Midterm: Evolution of Stellar Populations—Solution

Structure and Dynamics of Galaxies, Ay 124, Winter 2009

February 3, 2009

1. Consider a rotating bar in a galaxy. Let the z axis coincide with the principal axis of the tensor $\mathbf{I}$ and assume that the density distribution of the bar is stationary in a frame which rotates with angular frequency Ω. Show that, at an instant when $I_{xy} = 0$, the term $\frac{1}{2}\mathrm{d}^2\mathbf{I}/\mathrm{d}t^2$ which appears in the tensor virial theorem is a diagonal tensor with components $(I_{yy} - I_{xx})$, $(I_{xx} - I_{yy})$ and 0. Hence show that

$$\Omega^2 = \frac{(W_{xx} - W_{yy}) + 2(T_{xx} - T_{yy}) + (\Pi_{xx} - \Pi_{yy})}{2(I_{xx} - I_{yy})}, \tag{1}$$

and, if $T_{zz} = 0$,

$$\frac{v_0^2}{\sigma_0^2} = (1 - \delta)\frac{W_{xx} + W_{yy}}{W_{zz}} - 2, \tag{2}$$

where $v_0^2 \equiv 2(T_{xx} + T_{yy})/M$, $\sigma_0^2 \equiv (\Pi_{xx} + \Pi_{yy})/2M$ and $(1 - \delta)(\Pi_{xx} + \Pi_{yy}) \equiv 2\Pi_{zz}$. (Hint: The tensor transformation between the rotating and non-rotating frames is

$$\widetilde{\mathbf{I}} = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{I} \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

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Solution: In the rotating frame, the moment of inertia tensor can be written as

$$\mathbf{I}_{\mathrm{rot}} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}. \tag{4}$$

Transforming this to the non-rotating frame using the given transformation gives

$$\widetilde{\mathbf{I}} = \begin{pmatrix} I_{xx}\cos^2\Omega t + I_{yy}\sin^2\Omega t & -\frac{1}{2}(I_{xx} - I_{yy})\sin 2\Omega t & 0 \\ -\frac{1}{2}(I_{xx} - I_{yy})\sin 2\Omega t & I_{xx}\sin^2\Omega t + I_{yy}\cos^2\Omega t & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}. \tag{5}$$

We know that the components I_{xx}, I_{yy} and I_{zz} are constant, so

$$\begin{aligned} \frac{1}{2}\mathrm{d}^2\,\widetilde{I}_{xx} &= \Omega^2\cos 2\Omega t(I_{yy} - I_{xx}) \\ \frac{1}{2}\mathrm{d}^2\,\widetilde{I}_{yy} &= \Omega^2\cos 2\Omega t(I_{xx} - I_{yy}) \\ \frac{1}{2}\mathrm{d}^2\,\widetilde{I}_{zz} &= 0. \end{aligned} \tag{6}$$

If $I_{12} = 0$ *then* $\sin 2\Omega = 0$ *which implies* $\cos 2\Omega t = 1$*, so*

$$\begin{aligned}
\frac{1}{2}\mathrm{d}^2 \stackrel{\sim}{I}_{xx} &= \Omega^2(I_{yy} - I_{xx}) \\
\frac{1}{2}\mathrm{d}^2 \stackrel{\sim}{I}_{yy} &= \Omega^2(I_{xx} - I_{yy}) \\
\frac{1}{2}\mathrm{d}^2 \stackrel{\sim}{I}_{zz} &= 0.
\end{aligned} \tag{7}$$

