Class #4: Elements of Stellar Dynamics

Structure and Dynamics of Galaxies, Ay 124, Winter 2009

January 20, 2009

Material in this class is taken mostly from Binney & Tremaine §4.1–4.3.

Most of Ay 124 is focussed on observational properties of galaxies (which is why Binney & Merrifield is the main text). The underlying theory of stellar/galactic dynamics requires an entire class (or more. . .) itself, so we can only explore a few aspects of it here. You're encouraged to read more of Binney & Tremaine (and try the problems in that book) to get a more extensive understanding of galactic dynamics.

The basic idea of stellar dynamics is to determine how systems of stars (a.k.a. galaxies) evolve under the effects of their own self-gravity (and, perhaps, some other source of gravitational potential, such as dark matter). We'll make two approximations[1]: 1) we'll assume that stars are collisionless (more of this in a minute) and 2) we'll assume that diffuse (i.e. not in stars) gas is negligible, so we can ignore hydrodynamics.

In many cases, we'll apply stellar dynamics to systems which are a) in equilbrium and b) are at least partially self-gravitating. Since any such equilbrium must be a dynamic equilbrium (i.e. the individual stars are always moving along their orbits even though the overall structure of the stellar system is unchanging) the main goal of stellar dynamics in such cases is to determine a self-consistent distribution of stars in phase space which maintains the equilibrium when evolved gravitationally.

1 Collisionless Boltzmann Equation [CBE]

Our first goal is to find an equation which describes the evolution of a stellar system as described above. This equation is known as the "Collisionless Boltzmann Equation". To see why it's "collisionless" we need to consider relaxation times in stellar systems.

[1] Actually more than two, but these are the two important ones.

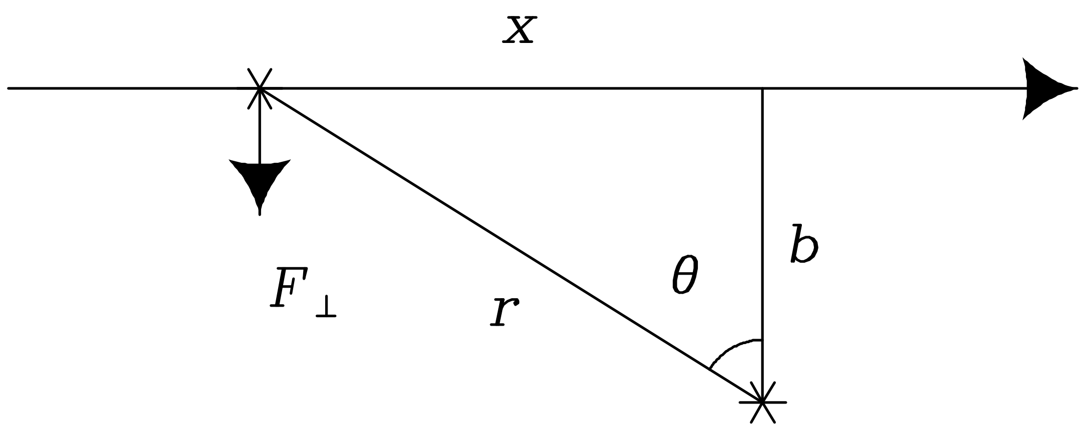

Figure 1: Geometry of a gravitational collision between two stars.

1.1 Relaxation Time

We want an order-of-magnitude estimate how long it takes for encounters with other stars to significantly change the energy of a star. Consider an encounter between two stars. Assuming the collision is a small perturbation to the motion of the star we can approximate the force experienced by the star as:

$$\dot{\mathbf{v}}_\perp = \frac{\mathrm{G}m}{b^2+x^2}\cos\theta = \frac{\mathrm{G}mb}{(b^2+x^2)^{3/2}} \approx \frac{\mathrm{G}m}{b^2}\left[1+\left(\frac{vt}{b}\right)^2\right]^{-3/2}. \tag{1}$$

