Class #6: Star Cluster Dynamics

Structure and Dynamics of Galaxies, Ay 124, Winter 2009

January 16, 2009

Material in this class is taken mostly from Binney & Tremaine Chapter 8.

Previously we derived the collisionless Boltzmann equation. The key assumption of that equation was that we could define a volume of phase space that contained $N \gg 1$ stars, and for which $\boldsymbol{\nabla}\Phi$ was the same for all stars. Eventually this must break down: if stars have close encounters then the accelerations of two stars can differ significantly even though they are very nearby. We previously derived the relaxation time for a system of stars. On timescales longer than this, the assumptions of the collisionless Boltzmann breakdown and stellar encounters become important. In systems such as globular clusters, which are as old as the Universe (more or less), the relaxation time is exceed (significantly).

1 Stellar "Encounters"

Stellar encounters can take numerous forms[1]:

Relaxation Stars perform a random walk away from their initial orbit due to large numbers of relatively weak encounters with other stars. The system evolves towards higher entropy (which for a gravitating system means a dense core with a diffuse halo, not a uniform density as in an ideal gas).

Equipartition From elementary kinetic theory we know that encounters tend to produce equipartition of energy. Consider a system of stars, with differing masses, formed through violent relaxation such that the position and velocity of stars is independent of mass. The more massive stars then have higher energy on average since $K = \frac{1}{2}mv^2$. Massive stars will therefore tend to lose energy to lower mass stars and they must therefore sink towards the center of the system (lower gravitational potential).

Escape Occasionally an encounter will leave a star with enough energy to escape to infinity. This leads to a slow but irreversible leakage of stars from the system—the only stable state immune from this is two stars in a Kepler orbit. Binney & Tremaine give a straightforward derivation of the timescale for such evaporation.

[1] Binney & Tremaine give significantly more discussion of these—you're encouraged to read it.

Inelastic encounters If two stars pass close enough to raise significant tides in each other, or, in extreme cases, to collide head on, kinetic energy can be dissipated—sometimes resulting in the formation of a binary system.

Binary formation in 3-body encounters A binary cannot form in an encounter between two stars (since it's just a Kepler problem with an initially unbound orbit and so the stars just move along hyperbolas), but with a third body nearby can leave two bodies bound with the third ejected with higher energy.

Interaction with primordial binaries Many stars probably form as part of a binary system. Stars encountering a binary can exchange energy with it, either increasing or decreasing the energy of the binary. Energy conservation therefore implies a flow of energy to or from the cluster as a whole. Since binaries are typically very tightly bound (compared to other stars in the system) they can significantly influence the structure of a cluster.

2 Exact Results in Kinetic Theory

If we drop the assumption of collisionless evolution there are still a few exact results that can be obtained (although in practice they're not all that useful...)

2.1 Virial Theorem

The virial theorem still holds for any system of N mutually gravitating particles. Suppose we have N particles with masses m_α and positions $\mathbf{x}^\alpha, \alpha = 1 \ldots N$. The moment of inertia tensor is

$$I_{jk} = \sum_{\alpha=1}^{N} m_\alpha x_j^\alpha x_k^\alpha, \tag{1}$$

and the 2$^{\mathrm{nd}}$ time derivative is

$$\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = \sum_{\alpha=1}^{N} m_\alpha (\ddot{x}_j^\alpha x_k^\alpha + 2\dot{x}_j^\alpha \dot{x}_k^\alpha + x_j^\alpha \ddot{x}_k^\alpha). \tag{2}$$

The acceleration of particle α is

$$\ddot{x}j^\alpha = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \frac{\mathrm{G} m_\beta (x_j^\beta - x_j^\alpha)}{|\mathbf{x}^\beta - \mathbf{x}^\alpha|^3}. \tag{3}$$

This allows us to write the moment of interia as

$$\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2\sum_{\alpha=1}^{N} m_\alpha \dot{x}_j^\alpha x_k^\alpha + \sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{N} \frac{\mathrm{G} m_\alpha m_\beta}{|\mathbf{x}^\beta - \mathbf{x}^\alpha|^3} \left\{ (x_j^\beta - x_j^\alpha) x_k^\alpha + (x_k^\beta - x_k^\alpha) x_j^\alpha \right\}. \tag{4}$$