Then, from the tensor virial theorem $\frac{1}{2}\mathrm{d}^2 \stackrel{\sim}{\mathbf{I}} /\mathrm{d}t^2 = 2\mathbf{T} + \mathbf{\Pi} + \mathbf{W}$ *we get*

$$\begin{aligned}
\Omega^2(I_{yy} - I_{xx}) &= 2T_{xx} + \Pi_{xx} + W_{xx} \\
\Omega^2(I_{xx} - I_{yy}) &= 2T_{yy} + \Pi_{yy} + W_{yy} \\
0 &= 2T_{zz} + \Pi_{zz} + W_z,
\end{aligned} \tag{8}$$

which lead to

$$\Omega^2 = \frac{(W_{xx} - W_{yy}) + 2(T_{xx} - T_{yy}) + (\Pi_{xx} - \Pi_{yy})}{2(I_{xx} - I_{yy})}. \tag{9}$$

Assuming $T_{zz} = 0$ *implies* $\Pi_{zz} = -W_{zz}$ *and so*

$$\frac{v_0^2}{\sigma_0^2} = 4\frac{T_{xx} + T_{yy}}{\Pi_{xx} + \Pi_{yy}}. \tag{10}$$

Adding together the xx *and* yy *components of the tensor virial theorem implies* $2T_{xx} + \Pi_{xx} + W_{xx} + 2T_{yy} + \Pi_{yy} + W_{yy} = 0$ *and so*

$$\begin{aligned}
\frac{v_0^2}{\sigma_0^2} &= -2\frac{\Pi_{xx} + W_{xx} + \Pi_{yy} + W_{yy}}{\Pi_{xx} + \Pi_{yy}} \\
&= -2 - 2\frac{W_{xx} + W_{yy}}{\Pi_{xx} + \Pi_{yy}} \\
&= -2 + (1 - \delta)\frac{W_{xx} + W_{yy}}{W_{zz}}.
\end{aligned} \tag{11}$$

2. a) By taking a suitable moment of the collisionless Boltzmann equation, show that in a steady-state axisymmetric galaxy

$$\frac{\partial(\nu\overline{v_R^2 v_\phi})}{\partial R} + \frac{\partial(\nu\overline{v_R v_z v_\phi})}{\partial z} - \frac{\nu}{R}\left(\overline{v_\phi^3} - \overline{v_\phi} R\frac{\partial\Phi}{\partial R}\right) + \frac{2\nu}{R}\overline{v_R^2 v_\phi} = 0. \tag{12}$$

b) Given that the system is symmetric in z, and that all odd moments of $v_\phi - \overline{v_\phi}$ vanish, so $\overline{v_R^2((\overline{v_\phi^3} - \overline{v_\phi}R\frac{\partial\Phi}{\partial R}))} = 0$ and $\overline{((\overline{v_\phi^3} - \overline{v_\phi}R\frac{\partial\Phi}{\partial R}))^2} = 0$, etc., show that at $z = 0$

$$\overline{v_R^2}\left(\frac{\partial\overline{v_\phi}}{\partial R} + \frac{\overline{v_\phi}}{R}\right) - \frac{2}{R}\overline{v_\phi}\overline{(v_\phi - \overline{v_\phi})^2} = 0. \tag{13}$$

Hence, using the Jeans equation

$$\frac{\partial(\nu\overline{v_R^2})}{\partial R} + \frac{\partial(\nu\overline{v_R v_z})}{\partial z} + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial\Phi}{\partial R}\right) = 0, \tag{14}$$

show that

$$\frac{\sigma_\phi^2}{\sigma_R^2} \equiv \frac{\overline{(v_\phi - \overline{v_\phi})^2}}{\overline{v_R^2}} \approx \frac{-B}{A-B}, \quad (15)$$

where $A = -\frac{1}{2}R\mathrm{d}\Omega\mathrm{d}R$ and $B = -(\Omega + \frac{1}{2}R\mathrm{d}\Omega/\mathrm{d}R)$ (with $\Omega^2 = R^{-1}(\partial\Phi/\partial R)_{z=0}$) are the Oort constants.