Integrating over the entire collision gives us the change in velocity of the star:

$$\mathbf{v}_\perp \approx \frac{\mathrm{G}m}{bv}\int_{-\infty}^{\infty}(1+s^2)^{-3/2}\mathrm{d}s = \frac{2\mathrm{G}m}{bv}, \tag{2}$$

whcih is approximately the force at closest approach times the duration of the encounter, b/v. The surface density of stars in a galaxy is of order $N/\pi R^2$, so in crossing the galaxy once, the star experiences

$$\delta n = \frac{N}{\pi R^2}2\pi b\mathrm{d}b = \frac{2N}{R^2}b\mathrm{d}b, \tag{3}$$

encounters with impact parameter between b and $b+\mathrm{d}b$. The encounters cause randomly oriented changes in velocity, so $\overline{\delta\mathbf{v}_\perp} = 0$, but there can be a net change in $v_\perp^2$:

$$\delta v_\perp^2 \approx \left(\frac{2\mathrm{G}m}{bv}\right)^2 \frac{2N}{R^2}d\mathrm{d}b. \tag{4}$$

Our perturbation approach breaks down if $\delta v_\perp \sim v_\perp$ which occurs is $b \lesssim b_{\min} = \mathrm{G}m/v^{2}$[2], so integrating over all impact parameters from $b_{\min}$ to R (the largest possible impact parameter):

$$\Delta v_\perp^2 = \int_{b_{\min}}^{R} \delta v_\perp^2 \approx 8N \left(\frac{\mathrm{G}m}{Rv}\right)^2 \ln\Lambda, \tag{5}$$

where $\Lambda = R/b_{\min}$ ("Coulomb logarithm"). The typical speed of a star in a self-gravitating galaxy is

$$v^2 \approx \frac{\mathrm{G}Nm}{R}. \tag{6}$$

Therefore, we find

$$\frac{\Delta v_\perp^2}{v^2} = \frac{8\ln\Lambda}{N}, \tag{7}$$

and the number of crossings required for order unity change in velocity is:

$$n_{\text{relax}} = \frac{N}{8\ln\Lambda}. \tag{8}$$

Since $\Lambda = R/b_{\min} \approx Rv^2/\mathrm{G}m \approx N$ we find $t_{\text{relax}} \approx [0.1N/\ln N]t_{\text{cross}}$.

Relaxation is important for systems up to globular cluster scales, but is entirely negligible for galaxies.

System	N	$t_{\text{relax}}/t_{\text{cross}}$	t_{relax}
Small stellar group	50	1.3	
Globular cluster	10^5	870	10^8yr
Galaxy	10^{11}	4×10^8	4×10^7Gyr

1.2 Deriving the CBE

We consider a large collection of stars moving under the influence of a smooth potential $\Phi(\mathbf{x}, t)$ (we can assume it to be smooth because of the collisionless approximation). The state of the system is fully specified by the *distribution function* $f(\mathbf{x}, \mathbf{v}, t)$ (a.k.a *phase space density*) which is defined such that $f(\mathbf{x}, \mathbf{v}, t)\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}$ gives the number of stars in a small volume. Given $f(\mathbf{x}, \mathbf{v}, t_0)$ we can compute the distribution at any later time from Newton's laws. It's convenient to write

$$(\mathbf{x}, \mathbf{v}) \equiv \mathbf{w} \equiv (w_1, \ldots, w_6), \tag{9}$$

which implies a 6-d flow velocity

$$\dot{\mathbf{w}} = (\dot{\mathbf{x}}, \dot{\mathbf{v}}) = (\dot{v}, -\boldsymbol{\nabla}\Phi). \tag{10}$$

Since stars don't appear or vanish (at least, not on the timescales we care about) this flow must conserve stars. This allows us to write a continuity equation for $f(\mathbf{w}, t)$:

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial(f\dot{w}_\alpha}{\partial\alpha} = 0. \tag{11}$$

[2]For a more careful treatment of small b encounters, take a look at the treatment of dynamical friction, §7.1 of Binney & Tremaine.

This is just like the usual mass continuity equation, but in six dimensions—it simply says that the change in phase space density in any volume is equal to the sum of net fluxes of density over the boundaries of that volume.

