Class #10: Dynamics of Disks and Spiral Structure

Structure and Dynamics of Galaxies, Ay 124, Winter 2009

February 5, 2009

Material in this class is taken mostly from Binney & Tremaine (2nd edition) §6.1 & §6.2.

1 Spiral Structure Basics

Spiral arms are a ubiquitous feature of disk galaxies and have therefore attracted many attempts to explain their origins. This turns out not to be too easy: many theories for their origins have come and gone (it was thought for a long time that they arose due to the influence of magnetic fields in the ISM). Bertil Lindblad was the first to realize that spiral arms arise because of interactions between the orbits and gravitational potentials of stars in the disk.

1.1 The Lin-Shu Hypothesis

Lin and Shu suggested that the spiral arms can be thought of in terms of *density waves*: compressions and rarefactions in the distribution of stars. Coupling this with Lindblad's ideas and the hypothesis that spiral patterns are long lived lead to the *Lin-Shu hypothesis* that spiral structure is just a stationary density wave. This allowed theorists to bring all the machinery of wave mechanics to bear on the problem. Unfortunately, Lin & Shu were only half right: spiral patterns are density waves, but they're definately not stationary!

1.2 Geometry

We can characterize spiral structure in terms of rotational symmetry. If $I(R, \phi)$ is the osberved intensity distribution in a disk then if $I(R, \phi + 2\pi/m) = I(R, \phi)$ (i.e. a rotation by $2\pi/m$ radians leaves the galaxy looking the same) then the galaxy (and spiral pattern) is said to have m-fold symmetry and m arms ($m > 0$). Most galaxies have $m = 2$—the predominance of two-armed galaxies is something that any good theory of spiral structure should be able to explain.

1.2.1 Leading and Trailing Arms

A trailing spiral arm has its tip pointing in the direction opposite to galactic rotation, while a leading arm has its tip point in the direction of galactic rotation. Observationally its difficult to figure out which is which because the sense (clockwise or counter-clockwise) of galactic rotation is uncertain unless the inclination of the galaxy is know—we need to know which side of the galaxy is closest to us to figure out which way it is rotating. This can be figured out if we can see, for example, dust obscuration features for example, but is difficult. For the few cases where the sense of rotation is known unambiguously the arms are trailing.

1.2.2 The Winding Problem

The *pitch angle* at some radius R is defined as the angle between a tangent to the spiral arm and the circle R =constant (see Fig. 1). It's useful to define a function describing a mathematical curve which runs along the center of an arm. If we have m arms we can write this as

$$m\phi + f(R,t) = \text{constant (modulo } 2\pi). \tag{1}$$

The function $f(R,t)$ is known as the shape function and allows us to define a radial wavenumber

$$k(R,t) \equiv \frac{\partial f(R,t)}{\partial R}. \tag{2}$$

The sign of k determines whether an arm is leading or trailing: if $m > 0$ (which we'll always assume) and the galaxy rotates in the direction of increasing ϕ then $k > 0$ corresponds to a trailing arm.

The pitch angle is given by

$$\cot\alpha = \left| R\frac{\partial \phi}{\partial R} \right|, \tag{3}$$

with the partial derivative evaluated along the curve and so $\cot\alpha = |kR/m|$. Typical galaxies have $\alpha \approx 10°–15°$. This is a problem: Suppose we begin, at time $t = 0$ with a radial arm, $f(R,t) = 0$, and "tag" the stars in that arm. The disk rotates with angular speed $\Omega(R)$. Since $\Omega(R)$ is a function of radius, the tagged stars do not remain radially aligned. At some later time, our tagged stars will actually lie along

$$\phi(R,t) = \phi_0 + \Omega(R)t, \tag{4}$$

with pitch angle

$$\cot\alpha = Rt\left|\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right|. \tag{5}$$

For a galaxy with a flat rotation curve, $\Omega(R)R = 200$km/s (for example) at $R = 5$kpc and after 10 Gyr the pitch angle would be $\alpha = 0.14°$. This is much smaller than any observed galaxy and is known as the *winding problem*—the material originally making up a spiral arm will be wound up into an ever tighter spiral by the differential rotation of the disk. We may be able to escape this problem by postulating that spiral arms are not long-lived (but that is implausible for grand design spirals) or that they're not material but instead are stationary density waves.

