Ay 124 — Midterm (Theoretical Part)

Posted on Tuesday, Feb, 3—Due by 5 pm on Wednesday, Feb 11 (directly to the TA)

Note:

You have up to 3 hours to do this part, and it counts for 60% of your midterm grade. Then you have up to 2 hours to do the observational part, which counts for 40%. You can do them in any order, with a break in between. The entire exam should be taken in a single, contiguous period of up to 8 hours (including both work parts and any breaks you may wish to take). Please mark your exams with the start and end times.

The Rules:

Closed book, closed notes, etc. No collaboration. You cannot discuss the exam with anyone, until everyone in the class has turned in their test.

Good Luck!

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1. a) [15 points] Consider a rotating bar in a galaxy. Let the z axis coincide with the principal axis of the tensor $\mathbf{I}$ and assume that the density distribution of the bar is stationary in a frame which rotates with angular frequency Ω. Show that, at an instant when $I_{xy} = 0$, the term $\frac{1}{2}\mathrm{d}^2\mathbf{I}/\mathrm{d}t^2$ which appears in the tensor virial theorem is a diagonal tensor with components $(I_{yy} - I_{xx})$, $(I_{xx} - I_{yy})$ and 0. Hence show that

$$\Omega^2 = \frac{(W_{xx} - W_{yy}) + 2(T_{xx} - T_{yy}) + (\Pi_{xx} - \Pi_{yy})}{2(I_{xx} - I_{yy})}. \tag{1}$$

b) [10 points] If $T_{zz} = 0$, show that

$$\frac{v_0^2}{\sigma_0^2} = (1-\delta)\frac{W_{xx} + W_{yy}}{W_{zz}} - 2, \tag{2}$$

where $v_0^2 \equiv 2(T_{xx} + T_{yy})/M$, $\sigma_0^2 \equiv (\Pi_{xx} + \Pi_{yy})/2M$ and $(1-\delta)(\Pi_{xx} + \Pi_{yy}) \equiv 2\Pi_{zz}$. (Hint: The tensor transformation between the rotating and non-rotating frames is

$$\mathbf{I}_{\mathrm{non-rot}} = \begin{pmatrix} \cos\Omega t & \sin\Omega t & 0 \\ -\sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{I}_{\mathrm{rot}} \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

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2. a) [10 points] By taking a suitable moment of the collisionless Boltzmann equation, show that in a steady-state axisymmetric galaxy

$$\frac{\partial(\nu\overline{v_R^2 v_\phi})}{\partial R} + \frac{\partial(\nu\overline{v_R v_z v_\phi})}{\partial z} - \frac{\nu}{R}\left(\overline{v_\phi^3} - \overline{v_\phi}R\frac{\partial\Phi}{\partial R}\right) + \frac{2\nu}{R}\overline{v_R^2 v_\phi} = 0. \tag{4}$$

b) [15 points] Given that the system is symmetric in z, and that all odd moments of $v_\phi - \overline{v_\phi}$ vanish, so $\overline{v_R^2(\overline{v_\phi^3} - \overline{v_\phi}R\frac{\partial\Phi}{\partial R})} = 0$ and $\overline{(\overline{v_\phi^3} - \overline{v_\phi}R\frac{\partial\Phi}{\partial R})^2} = 0$, etc., show that at $z = 0$

$$\overline{v_R^2}\left(\frac{\partial\overline{v_\phi}}{\partial R} + \frac{\overline{v_\phi}}{R}\right) - \frac{2}{R}\overline{v_\phi}\overline{(v_\phi - \overline{v_\phi})^2} = 0. \tag{5}$$

Hence, using the Jeans equation

$$\frac{\partial(\nu\overline{v_R^2})}{\partial R} + \frac{\partial(\nu\overline{v_R v_z})}{\partial z} + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial\Phi}{\partial R}\right) = 0, \tag{6}$$

show that

$$\frac{\sigma_\phi^2}{\sigma_R^2} \equiv \frac{\overline{(v_\phi - \overline{v_\phi})^2}}{\overline{v_R^2}} \approx \frac{-B}{A-B}, \tag{7}$$

where $A = -\frac{1}{2}R\mathrm{d}\Omega\mathrm{d}R$ and $B = -(\Omega + \frac{1}{2}R\mathrm{d}\Omega/\mathrm{d}R)$ (with $\Omega^2 = R^{-1}(\partial\Phi/\partial R)_{z=0}$) are the Oort constants.

3. [10 points] Prove that a system of N self-gravitating point masses with positive energy must disrupt, in the sense that at least one star must escape. Hint: use the virial theorem, and prove that the moment of inertia must increase without limit.