Ay 124 - HW #4

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1)

Clearly, for $t < \tau_{MS}$, the observed stellar mass distribution will be equal to the IMF:

$$\frac{d\phi}{dt}[t] = \beta\, e^{-\alpha t}\, \phi_0[M]$$

$$\phi[M, t] = \int_0^t \frac{d\phi}{dt}[M, t]\, dt = \beta \int_0^t e^{-\alpha t}\, dt\, \phi_0[M]$$

$$\phi[M, t] = \beta\, \alpha^{-1} \left(1 - e^{-\alpha t}\right) \phi_0[M]$$

For times $t > \tau_{MS}$, stars will begin to move off of the main sequence. Assuming every star of mass M moves off the main sequence after a given time $\tau_{MS}[M]$, the removal rate of such stars from the main sequence will be equal to the formation rate of the stars at the time time $t - \tau_{MS}$:

$$\frac{d\phi}{dt}[M, t] = \beta \left(e^{-\alpha t} - e^{-\alpha (t-\tau_{MS})}\right) \phi_0[M]$$

$$\phi[M, t] = \int_0^t \frac{d\phi}{dt}[M, t]\, dt = \beta \int_0^{\tau_{MS}} e^{-\alpha t}\, \phi_0[M]\, dt + \beta \int_{\tau_{MS}}^t \left(e^{-\alpha t} - e^{-\alpha (t-\tau_{MS})}\right) \phi_0[M]\, dt$$

$$= \beta \left(\int_0^t e^{-\alpha t}\, dt - \int_{\tau_{MS}}^t e^{-\alpha (t-\tau_{MS})}\, dt\right) \phi_0[M]$$

$$= \beta \left(-\alpha^{-1} \left(e^{-\alpha t} - 1\right) + \alpha^{-1} e^{\alpha \tau_{MS}} \left(e^{-\alpha t} - e^{-\alpha \tau_{MS}}\right)\right) \phi_0[M]$$

$$\phi[M, t] = \beta\, \alpha^{-1}\, e^{-\alpha t}\, (-1 + e^{\alpha \tau_{MS}})\, \phi_0[M]$$

Hence we can derive and expression for the initial mass fraction in terms of the observed mass fraction:

$$\phi_0[M] = \frac{\alpha\, e^{\alpha t}}{\beta\, (e^{\alpha \tau_{MS}} - 1)}\, \phi[M, t]$$

for $t > \tau_{MS}$, and $\phi_0[M] = \frac{\alpha}{\beta} (1 - e^{-\alpha t})^{-1}\, \phi[M]$ for $t < \tau_{MS}$

2)

a)

The general solution for the evolution of the metallicity for an accreting-box model of star formation is given by Eq. 5.57:

$$Z = p \left(1 - C\, e^{-u} - e^{-u} \int_0^u e^u \frac{d\, \mathrm{Log}[M_g]}{du}\, du\right)$$

For our model, the change in gas mass is given by a fraction q of the accreted mass, which is equal to the change in the total mass: $\delta M_g = q\, \delta M_t$. Defining a parameter u such that $\delta u = \delta M_t / M_g$, we have:

$$\delta M_g = q\, M_g\, \delta u$$

$$M_g = M_{g0}\, e^{q\, u}$$

Plugging this expression into our formula for Z, we have:

$$Z = p\left(1 - C\, e^{-u} - e^{-u} \int_0^u e^u \frac{d\left(\mathrm{Log}\left[M_{g0}\right] + q\, u\right)}{du}\, du\right) = p\left(1 - C\, e^{-u} - q\, e^{-u} \int_0^u e^u\, du\right)$$

$$= p\,(1 - C\, e^{-u} - q\, e^{-u}\,(e^u - 1)) = p\,(1 - C\, e^{-u} - q\,(1 - e^{-u}))$$