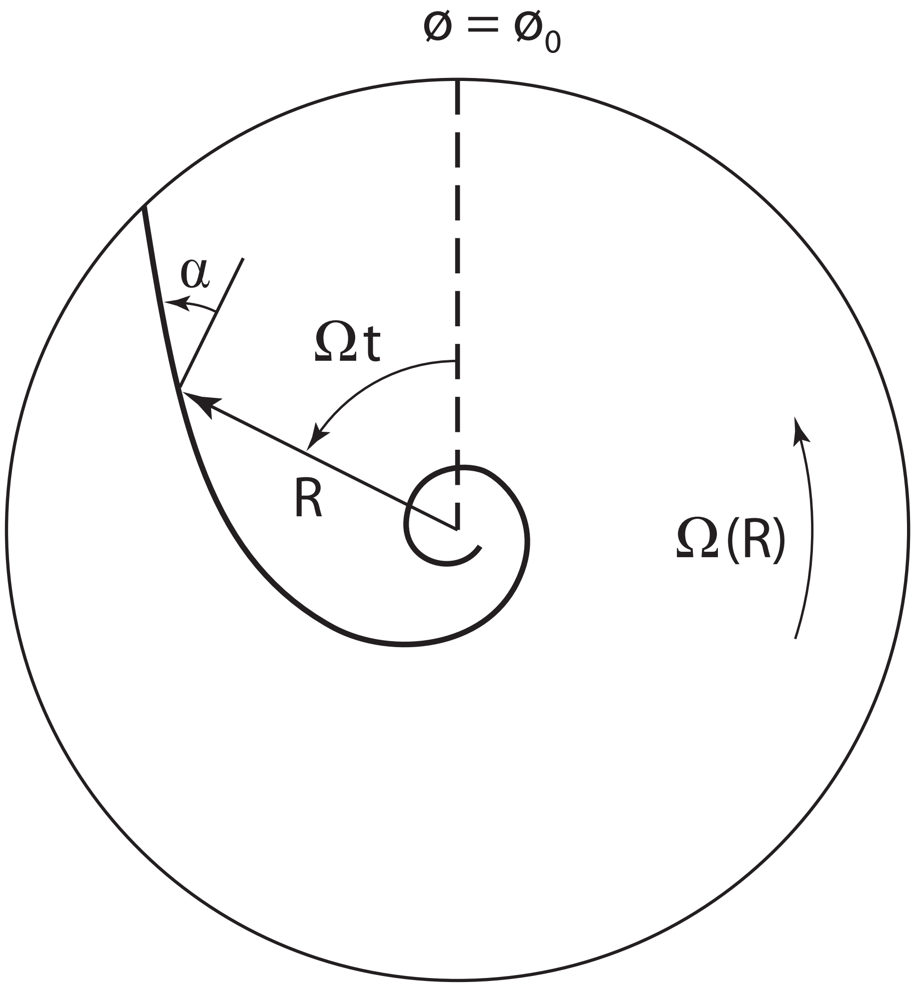


Figure 1: Winding up of a material arm with $\Omega(R) \propto R^{-1}$.

1.2.3 The Pattern Speed

In the Lin-Shu hypothesis, the sprial arms are a density wave pattern that rotates rigidly. We can then always move to a rotating frame with some angular frequency $\Omega_{\rm p}$ in which the pattern remains stationary. This *pattern speed* is not the same as the rotational frequency of the disk. The radius at which $\Omega_{\rm p} = \Omega(R)$ is known as the corotation radius. At smaller radii, $\Omega(R) > \Omega_{\rm p}$. Observationally, dust lanes are seen to lie on the inside of arms as defined by bright stars. If this reflects a time lag between the point of maximum compression of gas and the formation of stars it suggests that gas is flowing into arms from the inside. Since most arms are trailing this implies that the gas is rotating faster than the spiral pattern. Therefore, spiral patterns (at least those in grand design spirals) must be inside corotation.

Measuring the pattern speed is, in general, difficult (its a pattern, not a physical object) and relies on the assumption that there is in fact a well-defined pattern speed.