The first term on the right is $4K_{jk}$ where $\boldsymbol{K}$ is the kinetic energy tensor. The second sum is related to the potential energy tensor $\boldsymbol{W}$:

$$W_{jk} = \mathrm{G}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} m_\alpha m_\beta \frac{x_j^\alpha(x_k^\beta - x_k^\alpha)}{|\mathbf{x}^\beta - \mathbf{x}^\alpha|^3}$$

$$= -\frac{1}{2}\mathrm{G}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} m_\alpha m_\beta \frac{(x_j^\alpha - x_j^\beta)(x_k^\beta - x_k^\alpha)}{|\mathbf{x}^\beta - \mathbf{x}^\alpha|^3}, \tag{5}$$

where we got the second line by switching dummy indices in the first line and adding it to the first line (factor of $\frac{1}{2}$ is inserted to avoid double counting). Putting this all together gives us

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2K_{jk} + W_{jk}. \tag{6}$$

Assuming an equilibrium system (time derivatives go to zero) and taking the trace of this equation will give the usual scalar virial theorem.

2.2 Liouville's Theorem

Given a system of N stars we can describe its state at any time by a single point in a $6N$ dimensionless space known as Γ-space. This fully specifies the positions and velocities of all stars. This state is known as the *microstate*. In practice, we usually don't care about the microstate—instead we care only about more macroscopic properties. Formally, we can imagine specifying some system with known macroscopic properties (density distribution, velocity distribution, etc.) and making many realizations of this macrostate using N particles, each giving a different microstate. This collection of microstates is known as an *ensemble*, and can be described by the density of Γ-points in Γ-space.

If a single particle has position in 6-dimensional phase space $\mathbf{w}_\alpha \equiv (\mathbf{x}_\alpha, \mathbf{v}_\alpha)$ then the Γ-point for a collection of such particle is specified by the N vectors $\mathbf{w}_1, \ldots, \mathbf{w}_N$. We can define a distribution function giving the probability that a Γ-point is found in a unit volume of Γ-space at time t by $f^{(N)}(\mathbf{w}_1, \ldots \mathbf{w}_N, t)$, normalized such the its integral over the $6N$-dimensional Γ-space is unity. The evolution of this N-particle distribution function is governed by continuity equation, since a Γ-point must drift smoothly through Γ-space:

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{\alpha=1}^{N}\left\{\frac{\partial}{\partial \mathbf{x}_\alpha}\left[f^{(N)}\frac{\mathrm{d}\mathbf{x}_\alpha}{\mathrm{d}t}\right] + \frac{\partial}{\partial \mathbf{v}_\alpha}\left[f^{(N)}\frac{\mathrm{d}\mathbf{v}_\alpha}{\mathrm{d}t}\right]\right\} = 0. \tag{7}$$

To simplyify note that $\mathrm{d}\mathbf{x}_\alpha/\mathrm{d}t = \mathbf{v}_\alpha$, $\partial\mathbf{v}_\alpha/\partial\mathbf{x}_\alpha = 0$ (indepdent coordinates) and, for conservative forces, $\frac{\mathrm{d}\mathbf{v}_\alpha/\mathrm{d}t = -\partial\Phi_\alpha}{\partial\mathbf{x}_\alpha}$ so that $\partial(\mathrm{d}\mathbf{v}_\alpha/\mathrm{d}t)/\partial\mathbf{v}_\alpha = 0$. This leaves us with

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{\alpha=1}^{N}\left[\mathbf{v}_\alpha \cdot \frac{\partial f^{(N)}}{\partial \mathbf{x}_\alpha} - \frac{\partial \Phi_\alpha}{\partial \mathbf{x}_\alpha}\cdot\frac{\partial f^{(N)}}{\partial \mathbf{v}_\alpha}\right] = 0, \tag{8}$$

which, in terms of the Lagrangian derivative becomes

$$\frac{\mathrm{d}f^{(N)}}{\mathrm{d}t} = 0. \tag{9}$$

So, flow of Γ-points through Γ-space is incompressible—this is *Liouville's theorem*.