If we associate M_{g0} with the initial gas mass then u=0 initially. As the initial metallicity is zero, we must have Z(u=0)=0, so C=1 and

$$Z = p\,(1 - q)\,(1 - e^{-u})$$

b)

$$\delta M_s = (1 - q)\, \delta M_t = (1 - q)\, M_g\, \delta u = (1 - q)\, M_{g0}\, e^{q\, u}\, \delta u$$

$$M_s[u] = \frac{1 - q}{q} M_{g0}\, (e^{q\, u} - 1)$$

$$M_s[u_2] = \frac{1 - q}{q} M_{g0}\, (e^{q\, u_1} - 1)$$

$$M_g[u_0] = M_{g0}\, e^{q\, u_0}$$

$$\frac{M_s[u_2]}{M_g[u_0]} = \frac{\frac{1-q}{q} M_{g0}\, (e^{q\, u_1} - 1)}{M_{g0}\, e^{q\, u_0}}$$

$$\frac{M_s[u_1]}{M_g[u_0]} = \frac{1 - q}{q}\, (e^{q\, u_1} - 1)\, e^{-q\, u_0}$$

Locally, there is very little ongoing star formation, so we can approximate $M_s[u] \approx$ constant and take $u_1 = u_0$

$$\frac{M_s}{M_g} = \frac{1 - q}{q}\, (e^{q\, u} - 1)\, e^{-q\, u} = \frac{1 - q}{q}\, (1 - e^{-q\, u})$$

In the solar neighborhood, the mass of stars dominates over the gas mass, so we should expect $M_s >> M_g$. As $1 - e^{-q\, u} \leq 1$ for all positive q and u, this implies that $\frac{1-q}{q} >> 1$ and hence that $q << 1$ in the solar neighborhood.

c)

For $u_0 >> 1$, the metallicity is approximately

$$Z_0 = p\,(1 - q)\,(1 - e^{-u_0}) \approx p\,(1 - q)$$

The ratio of metallicities at u_0 and u_1 is then

$$\frac{Z_1}{Z_0} = \frac{p\,(1-q)\,(1-e^{-u_1})}{p\,(1-q)} = 1 - e^{-u_1}$$

$$e^{-u_1} = 1 - \frac{Z_1}{Z_0}$$

Hence we have an expression for u_1 in the limit of large u_0:

$$u_1 = -\mathrm{Log}\left[1 - \frac{Z_1}{Z_0}\right]$$

Thus in the limit of $Z_1 << Z_0$ we have

$$u_1 = -\mathrm{Log}\left[1 - \frac{Z_1}{Z_0}\right] \approx \frac{Z_1}{Z_0} << 1$$

We can now rewrite our expression for the gas/star mass ratio at times 1 and 0, given $u_1 << 1$:

$$\frac{M_s[u_1]}{M_g[u_0]} = \frac{1-q}{q}\,(e^{q\,u_1} - 1)\,e^{-q\,u_0} \approx \frac{1-q}{q}\,(1 + q\,u_1 - 1)\,e^{-q\,u_0} = (1-q)\,u_1\,e^{-q\,u_0}$$

Replacing u_1 with the expression above, and noting that q<<1 in galaxies like our own, we have:

$$\frac{M_s[u_1]}{M_g[u_0]} \approx -\mathrm{Log}\left[1 - \frac{Z_1}{Z_0}\right] e^{-q\,u_0}$$

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3)

$$\Sigma[R] = \Sigma_0\, e^{-R/R_d}$$

$$V[R] = 200\ \mathrm{km/s}$$

$$Q = 0.75$$

The dispersion relation of waves in a fluid disk is given by

$$\omega^2 = \kappa^2 - 2\pi G\Sigma\,|k| + v_s{}^2 k^2$$

Oscillatory in the disk will be stable if ω is real, so for the disk to be unstable we must have

$$\kappa^2 - 2\pi G\Sigma\,|k| + v_s{}^2 k^2 < 0$$

We can solve this equation as a quadratic in the wave number k:

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Solve[κ^2 - 2 π G Σ Abs[k] + vs^2 k^2 == 0, k]
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$$\left\{\left\{k \to \frac{-G\pi\Sigma - \sqrt{G^2\pi^2\Sigma^2 - \kappa^2 v_s^2}}{v_s^2}\right\}, \left\{k \to \frac{G\pi\Sigma - \sqrt{G^2\pi^2\Sigma^2 - \kappa^2 v_s^2}}{v_s^2}\right\}, \left\{k \to \frac{-G\pi\Sigma + \sqrt{G^2\pi^2\Sigma^2 - \kappa^2 v_s^2}}{v_s^2}\right\}, \left\{k \to \frac{G\pi\Sigma + \sqrt{G^2\pi^2\Sigma^2 - \kappa^2 v_s^2}}{v_s^2}\right\}\right\}$$