1.3 The Anti-Spiral Theorem

Since Newton's laws of motion and gravitation are time-reversible we know that for any stationary solution to the equations of motion in a disk which describes a trailing spiral there must be a corresponding solution describing a leading spiral. (To see this, just reverse the velocities of all particles at a given instant—this must correspond to the time-reversed solution and so must be valid.) Given the observed preponderance of trailing arms, this *anti-spiral theorem* implies that spiral patterns cannot be understood simply as steady-state solutions to the collisionless Boltzmann equation and Newtonian gravity. This implies that either i) the solutions are not steady-state (e.g. they could be triggered by a recent disturbance) or, ii) some non-time-reversible physics (e.g. dissipation in the ISM) is involved.

1.4 Angular Momentum Transport by Spiral Arm Torques

2 Wave Mechanics of Differentially Rotating Disks

2.1 Kinematic Density Waves

Any particle orbitting in an axissymmetric galaxy will execute a periodic orbit with some well defined period T_r. During this time, the azimuthal angle will increase by some amount $\Delta\phi$ (not necessarily equal to 2π). The corresponding radial and aziumthal frequencies are $\Omega_r = 2\pi/T_r$ and $\Omega_\phi = \Delta\phi/T_r$. In general $\Delta/2\pi$ will be irrational resulting in a "rosette" type orbit.

We can consider this orbit in a frame which rotates with frequency $\Omega_{\rm p}$. In this frame, the azimuthal position of the particle if $\phi_{\rm p} = \phi - \Omega_{\rm p} t$ and so in one radial period increases by $\Delta\phi_{\rm p} = \Delta\phi - \Omega_{\rm p} T_r$. We can therefore choose $\Omega_{\rm p}$ so that we have a closed orbit in this rotating frame. Specifically, if