Solution: Begin with the collisionless Boltzmann equation in cylindrical coordinates

$$v_R\frac{\partial f}{\partial R} + v_z\frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{R} - \frac{\partial\Phi}{\partial R}\right)\frac{\partial f}{\partial v_R} - \frac{v_R v_\phi}{R}\frac{\partial f}{\partial v_\phi} - \partial\Phi\partial z\frac{\partial f}{\partial v_z} = 0, \quad (16)$$

multiply by $v_R v_\phi$ and integrate over all velocities to obtain

$$\frac{\partial(\nu\overline{v_R^2 v_\phi})}{\partial R} + \frac{\partial(\nu\overline{v_R v_\phi v_z})}{\partial z} - \frac{\nu}{R}\left(\overline{v_\phi^3} - \overline{v_\phi}R\frac{\partial\Phi}{\partial R}\right) + \frac{2\nu}{R}\overline{v_R^2 v_\phi^2} = 0, \quad (17)$$

where we've used the fact that

$$\int v_j \frac{\partial f}{\partial v_i}\mathrm{d}^3\mathbf{v} = -\int \partial v_j \partial v_i f \mathrm{d}^3\mathbf{v} = -\delta_{ij}\nu. \quad (18)$$

Since we're told that the odd moments of $v_\phi - \overline{v_\phi}$ vanish we have that $\overline{v_R^2 v_\phi} = \overline{v_R^2}\,\overline{v_\phi}$ and $\overline{v_R v_z v_\phi} = \overline{v_R v_z}\,\overline{v_\phi}$. Using these in the above equation and subtracting $\overline{v_\phi}$ times the given Jeans equation then leaves

$$\nu\overline{v_R^2}\frac{\partial\overline{v_\phi}}{\partial R} + \nu\overline{v_r v_z}\frac{\partial\overline{v_\phi}}{\partial z} - \frac{\nu}{R}\left(\overline{v_\phi}^3 - \overline{v_\phi}\,\overline{v_\phi^2}\right) + \frac{\nu}{R}\overline{v_R^2}\,\overline{v_\phi} = 0. \quad (19)$$

Again using the fact that odd moments vanish $(\overline{v_\phi}^3 - \overline{v_\phi}\,\overline{v_\phi^2}) = 2\overline{v_\phi}\overline{(v_\phi - \overline{v_\phi})^2}$ and dividing through by ν leaves

$$\overline{v_R^2}\left(\frac{\partial\overline{v_\phi}}{\partial R} + \frac{\overline{v_\phi}}{R}\right) - \frac{2}{R}\overline{v_\phi}\overline{(v_\phi - \overline{v_\phi})^2} = 0. \quad (20)$$

Assuming small asymmetric drag, $\overline{v_\phi}^2 \approx v_c^2 = -R\partial\Phi/\partial R$ which gives

$$\frac{\sigma_\phi^2}{\sigma_R^2} \equiv \frac{\overline{(v_\phi - \overline{v_\phi})^2}}{\overline{v_R^2}} \approx \frac{-B}{A-B}. \quad (21)$$

3. Prove that a system of N self-gravitating point masses with positive energy must disrupt, in the sense that at least one star must escape. Hint: use the virial theorem, and prove that the moment of inertia must increase without limit.

   Solution: We know that such a system of particles obeys the tensor virial theorem. We can therefore take the trace of that equation and apply the scalar virial theorem $\frac{1}{2}\mathrm{d}^2I/\mathrm{d}t^2 = 2K + W$. In terms of the total energy, $E = K + W$, this becomes $\frac{1}{2}\mathrm{d}^2I/\mathrm{d}t^2 = K + E$. Since $E > 0$ for a positive energy system and since the kinetic energy is always positive this implies that $\mathrm{d}^2I/\mathrm{d}t^2 > 0$ always. The curve of I vs. t must therefore be concave upwards and I must therefore increase without bound as t increases. Since $I_{ij} = \sum_{\alpha=1}^N m^\alpha x_i^\alpha x_j^\alpha$ this implies that at least one star must reach an arbitrarily large coordinate value (i.e. be ejected to infinity).