2.3 BBGKY Hierarchy

The Liouville and collisionless Boltzmann equations are related. To see this, integrate the N-particle distribution function over all $\mathbf{w}_\alpha$ except one (it doesn't matter which one, they're all equivalent):

$$f^{(1)}(\mathbf{w}_1,t) = \int f^{(N)}(\mathbf{w}_1,\ldots \mathbf{w}_N,t)\mathrm{d}^6\mathbf{w}_2\ldots\mathrm{d}^6\mathbf{w}_N, \tag{10}$$

then do the same for eqn. (8)

$$\frac{\partial f^{(1)}(\mathbf{w}_1,t)}{\partial t} + \mathbf{v}_1\cdot\frac{\partial f^{(1)}(\mathbf{w}_1,t)}{\partial \mathbf{x}_1} = \int \frac{\partial \Phi_1}{\partial \mathbf{x}_1}\cdot\frac{\partial f^{(N)}}{\partial \mathbf{v}_1}\mathrm{d}^6\mathbf{w}_2\ldots\mathrm{d}^6\mathbf{w}_N, \tag{11}$$

where we've assumed that $f^{(N)}\rightarrow 0$ as $|\mathbf{x}_\alpha|\rightarrow\infty$. Assuming that $f^{(N)}$ is a symmetric function of $\mathbf{w}_1,\ldots,\mathbf{w}_N$ (which is true if all the stars have the same mass—we could solve this in the more general case if we really wanted to) then

$$\Phi_\alpha = \sum_{\beta\neq\alpha}\Phi_{\alpha\beta}, \mathrm{where}\Phi_{\alpha\beta} = -\frac{\mathrm{G}m}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|}. \tag{12}$$

This let's us write

$$\frac{\partial f^{(1)}(\mathbf{w}_1,t)}{\partial t} + \mathbf{v}_1\cdot\frac{\partial f^{(1)}(\mathbf{w}_1,t)}{\partial \mathbf{x}_1} = (N-1)\int \frac{\partial \Phi_{12}}{\partial \mathbf{x}_1}\cdot\frac{\partial f^{(N)}}{\partial \mathbf{v}_1}\mathrm{d}^6\mathbf{w}_2\ldots\mathrm{d}^6\mathbf{w}_N. \tag{13}$$

We can re-write this in terms of the 2-particle distribution function:

$$f^{(2)}(\mathbf{w}_1,\mathbf{w}_2,t) = \int f^{(N)}(\mathbf{w}_1,\ldots,\mathbf{w}_N,t)\mathrm{d}^6\mathbf{w}_3\ldots\mathrm{d}^6\mathbf{w}_N, \tag{14}$$

giving

$$\frac{\partial f^{(1)}}{\partial t} + \mathbf{v}_1\cdot\frac{\partial f^{(1)}}{\partial \mathbf{x}_1} = (N-1)\int \frac{\partial \Phi_{12}}{\partial \mathbf{x}_1}\cdot\frac{\partial f^{(2)}}{\partial \mathbf{v}_1}\mathrm{d}^6\mathbf{w}_2. \tag{15}$$

We could use a similar approach to express the evolution of $f^{(2)}$ in terms of the 3-particle distribution function, $f^{(3)}$ and the evolution of $f^{(n)}$) in terms of $f^{(n+1)}$. This sequence of equations is known as the BBGKY[2] hierarchy. It's no easier to solve than the Liouville equation unless we can truncate the hierarchy by guessing an approximate form for some $f^{(n+1)}$. For example, writing

$$f^{(2)}(\mathbf{w}_1,\mathbf{w}_2,t) = f^{(1)}(\mathbf{w}_1,t)f^{(1)}(\mathbf{w}_2,t) + g(\mathbf{w}_1,\mathbf{w}_2,t), \tag{16}$$