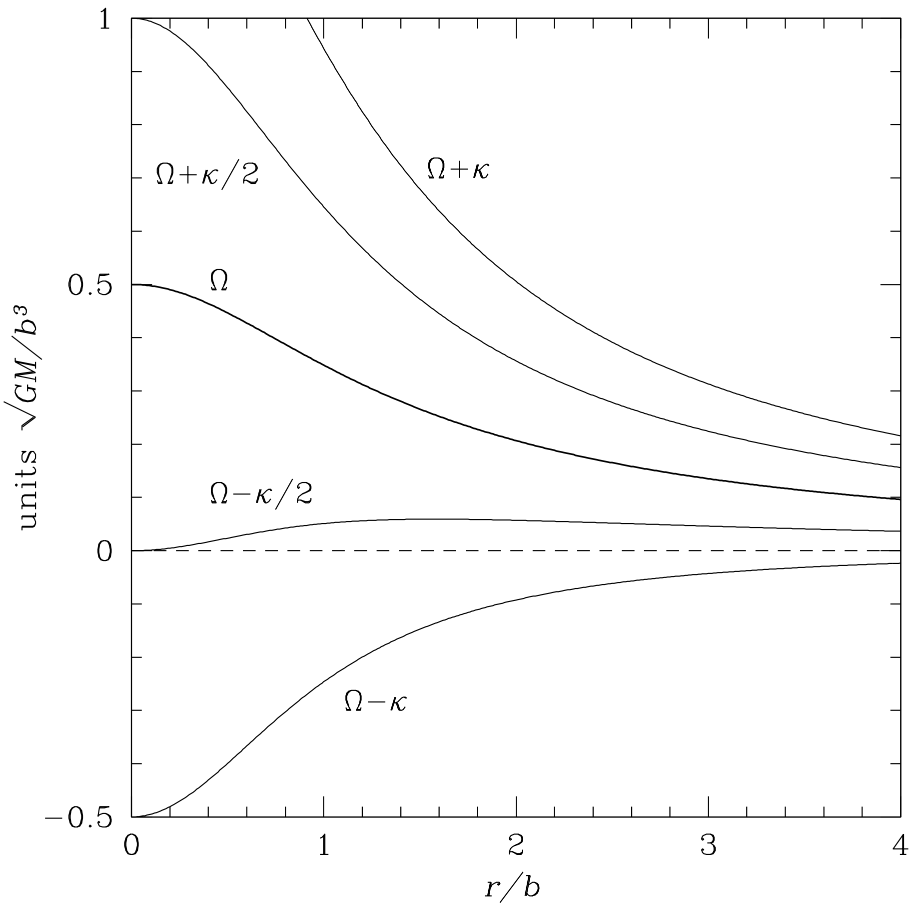


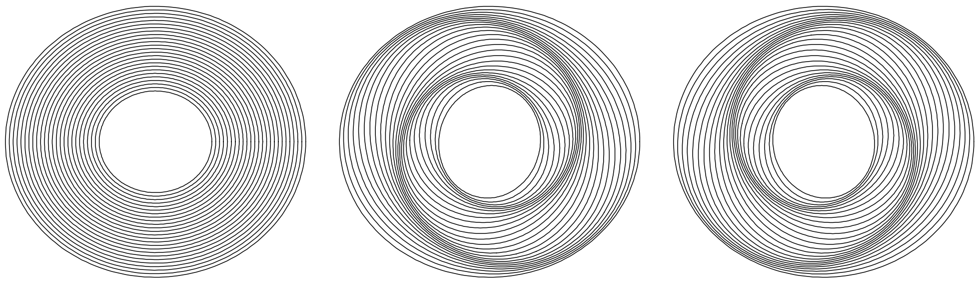
Figure 2: Behavior of $\Omega - n\kappa/m$ with radius for a model of our Galaxy.

$\Delta\phi 2\pi n/m$ for integers m and n we can make the orbit close after m radial oscillations by choosing

$$\Omega_{\rm p} = \Omega_\phi - \frac{n\Omega_r}{m} \approx \Omega - \frac{n\kappa}{m}, \tag{6}$$

where we have approximated Ω_ϕ and Ω_r by the value for circular orbits Ω and the epicyclic frequency $\kappa = (r\mathrm{d}\Omega^2/\mathrm{d}r + 4\Omega^2)^{1/2}$ (which is the frequency for small radial perturbations). In general, this function will vary with radius (determined by the rotation curve of the galaxy). Figure 2 plots a few such curves computed using a model of our own Galaxy. As first noticed by Lindblad, while most of these curves vary rapidly with radius, the $n = 1$, $m = 2$ (or $n = 2$, $m = 4$ etc.) curve varies only slowly. Imagine that it was precisely constant with radius. Then, orbits of this type (those which close after two radial oscillations) would be stationary in our rotating frame at all radii. By populating such orbits with stars we can create a stationary pattern in the rotating frame such as a bar or leading or trailing spiral arms (depending on how the axes of the orbital ellipses vary with radius). This is a rotating density wave—individual stars are moving in and out of the density wave as they orbit, but the pattern itself remains fixed (kind of like traffic on the LA freeways—there's *always* a density enhancement of cars in downtown, but individual cars move in and out of it. . .).

In a real galaxy $\Omega - n\kappa/m$ is never precisely constant so that orbits move at slightly different frequencies at each radius and any pattern will therefore wind up. We can compute the rate of



(a) (b) (c)

Figure 3: Examples of spiral patterns constructed from orbits which have $\Omega - \frac{1}{2}\kappa$ constant with radius.

winding by finding the pitch angle of the pattern. Let $\phi_\mathrm{p}(R,t)$ be the angle of the major axis of the pattern viewed in our rotating frame and let the major axes of the pattern are aligned at $t=0$ so that $\phi_\mathrm{p}(R,0)=\phi_0$. The drift rate is $\frac{\partial\phi_\mathrm{p}}{\partial t=\Omega-\frac{1}{2}\kappa-\Omega_\mathrm{p}}$; so

$$\phi_\mathrm{p}(R,t) = \phi_0 + [\Omega - \frac{1}{2}\kappa - \Omega_\mathrm{p}]t. \tag{7}$$

From our previous expression for pitch angle we then have

$$\cot\alpha = Rt\left|\frac{\mathrm{d}(\Omega-\frac{1}{2}\kappa)}{\mathrm{d}R}\right|. \tag{8}$$

For our Galaxy, $R\left|\frac{\mathrm{d}(\Omega-\frac{1}{2}\kappa)}{\mathrm{d}R}\right|$ is about 7 km/s/kpc so after 10 Gyr the pitch angle would be about $\alpha=0.8^\circ$—still much less than observed values but larger than for material arms. If we can slow down the winding some more this might start to look like a two-armed spiral structure and thereby explain the prevalence of such structure in observed galaxies.