We can write

$$\sum_{\alpha=1}^{6}\frac{\partial\dot{w}_\alpha}{\partial w_\alpha}=\sum_{i=1}^{3}\left(\frac{\partial v_i}{\partial x_i}+\frac{\partial\dot{v}_i}{\partial v_i}\right)=\sum_{i=1}^{3}-\frac{\partial}{\partial v_i}\left(\frac{\partial\Phi}{\partial x_i}\right)=0, \tag{12}$$

where we've used the fact that $(\partial v_i/\partial x_i)=0$ because v_i and x_i are independent coordinates and $\boldsymbol{\nabla}\Phi$ does not depend on velocities. Using eqn. (12) to simplify eqn. (11) we get the collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t}+\sum_{\alpha=1}^{6}\dot{w}_\alpha\frac{\partial f}{\partial w_\alpha}=0, \tag{13}$$

i.e.

$$\frac{\partial f}{\partial t}+\sum_{i=1}^{3}\left(\dot{v}_i\frac{\partial f}{\partial x_i}+\partial\Phi\partial x_i\frac{\partial f}{\partial v_i}\right)=0. \tag{14}$$

or

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}f-\boldsymbol{\nabla}\Phi\cdot\frac{\partial f}{\partial\mathbf{v}}=0 \tag{15}$$

This importance of this is clear if we look at the Langrangian derivative which we define to be:

$$\frac{\mathrm{d}f}{\mathrm{d}t}\equiv\frac{\partial f}{\partial t}+\sum_{\alpha=1}^{6}\dot{w}_\alpha\frac{\partial f}{\partial w_\alpha}. \tag{16}$$

The collisionless Boltzmann equation then implies:

$$\frac{\mathrm{d}f}{\mathrm{d}t}=0, \tag{17}$$

i.e. the flow of stars through phase space is incompressible—the phase space density around a given star always remains the same.

Collisions would invalidate this equation and lead to an additional term appearing on the right side of these equations. We'll come back to this issue when we study the Fokker-Planck equation. It's worth knowing that the phase space density and the coordinate density are the same in any canonical coordinate system (not just the Cartesian coordinates we've considered here—see p. 193 of Binney & Tremaine for more discussion of this point).

1.2.1 CBE in Arbitrary Coordinates

We can use the fact that f is constant along the trajectories of stellar phase points to derive the collisionless Boltzmann equation in other coordinates. For example, in cylindrical coordinates:

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+\dot{R}\frac{\partial f}{\partial R}+\dot{\phi}\frac{\partial f}{\partial\phi}+\dot{z}\frac{\partial f}{\partial z}+\dot{v_R}\frac{\partial f}{\partial v_R}+\dot{v}_\phi\frac{\partial f}{\partial v_\phi}+\dot{v}_z\frac{\partial f}{\partial v_z}=0 \tag{18}$$

which, using,

$$\dot{v}_R=-\frac{\partial\Phi}{\partial R}+\frac{v_\phi^2}{R};\dot{v}_\phi=-\frac{1}{R}\frac{\partial\Phi}{\partial\phi};\dot{v}_z=-\frac{\partial\Phi}{\partial z}, \tag{19}$$

simplifies to

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + \frac{v_\phi}{R}\frac{\partial f}{\partial \phi} + v_z \frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial \Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \frac{1}{R}\left(v_R v_\phi + \frac{\partial \Phi}{\partial \phi}\right)\frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z} = 0. \quad (20)$$

A similar procedure works in any coordinate system

1.3 Coarse-grained Distribution Function

Obviously to define a density of discrete objects such as stars we have to average over sufficiently large volumes that contain many stars. It's therefore useful to interpret f as a probability density: i.e. if there is some probability that a region of phase space D_0 contains a star at time $t = 0$ then at a later time t the probability is the same for the region of phase space D_t to which D_0 has been propagated by Newton's laws. The phase space density therefore plays a similar role as the wave function in quantum mechanics.

As an example, suppose that f_{M} is the distribution function of M-dwarf stars. We can calculate the *probability* that an M-dwarf lies within 1pc of the Sun using:

$$\begin{aligned} P &= \int \mathrm{d}^3\mathbf{v} \int_{|\mathbf{x}-\mathbf{x}_\odot|<1\mathrm{pc}} f_M(\mathbf{x},\mathbf{v})\mathrm{d}^3\mathbf{x} \\ &= \int \mathrm{d}^3\mathbf{v} \int Q_1(\mathbf{x},\mathbf{v}) f_M(\mathbf{x},\mathbf{v})\mathrm{d}^3\mathbf{x} = \langle Q_1 \rangle. \end{aligned} \quad (21)$$

We sometimes define a *coarse-grained distribution function* $\bar{f}$ by averging f over some small volume of phase space. It's important to note that $\bar{f}$ *does not* satisfy the collisionless Boltzmann equation.

