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Basic Probability
Part 1

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Outline

Introduce basic concepts of probability and some mathematical machinery:

- What is probability?
- Sample spaces and events.
- Axioms of Probability and some corollaries.
- Joint and conditional probabilities.

What is probability?

- We all have an idea of what probability is:
  - the probability it will rain today,
  - the probability I will roll double sixes,
  - the probability the freeway is jammed,
  - the probability global warming is real.
- Probability can be defined as long-run relative frequency.
  - I flip a coin. What is the probability I get a head?
  - Answer: $1/2$ since physics tells me that if I flipped the coin 1000 times I should get about 500 heads; 10,000 times I should get 5000 heads, etc.
- Mathematically, probability is a set function: it assigns a number (between zero and one, in this case) to a set.

Sample spaces and events

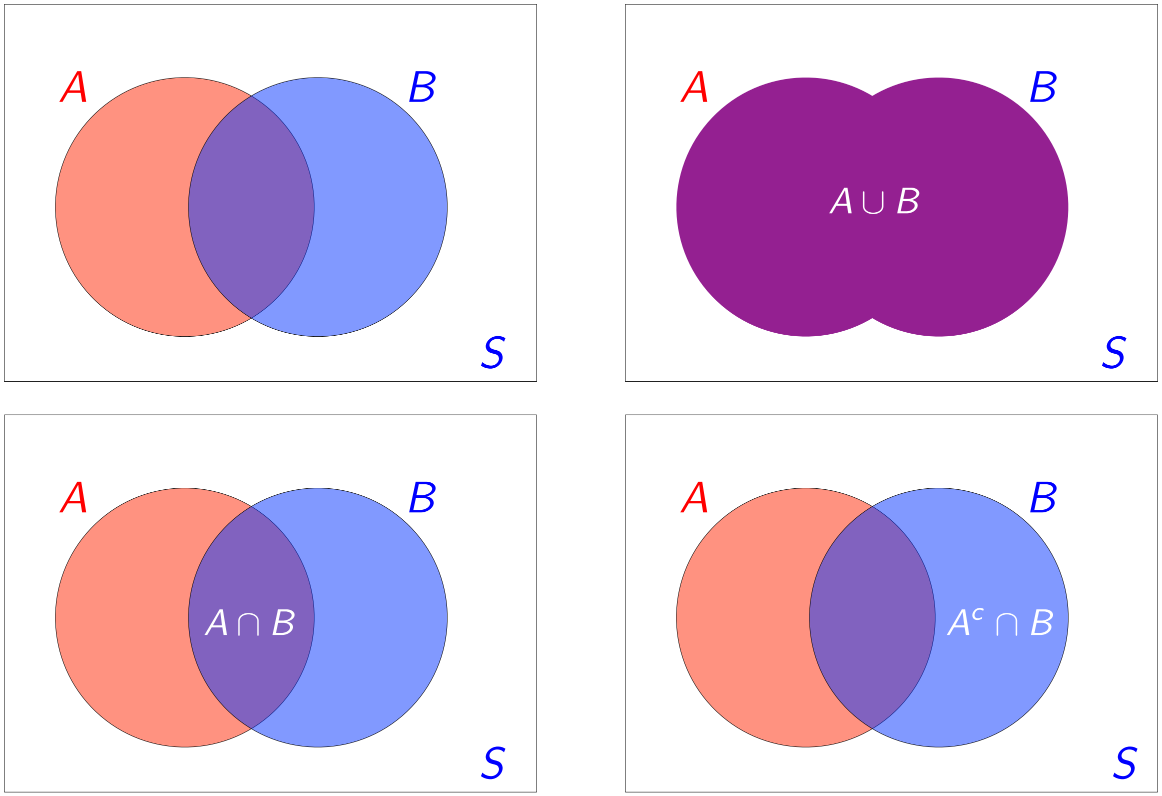
- Define a trial to be a measurement or observation of some phenomenon.

- The set of all possible outcomes of a trial is called the sample space, S.
  - sex of a newborn baby: $S = \{\text{boy}, \text{girl}\}$,
  - minutes to wait to get on the freeway this morning: $S = \{x : 0 \leq x < \infty\}$.

- An event is a subset of the sample space.
  - $E = \{\text{boy}\}$,
  - five minutes or less: $E = \{x : 0 \leq x \leq 5\}$.

Sample spaces and events

Venn diagrams:



$\cap$ = intersection; $\cup$ = union; ^{C} = complement.

Sample spaces and events

Handy "laws" of set operations:

Commutative:	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative:	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive:	$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$	$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

DeMorgan's laws:

$$\left(\bigcup_{i=1}^{n} A_i\right)^C = \bigcap_{i=1}^{n} A_i^C, \qquad \left(\bigcap_{i=1}^{n} A_i\right)^C = \bigcup_{i=1}^{n} A_i^C.$$

Axioms of Probability and some corollaries

Three axioms:

Axiom 1: $0 \leq P(A) \leq 1$.

Axiom 2: $P(S) = 1$.