These structures are known as *kinematic density waves* as they involve only the kinematics of orbits in an axisymmetric potential. The density wave itself will, however, produce a non-axisymmetric component to the potential—a major goal of spiral structure theory is to figure out if this non-axisymmetric potential can help reduce the rate of winding up of the arms.

2.2 Dispersion Relation

To study the behaviour of density waves we need a three step procedure:

1. Solve Poisson's equation to find the potential from the density waves;
2. Determine how this potential affects the orbits of the stars;
3. Match the response in stellar density with the input surface density to get a self-consistent solution.

To make the problem (somewhat) easier we can assume a razor-thin disk (2 dimensions are easier than 3) and consider small perturbations so that we can apply a linear perturbation analysis. Self-consistent density waves are then just the natural linear modes of a disk which can be computed (numerically at least).

2.2.1 Tight Winding Approximation

The big problem in computing modes of disks is that gravity is a long-range force so perturbations in all parts of the disk are coupled. However, Kalnajs, Lin & Toomre realised that for tightly wound disks the long-range coupling is negligible which makes things much easier. This *tight-winding*, *short-wavelength* or *WKB approximation* is very useful for understanding the properties of density waves in disks.

Suppose a pattern is describe by a shape function $f(R,t)$ (which gives the azimuthal coordinate of a curve following the pattern). The difference between adjacent arms at the same azimuthal position is ΔR where $|f(R+\Delta R,t) - f(R,t)| = 2\pi$. For tightly wound arms we can approximate this to get $\Delta R \frac{\partial f}{\partial R} = 2\pi$ or

$$\Delta R = \frac{2\pi}{|k|} = \frac{2\pi R}{m} \tan\alpha, \tag{9}$$

where $k = \partial f/\partial R$ is the wavenumber of the pattern. For pitch angles of between 10° and 15° this implies $|kR| \approx 7$–11 (for two-armed spirals). The WKB approximation requires $|kR| \gg 1$ but is often a reasonable approximation for smaller $|kR|$. So, we're going to use the WKB approximation but should treat it with some caution.[1]

2.2.2 Potential of the Spiral Pattern

We can split the surface density into unperturbed, $\Sigma_0(R)$ and perturbed, $\Sigma_1(R,\phi,t)$, parts and can split the perturbed part into a term describing the rapid variation in density in azimuth associated with arms and a term describing the slow variation in density along an arm:

$$\Sigma_1(R,\phi,t) = H(R,t)\mathrm{e}^{\mathrm{i}[m\phi + f(R,t)]}, \tag{10}$$

where $f(R,t)$ is the shape function and $H(R,t)$ describes the variation in density along an arm.[2] This assumes a sinusoidal variation in density with radius, which turns out to be correct in the linear perturbation regime.

To figure out the potential from this perturbation we first note that the rapid oscillation will mean that the contributions from distant parts of the perturbation will cancel out. So, we can just

[1] We need $|kR| \gg 1$ for this to work. Axisymmetric, $m = 0$, modes have zero pitch angle and so are technically always tightly wound, but don't necessarily have short wavelengths. In addition, modes with large $m \gg 1$ can be tightly wound even though their radial wavelength isn't short. We have to keep these caveats in mind also when applying the WKB approximation.

[2] The physical density is just the real part of Σ_1 of course.

consider the contribution from the few nearby oscillations. It's therefore useful to replace the shape function with a Taylor series about some point (R_0, ϕ_0):

$$\Sigma_1(R, \phi, t) \approx \Sigma_a \mathrm{e}^{\mathrm{i}k(R_0,t)(R-R_0)}, \tag{11}$$

wher

$$\Sigma_a = H(R_0, t)\mathrm{e}^{\mathrm{i}[m\phi_0 + f(R_0,t)]}. \tag{12}$$

(We've neglected variations with ϕ as they're much slower than radial variations for a tightly wound pattern.) This looks like a plane wave with wavevector $\mathbf{k} = k\hat{\mathbf{e}}_R$. The potential of a plane wave in a razor-thin disk can be computed analytically (see Binney & Tremaine, p. 441 if you want to run through the derivation of this):

$$\Phi_1(R, \phi, z, t) \approx \Phi_a \mathrm{e}^{\mathrm{i}k(R_0,t)(R-R_0) - |k(R_0,t)z|}, \tag{13}$$

where $\frac{\Phi_a = -2\pi \mathrm{G}\Sigma_a}{|k|}$. We can then choose $R = R_0$, $\phi = \phi_0$ and $z = 0$ to obtain a result for the disk plane

$$\Phi_1(R, \phi, t) = -\frac{2\pi \mathrm{G}}{|k|} H(R, t)\mathrm{e}^{\mathrm{i}[m\phi + f(R,t)]}. \tag{14}$$