2 The Jeans Equation

Since the Boltzmann equation is a function of seven variables its difficult to work with. We can simplify it by taking moments of this equation. For example, integrating over all velocities gives (using summation convention):

$$\int \frac{\partial f}{\partial t}\mathrm{d}^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i}\mathrm{d}^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i}\mathrm{d}^3\mathbf{v} = 0 \quad (22)$$

The range of velocities over which we integrate is independent of time so we can move the $\partial/\partial f$ outside of the integral. Similarly, since v_i does not depend on x_i we can take $\partial/\partial x_i$ outside of the integral. Also, the final term vanishes if we apply the divergence theorem and use the fact that $f(\mathbf{x},\mathbf{v},t) = 0$ for sufficiently large v. So, if we define

$$\nu \equiv \int f \mathrm{d}^3\mathbf{v}; \bar{v}_i \equiv \frac{1}{\nu}\int f v_i \mathrm{d}^3\mathbf{v} \quad (23)$$

then we find:

$$\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \bar{v}_i)}{\partial x_i} = 0, \quad (24)$$

which has the form of a continuity equation. Multiplying the collisionless Boltzmann equation instead by v_j and integrating over all velocities gives

$$\frac{\partial}{\partial t}\int f v_j \mathrm{d}^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i}\mathrm{d}^3\mathbf{v} = 0. \tag{25}$$

The last term can be transformed by applying the divergence theorem and using the fact that f vanishes for large v:

$$\int v_j \frac{\partial f}{\partial v_i}\mathrm{d}^3\mathbf{v} = \int \frac{\partial (v_j f)}{\partial v_i}\mathrm{d}^3\mathbf{v} - \int \frac{\partial v_j}{\partial v_i}\mathrm{d}^3\mathbf{v} = -\int \frac{\partial v_j}{\partial v_i}\mathrm{d}^3\mathbf{v} = -\int \delta_{ij} f \mathrm{d}^3\mathbf{v} = -\delta_{ij}\nu. \tag{26}$$

Equation (25) can then be rewritten as

$$\frac{\partial(\nu \bar{v}_j}{\partial t} + \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} + \nu\frac{\partial \Phi}{\partial x_j} = 0, \tag{27}$$

where

$$\overline{v_i v_j} = \frac{1}{\nu}\int v_i v_j f \mathrm{d}^3\mathbf{v}. \tag{28}$$

Subtracting $\bar{v}_j$ times the continuity equation leaves

$$\nu\frac{\partial \bar{v}_j}{\partial t} - \bar{v}_j\frac{\partial(\nu \bar{v}_i)}{\partial x_i} + \frac{\partial(\nu v_i \bar{v}_j)}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j}. \tag{29}$$

The, breaking $v_i v_j$ into the streaming motion part, $\bar{v}_i\bar{v}_j$, and the random motion part

$$\sigma_{ij}^2 = \overline{(v_i - \bar{v}_i)(v_j - \bar{v}_j)} = \overline{v_i v_j} - \bar{v}_i\bar{v}_j. \tag{30}$$

Using this in eqn. (29) gives the analogue of Euler's equation:

$$\nu\frac{\partial \bar{v}_j}{\partial t} + \nu\bar{v}_i\frac{\partial \bar{v}_j}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j} - \frac{\partial(\nu\sigma_{ij}^2)}{\partial x_i}. \tag{31}$$

The left side and first term on the right are analagous to the normal Euler equation of fluid flow. The final term on the right is similar to the pressure term $-\boldsymbol{\nabla} p$—more precisely, $-\nu\sigma_{ij}^2$ is a *stress tensor* which describes an anisotropic pressure. These equations are collectively known as Jeans equations. The stress tensor is symmetric so we can, at any point, transform to a coordinate system where the tensor is diagonal. The ellipsoid with axes aligned with this coordinate system and with principle axes σ_{11}, σ_{22} and σ_{33} is called the *velocity ellipsoid.*

While this equation is useful for connecting to observable properties (e.g. streaming velocities) it has a severe problem: we have no analogue to the equation of state in a fluid system to relate the six components of the stress tensor, $\boldsymbol{\sigma}^2$, to the density ν. While we could use higher moments of the Boltzmann equation to find expressions for the components of this tensor, they would depend on even higher moments. This process would continue *ad infinitum.* Therefore, we have to make some physical assumption about the form of $\boldsymbol{\sigma}^2$.