We can rewrite these solutions by taking the Toomre parameter $Q \equiv \frac{v_s \kappa}{\pi G \Sigma}$:

$$|k| = \frac{G \pi \Sigma \pm \sqrt{G^2 \pi^2 \Sigma^2 - \kappa^2 v_s^2}}{v_s^2} = \frac{1 \pm \sqrt{1 - \left(\frac{\kappa v_s}{G \pi \Sigma}\right)^2}}{v_s^2 / G \pi \Sigma} = \frac{1 \pm \sqrt{1 - Q^2}}{v_s^2 / G \pi \Sigma}$$

We can evaluate the two critical values of k between which the disk is unstable to oscillations, noting that our disk has Q=0.75, $\Sigma[R_0] = \Sigma_0\, e^{-R_0/R_d}$, and taking a typical sound speed of 10 km/s:

$$\Sigma \to \left(2.6\ 10^8\ M_{sun} / kpc^2\right) e^{-(8\ kpc)/(3.5\ kpc)}$$

$$\Sigma \to \frac{2.64424 \times 10^7\ M_{sun}}{kpc^2}$$

The critical wavenumbers for instability are then:

$$k \to \frac{1 \pm \sqrt{1 - Q^2}}{v_s^2 / (G \pi \Sigma)}\ cm^{-1}\ /.\ \left\{Q \to 0.75,\ v_s \to 10^6,\ G \to 6.67\ 10^{-8},\ \Sigma \to 2.64\ 10^7\ M_{sun} / kpc^2\right\}\ /.\ \left\{M_{sun} \to 1.99\ 10^{33},\ kpc \to 3.086\ 10^{21}\right\}$$

$$k \to \frac{1.15595 \times 10^{-21}\ (1 \pm 0.661438)}{cm}$$

Hence the disk will be unstable to wavenumbers in the range:

$$\frac{1.21}{kpc} < |k| < \frac{5.93}{kpc}$$

or

$$1.06\ kpc < \lambda < 5.19\ kpc$$

As stable perturbations oscillate as $A \propto e^{i \omega t}$, unstable perturbations (with imaginary values of ω) will grow as $A \propto e^{|\omega| t}$, where $|\omega| = i\,\omega$ for imaginary ω, and hence perturbations will grow on a timescale $1/\omega$. Thus the timescale of the growth is (for a typical wavenumber of ~4/kpc):

$$\kappa \to Q \pi G \Sigma / v_s\ s^{-1}\ /.\ \left\{Q \to 0.75,\ v_s \to 10^6,\ G \to 6.67\ 10^{-8},\ \Sigma \to 2.64\ 10^7\ M_{sun} / kpc^2\right\}\ /.\ \left\{M_{sun} \to 1.99\ 10^{33},\ kpc \to 3.086\ 10^{21}\right\}$$

$$\kappa \to \frac{8.66966 \times 10^{-16}}{s}$$

$$\omega^2 \to \left(\kappa^2 - 2 \pi G \Sigma\ k + v_s^2 k^2\right) s^{-2}\ /.\ \left\{\kappa \to 8.67*\hat{}-16,\ v_s \to 10^6,\ G \to 6.67\ 10^{-8},\ \Sigma \to 2.64\ 10^7\ M_{sun} / kpc^2,\ k \to 4\ kpc^{-1}\right\}\ /.\ \left\{M_{sun} \to 1.99\ 10^{33},\ kpc \to 3.086\ 10^{21}\right\}$$

$$\omega^2 \to -\frac{5.64881 \times 10^{-31}}{s^2}$$

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τ_growth → (-ω^2)^(-1/2) s /. {ω^2 → -5.65*^-31, s → (3.15 10^7)^(-1) yr}
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τ_growth → 4.22343 × 10^7 yr
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Thus the disk instabilities would grow exponentially on a timescale of tens of millions of years.