Since we obtained this by Taylor expansion in $|kR|^{-1}$ the error in the expression is $\mathcal{O}(|kR|^{-1})$. Taking the derivative of this with respect to R and ignoring the slow variation of the $H(R, t)$ term we find

$$\frac{\partial}{\partial R}\Phi_1(R, \phi, t) = 2\pi \mathrm{G}\mathrm{sgn}(k)H(R, t)\mathrm{e}^{\mathrm{i}[m\phi + f(R,t)]} = 2\pi G\mathrm{i}\mathrm{sgn}(k)\Sigma_1(R, \phi, t), \tag{15}$$

or

$$\Sigma_1(R, \phi, t) = \frac{\mathrm{i}\mathrm{sgn}(k)}{2\pi G}\frac{\partial}{\partial R}\Phi_1(R, \phi, t). \tag{16}$$

2.2.3 Dispersion Relations

Our remaining tasks are to find the response of the disk to the perturbation and to match this to the input perturbation for a self-consistent response. It turns out to be easier to treat a fluid disk rather than a stellar disk[3]. The basic features are quite similar and we will gain some insight as a "cold" (i.e. zero velocity dispersion) stellar disk is dynamically equivalent to a zero pressure fluid disk.

We begin by writing Euler's equation in cyclindrical coordinates:

$$\begin{aligned}
\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} &= -\frac{\partial \Phi}{\partial R} - \frac{1}{\Sigma_\mathrm{d}}\frac{\partial p}{\partial R} \\
\frac{\partial v_\phi}{\partial t} + v_R\frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R}\frac{\partial v_\phi}{\partial \phi} - \frac{v_\phi v_R}{R} &= -\frac{1}{R}\frac{\partial \Phi}{\partial \phi} - \frac{1}{\Sigma_\mathrm{d} R}\frac{\partial p}{\partial \phi},
\end{aligned} \tag{17}$$

where we use a surface density Σ_d since we're working with a two-dimensional disk. We can choose a simple equation of state $p = K\Sigma_\mathrm{d}^\gamma$ which leads to sound waves propagating in a disk of surface density Σ_0 with speed

$$v_\mathrm{s}^2(\Sigma_0) = \left(\frac{\mathrm{d}p}{\mathrm{d}\Sigma}\right) = \gamma K \Sigma_0^{\gamma - 1}. \tag{18}$$

[3]The reason is that, in a stellar disk, stars passing through a given point at any time will have come from a range of radii due to radial perturbations in their orbits. This leads to an averaging over perturbation and tends to reduce its effects.

Using these expressions and replacing the pressure, p, by the specific enthalpy

$$h = \frac{\gamma}{\gamma - 1} K \Sigma_{\rm d}^{\gamma-1}, \tag{19}$$

we find

$$-\frac{\partial \Phi}{\partial R} - \frac{1}{\Sigma_{\rm d}} \frac{\partial p}{\partial R} = -\frac{\partial \Phi}{\partial R} - \gamma K \Sigma_{\rm d}^{\gamma-2} \frac{\partial \Sigma_{\rm d}}{\partial R} = -\frac{\partial}{\partial R}(\Phi + h), \tag{20}$$

with a similar expression for the ϕ Euler equation.

For a small spiral perturbation we can linearize these equations. Denoting the unperturbed solutions by subscript "0" we have

$$\frac{v_{\phi 0}^2}{R} = \frac{\rm d}{{\rm d}R}(\Phi_0 h_0) = \frac{{\rm d}\Phi_0}{{\rm d}R} + v_{\rm s}^2 \frac{\rm d}{{\rm d}R} \ln \Sigma_0, \tag{21}$$

which equates centripetal acceleration to the gravitational and pressure forces. For typical disks, $v_{\rm s} \ll v_{\phi 0}$ so we can ignore the pressure force leaving

$$v_{\phi 0} = \sqrt{R \frac{{\rm d}\Phi_0}{{\rm d}R}} = R\Omega(R), \tag{22}$$

as expected.