where $g(\mathbf{w}_1,\mathbf{w}_2,t)$ is known as the *two-particle correlation function* and measures the probability in excess of that expected from the 1-particle distribution functions of finding a particle at $\mathbf{w}_1$ due to a particle at $\mathbf{w}_2$, allows us to derive the following (see Binney & Tremaine for details):

$$\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} - \frac{\partial \Phi(\mathbf{x})}{\partial \mathbf{x}}\cdot\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{v}} = -N^2\mathrm{G}m\int\frac{\partial g(\mathbf{x},\mathbf{v},\mathbf{x}_2,\mathbf{v}_2,t)}{\partial \mathbf{v}}\cdot\frac{\partial}{\partial \mathbf{x}_2}\left(\frac{1}{|\mathbf{x}-\mathbf{x}_2|}\right)\mathrm{d}^3\mathbf{x}_2\mathrm{d}^3\mathbf{v}_2, \tag{17}$$

assuming $N\gg 1$. This shows that the collisionless Boltzmann equation follows from assuming that the two-particle correlation function is zero—this makes sense as a zero correlation function implies that individual particles to not influence other particles directly.

[2] Bogoliubov, Born, ,Green, Kirkwood & Yvon.

3 Meet the Fokker-Planck Equation

3.1 Master Equation

With the collisionless Boltzmann equation the Lagrangian derivative of the distribution function is zero. When collisions are important we can therefore write

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \Gamma[f], \tag{18}$$

where $\Gamma[f]$ is a *collision term* which describes the rate of change of f due to encounters. The value of the collision term is a function of $\mathbf{x}$, $\mathbf{v}$ and t and therefore of $f(\mathbf{x}, \mathbf{v}, t)$.

Assume that $\Psi(\mathbf{w}, \Delta\mathbf{w})\mathrm{d}^3\Delta\mathbf{w}\Delta t$ is the probability that a star with coordinates $\mathbf{w}$ is scattered into some new volume of phase space $\mathrm{d}^3\Delta\mathbf{w}$ centered on $\mathbf{w} + \Delta\mathbf{w}$ during a time interval Δt. Ψ (the scattering cross-section) includes the effects of encounters with other stars, but not of the smooth potential. We consider a test star being scattered by a bunch of field stars. Stars are scattered out of a unit volume of phase space at a rate

$$\left.\frac{\partial f(\mathbf{w})}{\partial t}\right|_{-} = -f(\mathbf{w}) \int \Psi(\mathbf{w}, \Delta\mathbf{w})\mathrm{d}^3\Delta\mathbf{w}, \tag{19}$$

and some other stars are scattered into this volume at a rate

$$\left.\frac{\partial f(\mathbf{w})}{\partial t}\right|_{+} = \int \Psi(\mathbf{w} - \Delta\mathbf{w}, \Delta\mathbf{w}) f(\mathbf{w} - \Delta\mathbf{w})\mathrm{d}^3\Delta\mathbf{w}. \tag{20}$$

The sum of these two is the collision term $\Gamma[f]$. This gives us the *master equation*

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \Gamma[f] = \int [\Psi(\mathbf{w} - \Delta\mathbf{w}, \Delta\mathbf{w}) f(\mathbf{w} - \Delta\mathbf{w}) - \Psi(\mathbf{w}, \Delta\mathbf{w}) f(\mathbf{w})]\mathrm{d}^3\Delta\mathbf{w}. \tag{21}$$

3.2 Fokker-Planck Equation

When we derived the relaxation timescale for a stellar system we found that the contribution to the mean square velocity perturbation from each logarithmic interval of impact parameter was the same (hence the appearance of the $\ln\Lambda = \ln R/b_{\min}$ term). Therefore, if $R \gg b_{\min}$ most of the velocity perturbation comes from weak encounters with $\delta v \ll v$. This dominance of weak encounters allows us to find a simplified form for the collision term by expanding in a Taylor series:

