Ay 124 – Problem Set # 2

1) CBE $$\frac{df}{dt} = 0 = \frac{\partial f}{\partial t} + \dot r \frac{\partial f}{\partial r} + \dot\phi \frac{\partial f}{\partial \phi} + \dot\theta \frac{\partial f}{\partial \theta} + \dot v_r \frac{\partial f}{\partial v_r} + \dot v_\phi \frac{\partial f}{\partial v_\phi} + \dot v_\theta \frac{\partial f}{\partial v_\theta}$$

where $v_r = \dot r$, $v_\phi = r\sin(\theta)\dot\phi$, $v_\theta = r\dot\theta$

$$\rightarrow \dot v_r = \ddot r, \quad \dot v_\phi = \dot r \sin(\theta)\dot\phi + r\cos(\theta)\dot\theta\dot\phi + r\sin(\theta)\ddot\phi, \quad \dot v_\theta = \dot r\dot\theta + r\ddot\theta$$

Write $\vec a = -\vec\nabla\Phi$ in spherical coordinates

$$-\frac{\partial\Phi}{\partial r} = \ddot r - r\dot\theta^2 - r\sin^2\theta\,\dot\phi^2$$
$$-\frac{1}{r\sin(\theta)}\frac{\partial\Phi}{\partial\phi} = r\sin(\theta)\ddot\phi + 2\sin(\theta)\dot r\dot\phi + 2r\cos(\theta)\dot\theta\dot\phi$$
$$-\frac{1}{r}\frac{\partial\Phi}{\partial\theta} = 2\dot r\dot\theta + r\ddot\theta - r\sin(\theta)\cos(\theta)\dot\phi^2$$

Rewrite $\dot v_r, \dot v_\theta, \dot v_\phi$ in terms of velocities and derivatives of Φ:

$$\dot v_r = -\frac{\partial\Phi}{\partial r} + r\dot\theta^2 + r\sin^2(\theta)\dot\phi^2 = -\frac{\partial\Phi}{\partial r} + \frac{v_\theta^2 + v_\phi^2}{r}$$
$$\dot v_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} - \dot r\dot\theta + r\sin(\theta)\cos(\theta)\dot\phi^2 = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} - \frac{v_r v_\theta}{r} + \frac{\cot(\theta)}{r} v_\phi^2$$
$$\dot v_\phi = -\frac{1}{r\sin(\theta)}\frac{\partial\Phi}{\partial\phi} - \dot r\sin(\theta)\dot\phi - r\cos(\theta)\dot\theta\dot\phi = -\frac{1}{r\sin(\theta)}\frac{\partial\Phi}{\partial\phi} - \frac{v_r v_\phi}{r} - \frac{\cot(\theta)}{r} v_\theta v_\phi$$

With these expressions the CBE becomes

$$\frac{df}{dt} = 0 = \frac{\partial f}{\partial t} + v_r\frac{\partial f}{\partial r} + \frac{v_\phi}{r\sin(\theta)}\frac{\partial f}{\partial\phi} + \frac{v_\theta}{r}\frac{\partial f}{\partial\theta} + \left(\frac{v_\theta^2 + v_\phi^2}{r} - \frac{\partial\Phi}{\partial r}\right)\frac{\partial f}{\partial v_r}$$
$$+ \frac{1}{r}\left(\cot(\theta)v_\phi^2 - v_r v_\theta - \frac{\partial\Phi}{\partial\theta}\right)\frac{\partial f}{\partial v_\theta} - \frac{1}{r}\left(v_\phi v_r + v_\phi v_\theta\cot(\theta) + \frac{1}{\sin(\theta)}\frac{\partial\Phi}{\partial\phi}\right)\frac{\partial f}{\partial v_\phi}$$

2) Write v_R and v_z in terms of v_r and v_θ:

$$v_R = v_r\sin(\theta) + v_\theta\cos(\theta)$$
$$v_z = v_r\cos(\theta) - v_\theta\sin(\theta)$$

For small $|z|/R$: $\theta \approx \frac{\pi}{2} - \frac{z}{R} \rightarrow \cos(\theta) \approx \frac{z}{R}$, $\sin(\theta) \approx 1$

$$\rightarrow v_R \approx v_r + \frac{z}{R}v_\theta, \quad v_z \approx \frac{z}{R}v_r - v_\theta$$

Since $v_R v_z \approx \frac{z}{R}v_r^2 - \frac{z}{R}v_\theta^2 - v_\theta v_r$, now average using $\overline{v_\theta v_r} = 0$

$\overline{v_R v_z} = \frac{z}{R}\left(\overline{v_r^2} - \overline{v_\theta^2}\right)$, and $\overline{v_R^2} = \overline{v_r^2} + 2\underbrace{\overline{v_r v_\theta}}_{=0} + \left(\frac{z}{R}\right)^2\overline{v_\theta^2} \approx \overline{v_r^2}$, for small

$$\Rightarrow \overline{v_R v_z} \approx \frac{z}{R}\left(\overline{v_R^2} - \overline{v_\theta^2}\right)$$, to leading order in $\frac{|z|}{R}$