For the first order perturbation we can write $v_R = \epsilon v_{R1}$, $v_\phi = v_{\phi 0} + \epsilon v_{\phi 1}$, $h = h_0 + \epsilon h_1$, $\Sigma_{\rm d} = \Sigma_0 + \epsilon \Sigma_{\rm d1}$, $\Phi = \Phi_0 + \epsilon \Phi_1$, where $\epsilon \ll 1$ and the subscript 1 quantities are the same order of magnitude as the subscript 0 quantities. Then, to first order in ϵ:

$$\begin{aligned} \frac{\partial v_{R1}}{\partial t} + \Omega \frac{\partial v_{R1}}{\partial phi} - 2\Omega v_{\phi 1} &= -\frac{\partial}{\partial R}(\Phi_1 + h_1), \\ \frac{\partial v_{\phi 1}}{\partial t} + \left[\frac{{\rm d}(\Omega R)}{{\rm d}R} + \Omega\right] v_{R1} + \Omega \frac{\partial v_{\phi 1}}{\partial \phi} = -1 over R \frac{\partial}{\partial \phi}(\Phi_1 + h_1). \end{aligned} \tag{23}$$

The square bracket we can write as $-2B(R)$ where

$$B(R) = -\left(\Omega + \frac{1}{2} R \frac{{\rm d}\Omega}{{\rm d}R}\right) = -\frac{\kappa^2}{4\Omega}. \tag{24}$$

Since these are linear equations any solution must be the sum of terms of the form

$$\begin{aligned} v_{R1} &= v_{Ra}(R) {\rm e}^{{\rm i}(m\phi - \omega t)} \\ v_{\phi 1} &= v_{\phi a}(R) {\rm e}^{{\rm i}(m\phi - \omega t)} \\ \Phi_1 &= \Phi_a(R) {\rm e}^{{\rm i}(m\phi - \omega t)} \\ h_1 &= h_a(R) {\rm e}^{{\rm i}(m\phi - \omega t)} \\ \Sigma_{d1} &= \Sigma_{da}(R) {\rm e}^{{\rm i}(m\phi - \omega t)} \end{aligned} \tag{25}$$

where $m \geq 0$ is an integer and the resulting perturbation has m-fold symmetry. (Note again that we take only the real part of these solutions.) Using these forms we can solve for the velocities:

$$\begin{aligned} v_{Ra}(R) &= \frac{\rm i}{\Delta} \left[(\omega - m\Omega) \frac{\rm d}{{\rm d}R}(\Phi_a + h_a) - \frac{2m\Omega}{R}(\Phi_a + h_a)\right] \\ v_{\phi a}(R) &= -\frac{1}{\Delta} \left[2B \frac{\rm d}{{\rm d}R}(\Phi_a + h_a) - \frac{m(\omega - m\Omega)}{R}(\Phi_a + h_a)\right], \end{aligned} \tag{26}$$

where

$$\Delta \equiv \kappa^2 - (\omega - m\Omega)^2, \tag{27}$$

and κ, Ω, Δ, Φ_a and h_a are all functions of radius. For ω real there may be radii where $\Delta = 0$ and the solution diverge. These correspond to (Lindblad) resonances and occur where a density wave can be supported even for vanishingly small forcing function (the stuff in square brackets). Our solution breaks down here (it can't be treated as a perturbation) and a more sophisticated approach is needed.