Axiom 3: $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, for a sequence of mutually exclusive events, $A_1, A_2, \ldots$ ($A_i \cap A_j = \emptyset$ for $i \neq j$).

All other rules of probability can be derived from these axioms.

Axioms of Probability and some corollaries

Useful:

- $P(A^c) = 1 - P(A)$.
- If $A \subset B$ (A is a subset of B), then $P(A) \leq P(B)$.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- More generally,

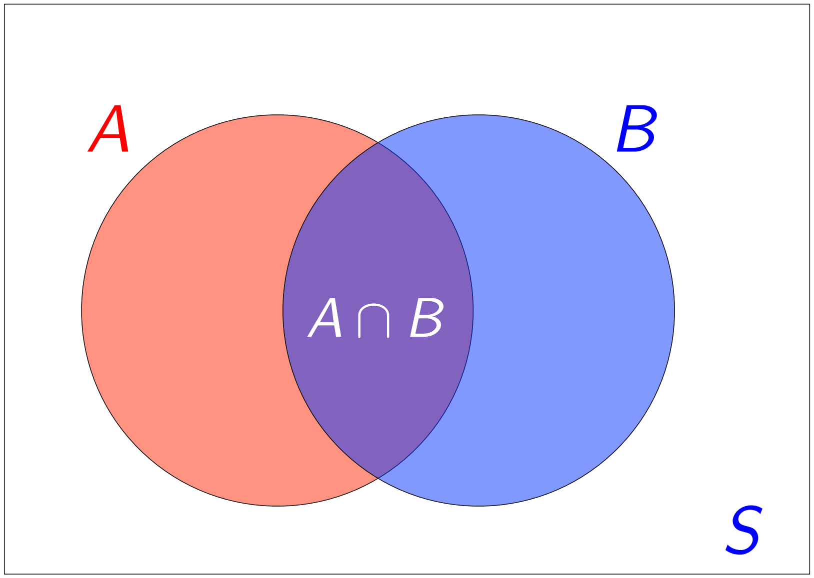
$$
\begin{aligned}
P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i_1 < i_2} P(A_{i_1} \cap A_{i_2}) + \ldots \\
+ (-1)^{r+1} \sum_{i_1 < i_2 \ldots < i_r} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) \\
+ \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cdots \cap A_n),
\end{aligned}
$$

where the summation on the second line is over all possible subsets of size r from the set of size n.

Joint and conditional probabilities

- The joint probability of events A and B is $P(A \cap B)$.
- The conditional probability of A given that event B occurs is,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

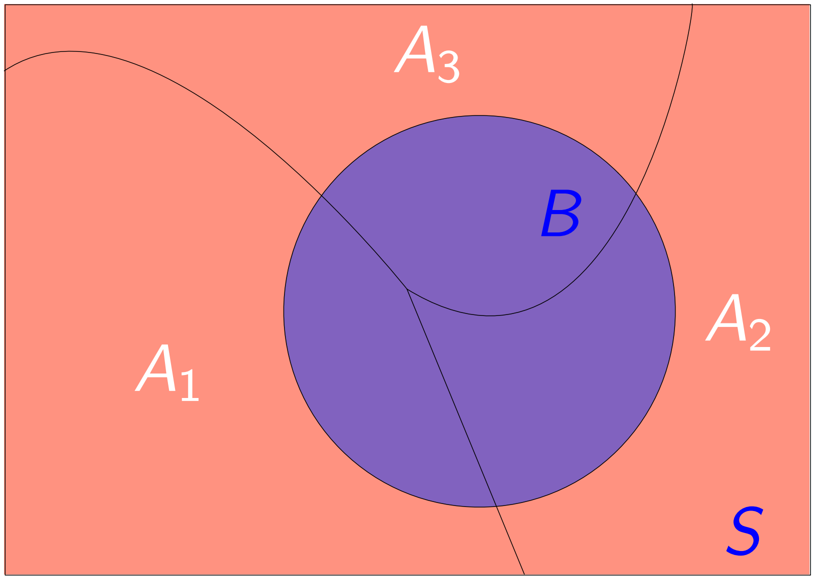


- The unconditional probability of A is $(P(A \cap S)/P(S))$.
- Definition of independence: $P(A \cap B) = P(A)P(B)$.
- Another equivalent definition of independence: $P(A|B) = P(A)$.

Joint and conditional probabilities

Law of Total Probability:

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B|A_i)P(A_i).$$



Joint and conditional probabilities

Bayes' Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}.$$

Bayes' Rule is especially useful because it allows us to express $P(A|B)$ in terms of $P(B|A)$, if we know something about the latter but not the former.

Example: Let A be the event that true CO2 concentration is greater than 400 ppm. Let B be the event that OCO-2 observes 398 ppm.

Bibliography

- *A First Course in Probability* by Sheldon Ross, Prentice Hall, 2010.
- *An Introduction to Probability Theory and Its Applications* Volumes 1 and 2, by William Feller, John Wiley and Sons, 1957.

Next

In the next module, we will discuss how these rules translate for settings in which we model numerical phenomena. In other words: data.