – Due Oct 8, 2007

Version 2 – Oct 5, 2007

These problems cover Shankar 1.1-1.6, up through operators.

v. 2: Clarification: For (1c), the hint in Shankar (Exercise 1.1.1) makes it clear that you need to also show $0|v\rangle = |0\rangle$. Shankar's hint for the proof of $0|v\rangle = |0\rangle$ assumes there exists a multiplicative identity element in the scalar field 1. *This assumption is unnecessary.* While it is certainly consistent with (1c) since the problem asks you to assume there is a 1 element, one can prove $0|v\rangle = |0\rangle$ more generally. A hint on proving $0|v\rangle = |0\rangle$: the definition of $|0\rangle$ has nothing to do with additive inverses; it defines $|0\rangle$ as the *additive identity* element.

Note also that our class definitions assume the uniqueness of additive identities and inverses; these do not need to be assumptions (Shankar does not assume them), but I figured there was no point in forcing you to prove these points since they are generic properties of identity and inverse elements in groups; they are not specific to vector spaces.

Sorry about the confusion; the 1st edition of Shankar, which I had in hand when I was writing the relevant notes and PS, had slightly different axioms and also did not have this misleading hint.

Clarification for (4a): You *may* assume the identity matrix has unit determinant.

1. Using the rules defining a linear vector space, show that

   (a) Scalar addition is commutative $\alpha + \beta = \beta + \alpha$ and associative $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

   (b) Scalar multiplication is associative, $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

   (c) If the field has an element 1 that is the identity for scalar multiplication, then the addition inverse of 1, denoted by -1, satisfies $(-1)|v\rangle = -|v\rangle$; *i.e.*, that -1 multiplying a vector $|v\rangle$ gives its vector addition inverse $-|v\rangle$.

   See Shankar for some hints. Be careful to avoid assuming what you want to prove.

2. Consider our example vector space of real antisymmetric 3×3 matrices from the lecture notes. Let there be a set of three vectors with the first two being the $|1\rangle$ and $|2\rangle$ vectors from the notes. Let there be a third matrix $|C\rangle$, whose elements are not yet determined. First, write down $|C\rangle$ in terms of the fewest possible parameters (remember, it is real and antisymmetric). Next, write down a set of conditions on these parameters to require that $|C\rangle$ be linearly independent of the first two. Then, assuming these conditions are satisfied, use Gram-Schmidt orthogonalization to obtain from $|C\rangle$ a vector orthogonal to $|1\rangle$ and $|2\rangle$. How is the new vector related to the $|3\rangle$ vector given in the lecture notes?

3. (Shankar 1.6.2): Assuming Λ and Ω are Hermitian operators, what can you say about

$$\Omega\Lambda \qquad \Omega\Lambda + \Lambda\Omega \qquad [\Omega, \Lambda] \qquad i\,[\Omega, \Lambda]$$

4. Determinant and trace identity fun:

   (a) (Shankar 1.6.4 and 1.7.2) Take it as given that the determinant of a matrix and its transpose are equal and that the determinant of a product of two matrices is the product of the individual determinants. Show that a unitary matrix has a determinant of unit modulus and that the determinant of any matrix is unaffected by a unitary transformation. (Hint: how does a determinant behave under complex conjugation?)

   (b) (Shankar 1.7.1) The trace of a matrix is defined to be the sum of its diagonal elements:

$$\mathrm{Tr}\,(\Omega) = \sum_i \Omega_{ii}$$

   Show the following:

$$\mathrm{Tr}(\Omega\Lambda) = \mathrm{Tr}(\Lambda\Omega)$$
$$\mathrm{Tr}(\Omega\Lambda\Theta) = \mathrm{Tr}(\Lambda\Theta\Omega) = \mathrm{Tr}(\Theta\Omega\Lambda)$$
$$\mathrm{Tr}(U^\dagger \Omega U) = \mathrm{Tr}(\Omega) \quad \text{where } U \text{ is unitary}$$

   (Recall that matrix multiplication is equivalent to $[\Omega\Lambda]_{ij} = \sum_k \Omega_{ik}\Lambda_{kj}$.)