The linearized equation of state is

$$h_a = \gamma K \Sigma_0^{\gamma-2} \Sigma_{\mathrm{da}} = v_{\mathrm{s}}^2 \Sigma_{\mathrm{da}} / \Sigma_0, \tag{28}$$

and the perturbde surface density can be found from the perturbed velocities using the continuity equation in cylindrical coordinates

$$\frac{\partial \Sigma_{\mathrm{d1}}}{\partial t} + \Omega \frac{\partial \Sigma_{\mathrm{d1}}}{\partial \phi} + \frac{1}{R}\frac{\partial}{\partial R}(R v_{R1} \Sigma_0) + \frac{\Sigma_0}{R}\frac{\partial v_{\phi 1}}{\partial \phi} = 0, \tag{29}$$

giving

$$-\mathrm{i}(\omega - m\Omega)\Sigma_{\mathrm{da}} + \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}(R v_{Ra} \Sigma_0) \frac{im\Sigma_0}{R} v_{\phi a} = 0. \tag{30}$$

This gives us four equations for five variables (the fifth being Φ_a) and so determine the linear response of the system to an imposed potential perturbation Φ_a. If the potential perturbation is caused by the density perturbation then we can use Poisson's equation to close this system of equations. In general, this requires a numerical solution.

For a simpler approach we can use the WKB approximation for local perturbations in which we assume that the $\mathrm{d}(\Phi_a + h_a)/\mathrm{d}R$ terms are much larger (by a factor of kR) than the $(\Phi_a + h_a)/R$ terms (i.e. the oscillations are in the short-wavelength regime). Thus

$$\begin{aligned} v_{Ra} &= -\frac{(\omega - m\Omega)}{\Delta} k(\Phi_a + h_a) \\ v_{\phi a} &= -\frac{2\mathrm{i}B}{\Delta} k(\Phi_a + h_a), \end{aligned} \tag{31}$$

and a similar approximation for Σ_{da}. The simplified continuity equation is then

$$-(\omega - m\Omega)\Sigma_{\mathrm{da}} + k\Sigma_0 v_{Ra} = 0. \tag{32}$$

Eliminating v_{Ra}, h_a and Φ_a from this leaves us with

$$\Sigma_{\mathrm{da}}(R) = \widetilde{P}_m\,(k, R, \omega)\Sigma_a(R), \tag{33}$$

where

$$\widetilde{P}_m\,(k, R, \omega) = \frac{2\pi \mathrm{G}\Sigma_0 |k|}{\kappa^2 - (\omega - m\Omega)^2 + v_{\mathrm{s}}^2 k^2}, \tag{34}$$

is the *polarization function* for tightly wound density waves. Σ_{da} here is the density response of the disk to an imposed potential cause by a density perturbation Σ_a. For a self-consistent response $\widetilde{P}_m\,(k, R, \omega) = 1$ which implies

$$(\omega - m\Omega)^2 = \kappa^2 - 2\pi \mathrm{G}\Sigma_0 |k| + v_{\mathrm{s}}^2 k^2, \tag{35}$$

which is the dispersion relation for the density waves.

2.3 Local Stability of Disks

The preceeding analysis allows us to ask about the stability of a disk to axisymmetric perturbations. Consider a cold disk, $v_{\rm s} = 0$, then

$$\omega^2 = \kappa^2 - 2\pi \mathrm{G}\Sigma|k|. \tag{36}$$

The right hand side is real, so *omega*^{2} must be real. If $\omega^2 > 0$ then ω is real and the disk is stable (perturbations oscillate) but if $\omega^2 < 0$ then ω is imaginary and the disk is unstable (perturbations grow exponentially). There is a critical wavenumber

$$k_{\rm crit} \equiv \frac{\kappa^2}{2\pi \mathrm{G}\Sigma}, \tag{37}$$

which delineates these regimes. The instability is violent: as the wavenumber becomes large (wavelength shrinks to zero) the growth rate ω grows without limit so a cold disk disintegrates on small scales in arbitrarily short time.

With a non-zero sound speed we can again have instability if $\omega^2 < 0$. The dispersion relation is now quadratic. It's easy to show that if

$$Q \equiv \frac{v_s \kappa}{\pi \mathrm{G}\Sigma} > 1, \tag{38}$$

then the disk is stable for all wavenumbers. If $Q < 1$ the disk is unstable for some range of wavenumbers. For a stellar disk the equivalent criterion is

$$Q \equiv \frac{\sigma_R \kappa}{3.36 \mathrm{G}\Sigma} > 1. \tag{39}$$

This Q is known as *Toomre's stability parameter*. It shows that "hot" disks (those with high velocity dispersion) are resistant to instability. In our local region of the Milky Way $Q_\star \approx 2.7$ and $Q_{\rm g} \approx 1.5$ so the disk is stable (really we should treat the stars and gas as a gravitationally coupled system).