The Jeans equations in other coordinate systems can be obtained by taking moments of the Boltzmann equation in those coordinate systems. For example, in cylindrical coordinates:

$$\frac{\partial \nu}{\partial t} + \frac{1}{R}\frac{\partial(R\nu\bar{v}_R)}{\partial R} + \frac{\partial(\nu\bar{v}_z)}{\partial z} = 0, \tag{32}$$

$$\frac{\partial(\nu\bar{v}_R)}{\partial t}+\frac{\partial(\nu\overline{v_R^2})}{\partial R}+\frac{\partial(\nu\overline{v_R v_z})}{\partial z}+\nu\left(\frac{\overline{v_R^2}-\overline{v_\phi^2}}{R}+\frac{\partial\Phi}{\partial R}\right)=0, \tag{33}$$

$$\frac{\partial(\nu\bar{v}_\phi)}{\partial t}+\frac{\partial(\nu\overline{v_R v_\phi})}{\partial R}+\frac{\partial(\nu\overline{v_\phi v_z})}{\partial z}+\frac{2\nu}{R}\overline{v_\phi v_R}=0. \tag{34}$$

and

$$\frac{\partial(\nu\bar{v}_z)}{\partial t}+\frac{\partial(\nu\overline{v_R v_z})}{\partial R}+\frac{\partial(\nu\overline{v_z^2})}{\partial z}+\frac{\nu\overline{v_\phi v_R}}{R}+\nu\frac{\partial\Phi}{\partial z}=0. \tag{35}$$

In spherical coordinates we similarly have:

$$\frac{\mathrm{d}(\nu\overline{v_r^2})}{\mathrm{d}r}+\frac{\nu}{r}\left[2\overline{v_r^2}-\left(\overline{v_\theta^2}+\overline{v_\phi^2}\right)\right]=-\nu\frac{\mathrm{d}\Phi}{\mathrm{d}r}. \tag{36}$$

2.1 Application: Asymmetric Drift

It is observed that stellar populations in our Galaxy which have a large mean square radial velocity rotate around Galactic Center more slowly than the local standard of rest (LSR). We can explain this phenomenon.

Define the *asymmetric drift*, v_a, as the difference between the LSR and the mean rotational velocity of the population. Taking the Jeans equations in cylindrical coordinates, and evaluating them at $z=0$ (since the Sun is close to the midplane) and assume that $\partial\nu/\partial z$ (i.e. ignore the vertical gradient in the disk) then:

$$\frac{R}{\nu}\frac{\partial(\nu\overline{v_R^2}}{\partial R}+R\frac{\partial(\overline{v_R v_z})}{\partial z}+\overline{v_R^2}-\overline{v_\phi^2}+R\frac{\partial\Phi}{\partial R}=0;(z=0) \tag{37}$$

Defining an *azimuthal velocity dispersion*:

$$\sigma_\phi^2=\overline{(v_\phi-\bar{v}_\phi)^2}=\overline{v_\phi^2}-\bar{v}_\phi^2 \tag{38}$$

and using $R(\partial\Phi/\partial R)=v_{\mathrm{c}}^2$, where v_{c} is the circular velocity, leaves us with

$$\begin{aligned}\sigma_\phi^2-\overline{v_R^2}-\frac{R}{\nu}\frac{\partial(\nu\overline{v_R^2})}{\partial R}-R\frac{\partial(\overline{v_r v_z})}{\partial z} &= v_c^2-\bar{v}_\phi^2\\ &= (v_c-\bar{v}_\phi)(v_c+\bar{v}_\phi)=v_a(2v_c-v_a).\end{aligned} \tag{39}$$

