Problem Set #4

Structure and Dynamics of Galaxies, Ay 124, Winter 2009

February 27, 2009

1. [Binney & Merrifield, Problem 5-8[1]; 20 points] Define $\phi(M)\mathrm{d}M$ to be the number of stars on the main sequence in the mass range M to $M + \mathrm{d}M$ as observed in some stellar system. Similarly, let $\phi_0(M)\mathrm{d}M$ be the initial number of stars in the same mass range just after stars are born (i.e. the initial mass function). Assuming that the stellar system has a star formation rate proportional to $\exp(-\alpha t)$ and that the first star was born at $t = 0$ show that at some later time t the initial mass function is related to the measureable quantity $\phi(M)$ by

$$\phi_0(M) = \begin{cases} \frac{\alpha \mathrm{e}^{-\alpha t}}{1-\mathrm{e}^{\alpha \tau_{\mathrm{MS}}(M)}}\phi(M) & \text{if } t > \tau_{\mathrm{MS}}(M) \\ \phi(M) & \text{if } t < \tau_{\mathrm{MS}}(M), \end{cases} \tag{1}$$

where $\tau_{\mathrm{MS}}(M)$ is the main sequence lifetime of a star of mass M.

2. [Binney & Merrifield, Problem 5.13; 30 points] Consider a modification of the accreting box model in which the gas is initially metal free and that for each unit of mass accreted a fraction $(1 - q)$ is locked up in stars (i.e. $\delta M_{\mathrm{s}} = (1 - q)\delta M_{\mathrm{a}}$ and, consequently, $\delta M_{\mathrm{g}} = q\delta M_{\mathrm{a}}$).

   a) [10 points] Show that a parametric solution for $Z(M_{\mathrm{g}})$ is

$$Z = p(1 - q)(1 - \mathrm{e}^{-u}); \quad M_{\mathrm{g}} = M_{\mathrm{g0}}\mathrm{e}^{qu}, \tag{2}$$

   where M_{g0} is the initial gas mass.

   b) [10 points] Show that the ratio of the stellar mass at t_1 to the mass in gas at the present time t_0 is

$$\frac{M_{\mathrm{s}}(u_1)}{M_{\mathrm{g}}(u_0)} = \frac{1-q}{q}(\mathrm{e}^{qu_1} - 1)\mathrm{e}^{-qu_0}, \tag{3}$$

   where u_i is the value of the parameter u at time t_i. Explain why this relation together with observations of our Galactic disk imply $q \ll 1$.

   c) [10 points] Now consider the case $u_0 \gg 1$, and let u_1 be an epoch at which the metallicity Z_1 was substantially lower than Z_0. Show (i) that the present metallicity $Z_0 \approx p(1 - q)$; (ii) that $u_1 \approx -\ln(1 - Z_1/Z_0) \ll 1$; (iii) that

$$\frac{M_{\mathrm{s}}(u_1)}{M_{\mathrm{g}}(u_0)} \approx -\ln\left(1 - \frac{Z_1}{Z_0}\right)\mathrm{e}^{-qu_0}. \tag{4}$$

[1] The problem in the book has an error. A list of errors in Binney & Merrifield is available at http://www-thphys.physics.ox.ac.uk/people/JamesBinney/bmerrors.pdf.

This differs from the equivalent expression for the simple accretion model only by a factor of e^{-qu_0}. Since this factor can take on a value of between 0 and 1 depending on q and u_0 it follows that in this modified accretion model the fraction of low metallicity stars can be made arbitrarily small.

3. [50 points] A simple model for the disk of the Milky Way consists of an exponential disk $\Sigma(R) = \Sigma_0 \exp(-R/R_\mathrm{d})$ with $R_\mathrm{d} = 3.5\mathrm{kpc}$ and $\Sigma_0 = 2.6 \times 10^8 M_\odot/\mathrm{kpc}^2$ and a flat rotation curve $V(R) = 200\mathrm{km/s}$. If the disk had a Toomre parameter of $Q = 0.75$ what range of wavelengths would be unstable at the Solar radius ($R = 8\mathrm{kpc}$)? (You may approximate the disk as a fluid disk. Hint: Use the dispersion relation, stability requires $\omega^2 > 0$ so that ω is real.) What would be the growth rate at a typical wavelength in this range? (Hint: For perturbations which grow as $\exp(\omega t)$ the timescale is $1/\omega$.)