$$\begin{aligned} \Psi(\mathbf{w} - \Delta\mathbf{w}, \Delta\mathbf{w}) f(\mathbf{w} - \Delta\mathbf{w}) &= \Psi(\mathbf{w}, \Delta\mathbf{w}) f(\mathbf{w}) - \sum_{i=1}^{6} \Delta w_i \frac{\partial}{\partial w_i}[\Psi(\mathbf{w}, \Delta\mathbf{w}) f(\mathbf{w})] \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{6} \Delta w_i w_j \frac{\partial^2}{\partial w_i \partial w_j}[\Psi(\mathbf{w}, \Delta\mathbf{w}) f(\mathbf{w})] + \mathcal{O}(\Delta\mathbf{v}^3). \end{aligned} \tag{22}$$

Truncating after the second-order terms is known as the *Fokker-Planck approximation.* Doing the integral over $\Delta\mathbf{w}$ then gives us:

$$\Gamma[f] = -\sum_{i=1}^{6} \frac{\partial}{\partial w_i}[f(\mathbf{w}) D(\Delta w_i) + \frac{1}{2} \sum_{i,j=1}^{6} \frac{\partial^2}{\partial w_i \partial w_j}[f(\mathbf{w}) D(\Delta w_i \Delta w_j)], \tag{23}$$

where the *diffusion coefficients* D are define as

$$D(\Delta w_i) \equiv \int \Delta w_i \Psi(\mathbf{w}, \Delta\mathbf{w}) \mathrm{d}^3\Delta\mathbf{w}, \tag{24}$$

and give the expectation for the change in w_i per unit time. (Higher order terms in the Taylor series could be kept but their diffusion coefficients are generally much smaller.) The useful feature of the Fokker-Planck equation is that the dependence on the field star distribution function is entirely within the diffusion coefficients. Once the diffusion coefficients are know the Fokker-Planck equation is a simple differential equation, rather than an integro-differential equation (which are, in general, bad news). Because of this, the Fokker-Planck equation (with a couple of additional approximations) is the main tool to study the slow evolution of stellar systems driven by encounters.

The diffusion coefficients can be evaluated (see Binney & Tremaine, Appendix 8.A):

$$\begin{aligned} D(\Delta v_i) &= 4\pi \mathrm{G}^2 m_a (m + m_a) \ln\Lambda \frac{\partial}{\partial v_i} h(\mathbf{v} \\ D(\Delta v_i \Delta v_j) &= 4\pi \mathrm{G}^2 m_a^2 \ln\Lambda \frac{\partial^2}{\partial v_i \partial v_j} g(\mathbf{v}, \end{aligned} \tag{25}$$

where the *Rosenbluth potentials* are:

$$h(\mathbf{v}) = \int \frac{f_a(\mathbf{v}_a)\mathrm{d}^3\mathbf{v}_a}{|\mathbf{v} - \mathbf{v}_a|}; g(\mathbf{v}) = \int f_a(\mathbf{v}_a)|\mathbf{v} - \mathbf{v}_a|\mathrm{d}^3\mathbf{v}_a, \tag{26}$$

where $f_a(\mathbf{v}_a)$ is the field star distribution function and

$$\Lambda = \frac{b_{\max} v_{\mathrm{typ}}^2}{\mathrm{G}(m + m_a)}. \tag{27}$$

The diffusion coefficients have the same form as Chandrasekhar's expression for dynamical friction—this is a good example of the fluctuation-dissipation theorem at work.

4 Evolution of Spherical Systems

Fokker-Planck codes have been applied extensively to the evolution of globular clusters and can determine (very accurately with modern calculations) the rates of various encounter-driven processes. The timescales for these processes all scale with the median relaxation time for the initial cluster. (Read Binney & Tremaine for a more detailed discussion of these estimates.)

4.1 Evaporation & Ejection

Ejection, in which a star is kicked up to above the escape velocity by a single encounter, turns out to have a much longer timescale than evaporation (a random walk up to escape velocity due to many weak encounters). Estimates of the timescales for these processes are:

$$t_{\mathrm{ej}} = 1.1 \times 10^3 \ln(0.4N) t_{\mathrm{rh}}, \tag{28}$$

and

$$t_{\rm evap} \approx 300 t_{\rm rh}. \tag{29}$$

Tidal fields (such as that from the Galaxy) can enhance the evaporation rate significantly (they effectively lower the potential barrier needed to escape the cluster).