Assuming $v_{\mathrm{a}}\ll 2v_{\mathrm{c}}$ then

$$2v_c v_a\approx\overline{v_R^2}\left[\frac{\sigma_\phi^2}{\overline{v_R^2}}-1-\frac{\partial\ln(\nu\overline{v_R^2})}{\partial\ln R}-\frac{R}{\overline{v_R^2}}\frac{\partial(\overline{v_R v_z})}{\partial z}\right]. \tag{40}$$

Observations of galaxies suggest that $\overline{v_z^2}\propto\nu$ so, assuming the shape of the velocity ellipsoid is constant, $[\partial\ln(\nu\overline{v_{\mathrm{R}}^2})/\partial\ln R]\approx 2(\partial\ln\nu/\partial\ln R)$. The second derivative is more difficult and depends on the alignment of the velocity ellipsoid at points just away from the midplane. Numerical studies show that the ellipsoid behavior is somewhere between remaining aligned with the cylindrical coordinate system and aligning with a spherical coordinate system. In the first case the derivative is zero, in the second we have $\overline{v_{\rm R}v_{\rm z}} \approx (\overline{v_{\rm R}^2} - \overline{v_{\rm z}^2})(z/R)$. Taking a case midway between these two gives:

$$\frac{2v_c v_a}{\overline{v_R^2}} \approx \left[\frac{\sigma_\phi^2}{\overline{v_R^2}} - \frac{3}{2} - 2\frac{\partial \ln \nu}{\partial \ln R} + \frac{1}{2}\frac{\overline{v_z^2}}{\overline{v_R^2}} \pm \frac{1}{2}\left(\frac{\overline{v_z^2}}{\overline{v_R^2}} - 1\right)\right], \tag{41}$$

where the sign ambiguity covers the range of possible behavior for the velocity ellipsoid. Assuming $\nu = \nu_0 \exp(-R/R_{\rm d})$, $\sigma_\phi^2 \approx \overline{v_{\rm z}^2} \approx 0.45\overline{v_{\rm R}^2}$, $R_0/R_{\rm d} = 2.4$ and $v_{\rm c} = 220$km/s, then we find $v_{\rm a} \approx \overline{v_{\rm R}^2}/110$km/s which agrees quite well with observations.

Also read: "Mass Density in the Solar Neighborhood", "Schwarzchild's Velocity Ellipsoid", "Velocity Dispersions in Spherical Systems" and "Spheroidal Components with Isotropic Velocity Dispersion" (B&T, p. 202–211)

3 Virial Equations

We can also take spatial moments of the Boltzmann equation to find equations governing the global properties of the system. Identifying ν with the mass density ρ, multiplying the Boltzmann equation by x_k and integrating over all spatial variables gives:

$$\int x_k \frac{\partial(\rho \mathbf{v_j})}{\partial \mathbf{t}} {\rm d}^3\mathbf{x} = -\int x_k \frac{\partial(\rho\overline{v_i v_j})}{\partial x_i} {\rm d}^3\mathbf{x} - \int \rho x_k \frac{\partial \Phi}{\partial x_k} {\rm d}^3\mathbf{x}. \tag{42}$$

The second term on the right is the potential energy tensor, $\mathbf{W}$ and the first term on the right can be rewritten as (using the divergence theorem and assuming that the density vanishes at infinity):

$$\int x_k \frac{\partial(\rho\overline{v_i v_j})}{\partial x_i} {\rm d}^3\mathbf{x} = -\int \delta_{ki}\rho\overline{v_i v_j} {\rm d}^3\mathbf{x} = -2K_{kj}, \tag{43}$$

where we've defined the *kinetic energy tensor* $\mathbf{K}$ as

$$K_{jk} \equiv \frac{1}{2}\int \rho\overline{v_j v_k} {\rm d}^3\mathbf{x}. \tag{44}$$

We can split this into ordered and random contributions:

$$K_{jk} = T_{jk} + \frac{1}{2}\Pi_{jk}, \tag{45}$$

where

$$T_{jk} \equiv \frac{1}{2}\int \rho\bar{v}_j\bar{v}_k {\rm d}^3\mathbf{x}; \Pi_{jk} \equiv \int \rho\sigma_{jk}^2 {\rm d}^3\mathbf{x}. \tag{46}$$

We can take the time derivative in eqn. (43) outside of the integral (since x_k does not depend on time) and then average over (k, j) and (j, k) components leaving

$$\frac{1}{2}\frac{\rm d}{{\rm d}t}\int \rho(x_k\bar{v}_j + x_j\bar{v}_k){\rm d}^3\mathbf{x} = 2T_{jk} + \Pi_{jk} + W_{jk}. \tag{47}$$

(We've made use of the symmetry of these tensors under exchange of indices.) Defining a *moment of interia tensor* $\mathbf{I}$:

$$I_{jk} \equiv \rho x_j x_k {\rm d}^3\mathbf{x}, \tag{48}$$

we find

$$\frac{1}{2}\frac{\mathrm{d}I_{jk}}{\mathrm{d}t} = \frac{1}{2}\int \frac{\partial \rho}{\partial t} x_j x_k \mathrm{d}^3\mathbf{x}. \tag{49}$$

Using the continuity equation (and divergence theorem) this becomes

$$-\frac{1}{2}\int \frac{\partial(\rho \bar{v}_i)}{\partial t} x_j x_k \mathrm{d}^3\mathbf{x} = \frac{1}{2}\int \rho \bar{v}_i (x_k \delta_{ji} + x_j \delta_{ki}) \mathrm{d}^3\mathbf{x}, \tag{50}$$

which implies

$$\frac{1}{2}\frac{\mathrm{d}I_{jk}}{\mathrm{d}t} = \frac{1}{2}\int \rho (\bar{v}_j x_k + \bar{v}_k x_j) \mathrm{d}^3\mathbf{x}. \tag{51}$$

Combining these results gives us the *tensor virial theorem*:

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2T_{jk} + \Pi_{jk} + W_{jk}. \tag{52}$$

Taking the trace of this equation and assuming a steady state ($\ddot{\mathbf{I}} = 0$) we find the scalar virial theorem:

$$2K + W = 0. \tag{53}$$

Expressed in terms of the system's mass, M, and rms velocity $\langle v^2 \rangle$:

$$\langle v^2 \rangle = \frac{|W|}{M} = \frac{\mathrm{G}M}{r_{\mathrm{g}}}. \tag{54}$$

We can also find:

$$E = K + W = -K = \frac{1}{2}W. \tag{55}$$

3.1 Application: Rotation of Elliptical Galaxies

Using the tensor virial theorem we can infer information about the internal motions of elliptical galaxies from their shapes and rotation speeds. We consider an axisymmetric system which rotates about its symmetry axis and is seen edge on. We can define the x-axis to lie along the line of sight. From the symmetry of this system we have:

$$W_{xx} = W_{yy}; W_{ij} = 0 (i \neq j), \tag{56}$$

and similar relations for $\mathbf{\Pi}$ and $\mathbf{T}$. The only nontrivial, independent virial equations are then:

$$2T_{xx} + \Pi_{xx} + W_{xx} = 0; 2T_{zz} + \Pi_{zz} + W_{zz} = 0. \tag{57}$$

Taking the ratio of these gives:

$$\frac{2T_{xx} + \Pi_{xx}}{2T_{zz} + \Pi_{zz}} = \frac{W_{xx}}{W_{zz}}. \tag{58}$$

If the only streaming motion is rotation about the z-axis, $T_{zz} =$, and

$$2T_{xx} = \frac{1}{2}\int \rho \bar{v}_\phi^2 \mathrm{d}^3_{\mathbf{x}} = \frac{1}{2} M v_0^2, \tag{59}$$