4.2 Core Collapse

Calculations of the evolution of spherical systems (such as globular clusters) all predict a runaway collapse at the center of the cluster such that the central density becomes infinite after finite time. This is known as *core collapse*. Core collapse is a two-stage process. Initially, evaporation causes some stars to gain energy an populate a halo. To conserve energy, the core of the cluster must contract. The second stage is a result of the gravothermal instability (see §8.2 of Binney & Tremaine)—the negative heat capacity of gravitational systems causes the hot core to lose energy to the cooler surrounding halo and thereby grow hotter still, leading to a runaway collapse.

4.3 Equipartition

Since no real star cluster will consists of stars all having the same mass, we should consider equipartition also. Equipartiation leads to lower mass stars having higher velocities than massive stars (on average), making them more prone to evaporation. Therefore, globular cluster tend to lose low mass stars preferentially. Since low mass stars have high mass-to-light ratios, their preferential loss decreases the mass-to-light ratio of the cluster as a whole, and may (partially) explain why globular clusters have lower mass-to-light ratios than comparably old galactic systems (e.g. elliptical galaxies).

4.4 Binary Stars

Binary stars can be separated into two classes, *soft* and *hard* based upon whether the magnitude of their energy is less than or greater than the typical kinetic energy of a field star. We may therefore expect that soft binaries are prone to being broken apart by a passing field star, while hard binaries should not be. This is (more or less) correct, and hard binaries instead tend to transfer energy to the passing star. A good rule of thumb (known as *Heggie's law*) is that "hard binaries get harder and soft binaries get softer" as a result of encounters. The rate of hardening of hard binaries is approximately

$$\langle \dot{E} \rangle = -5.1 \frac{\nu {\rm G}^2 m^3}{\sigma}. \tag{30}$$

To see that this makes sense, consider that the radius within which a field star must pass to significantly affect the binary is $\frac{r \sim {\rm G}M}{\sigma^2}$. The number of such encounters per unit time is therefore $\frac{\dot{N} = \nu \sigma r^2 = \nu {\rm G}^2 M^2}{\sigma^3}$. Finally, the change in energy will be of order the energy of the field star $m\sigma^2$ so $\langle \dot{E} \rangle = \dot{N} m \sigma^2 \sim \nu {\rm G}^2 m^3 / \sigma$. The average energy change per relaxation time is then $\langle \dot{E} \rangle t_{\rm relax} \approx 0.2 m \sigma^2$ which is indepdent of the hardness of the binary (harder binaries have smaller cross-sections for encounters, but the energy changes involved are larger—the two effects cancel).

4.5 Inelastic Encounters

Direct collisions between stars are not too well understood, but may be important in the centers of globular clusters during core collapse. If a collision is able to dissipate enough energy then the stars may actually coallesce. This can lead either to runaway growth, via further coallescence, if the collision timescale is shorter than the nuclear evolution timescale of the (now more massive) star, or to evolution into a neutron star/black hole if the nuclear evolution timescale gets reduced below the collision timescale.

For near-collisions, tidal forces can raise large tides on the stars, which dissipates some of the orbital energy. The effect of this process, repeated each time stars in a binary pass through pericenter, is to gradually circularize and harden the binary. This *tidal capture* process can be a dominant mechanism for forming binaries in globular clusters.

4.5.1 Binaries & Core Collapse

Hard binaries are likely responsible for stopping core collapse by providing a heat source in the cluster center. As core collapse proceeds, more and more binaries are formed through inelastic encounters. These hard binaries continue to get harder through encounters with field stars. This implies that the field stars are gaining energy at the expense of the binaries. This energy, once shared with the cluster as a whole (another equipartition process) acts as a heat source which cools (due to the negative heat capacity) the central core and helps halt the flow of energy to the surrounding halo, thus preventing the gravothermal instability.