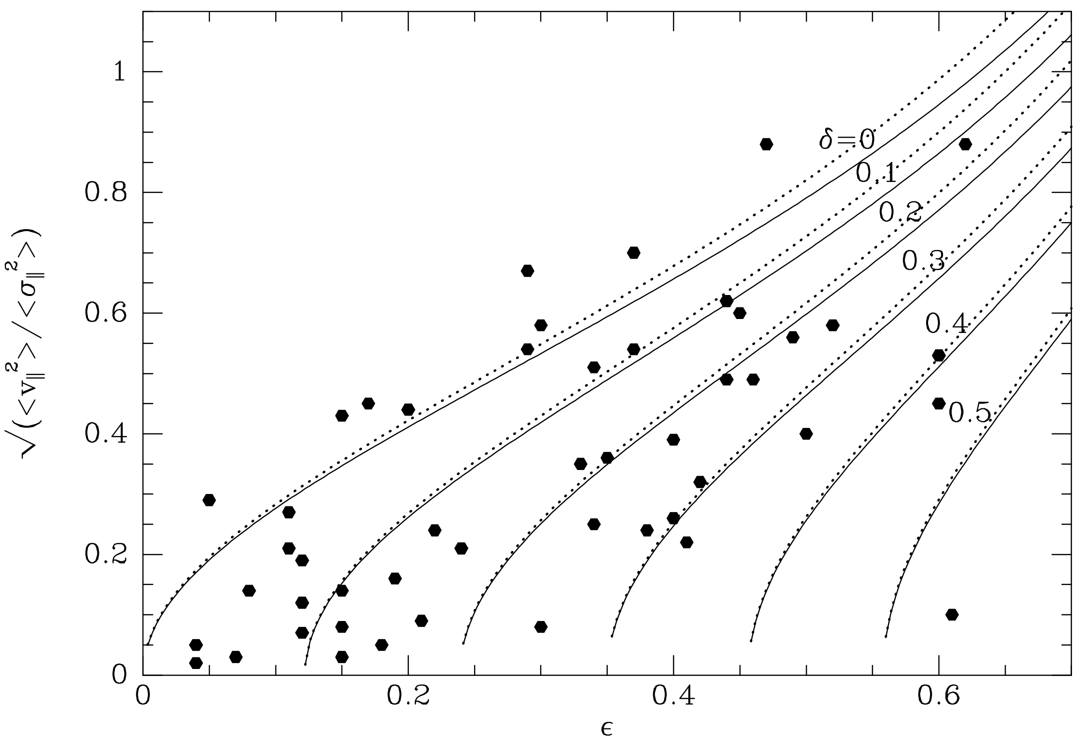

Figure 2: Rotation parameter, v/σ, as a function of ellipticity, ϵ. The second parameter, δ, is constant along each line. Dahsed lines show how points move in this plane as the galaxy is inclined to the line of sight.

where M is the mass of the system and v_0^2 is the mass-weighted mean-square rotation speed. We also have

$$\Pi_{xx} = M\sigma_0^2 \tag{60}$$

where σ_0^2 is the mass-weighted mean-square random velocity along the line of sight to the galaxy, and

$$\Pi_{zz} = (1-\delta)\Pi_{xx} = (1-\delta)M\sigma_0^2, \tag{61}$$

where $\delta < 1$ is a parameter that measures the degree of anisotropy in the galaxy's velocity dispersion tensor. With these definitions we find:

$$\frac{v_0^2}{\sigma_0^2} = 2(1-\delta)W_{xx}/W_{zz} - 2 \tag{62}$$

It can be shown (Binney & Tremaine, §2.5) that for a system whose isodensity surfaces are concentric ellipsoids the ratio W_{xx}/W_{zz} depends only on the ellipticity, ϵ, of those ellipsoids and not on the density profile. The above equation therefore states that the ratio v_0/σ_0 depends only on ϵ and δ.

In practice, we don't see galaxies edge on. At some inclination i we expect the observed rotation speed to be

$$\widetilde{v}\,(i) = v_0 \sin i, \tag{63}$$

while the observed line-of-sight velocity dispersion varies as:

$$\widetilde{\sigma}^2\,(i) = \sigma_0^2 \sin^2 i + (1-\delta)\sigma_0^2 \cos^2 i = \sigma_0^2(1-\delta\cos^2 i). \tag{64}$$

The apparent axial ratio will also vary as the galaxy is inclined:

$$(1-\epsilon_a)^2 = (1-\epsilon_t^2)\sin^2 i + \cos^2 i, \tag{65}$$

and so

$$\epsilon_a(2-\epsilon_a) = \epsilon_t(2-\epsilon_t)\sin^2 i. \tag{66}$$

Observationally, dwarf elliptical galaxies are found to follow the prediction for the $\delta = 0$ line quite well—this suggests that they are oblate spheroidal bodies supported by rotation. Giant elliptical galaxies however do not follow the $\delta = 0$ line, suggesting that they are not rotationally flattened—instead they must have anisotropic velocity dispersion tensors.

Also read: "Mass-to-light" ratios of spherical systems (B&T, p. 214)