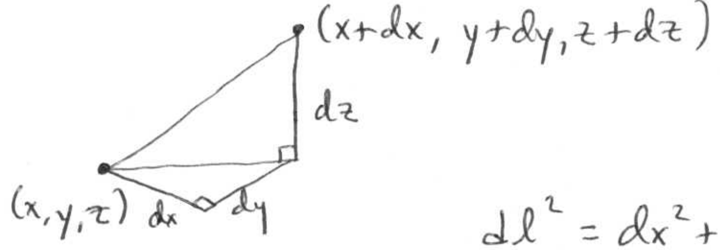
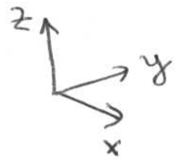
1) Homogeneous, Isotropic Spaces.

First possibility is flat 3D Euclidean space.
Distances given by Pythagorean theorem:



$$dl^2 = dx^2 + dy^2 + dz^2.$$



Can also write in spherical coordinates:

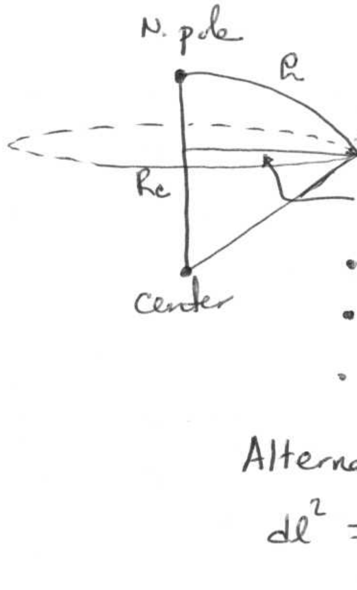
$$dl^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\phi^2.$$

Both useful:

- perturbation analyses, simulations easiest in Cartesian form.
- observations easiest in spherical form.

Other possibilities for isotropic space? Consider 3-sphere:

$$dl^2 = d\rho^2 + R_c^2 \sin^2\frac{\rho}{R_c}\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad (\text{"closed space"})$$



$R_c \sin\frac{\rho}{R_c}$. (one dimension suppressed).

- Finite. $V = 2\pi^2 R_c^4$.
- No special point on surface
- No special direction.

Alternate forms:

$$dl^2 = \frac{dx^2}{1 - R_c^{-2}x^2} + x^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad \text{sub. } x = R_c \sin\frac{\rho}{R_c}.$$

$$dl^2 = \frac{dx^2 + dy^2 + dz^2}{1 + \frac{1}{4R_c^2}(x^2 + y^2 + z^2)} \qquad (\text{stereographic projection}).$$

Also possible to take $R_c^2 \to -R_c^2$ (imaginary radius of curvature!)

$$\Rightarrow \quad d\ell^2 = d\rho^2 + R_c^2 \sinh^2 \frac{\rho}{R_c} \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad (\text{"open space"})$$

- Infinite!
- Still homogeneous & isotropic.

Fourth possibility? Projective sphere. Identify antipodal points on the sphere. Volume now $V = \pi^2 R_c^4$! But no different from sphere unless R_c is small enough to see the "same" object. So we won't talk about this more in this class.

Summary of spaces:

	Euclidean	Closed	Open
Volume	∞	$2\pi^2 R_c^4$	∞
Sum of angles in triangle.	π	$>\pi$	$<\pi$
Pythagorean theorem	$a^2+b^2=c^2$	$a^2+b^2>c^2$	$a^2+b^2<c^2$
Area of circle	$\pi\rho^2$	$<\pi\rho^2$	$>\pi\rho^2$
Volume of sphere	$\frac{4}{3}\pi\rho^3$	$<\frac{4}{3}\pi\rho^3$	$>\frac{4}{3}\pi\rho^3$

2) Robertson-Walker metric.

The Universe can *expand* while remaining homogeneous & isotropic. Suppose distances vary with time according to the scale factor $a(t)$:

$$dl \propto a(t).$$

$$\ell \propto a(t).$$

$$R_c \propto a(t).$$

Let's define:

$$\ell = r\, a(t) \qquad r = \text{comoving radial distance}$$

$$R_c = \mathcal{R}\, a(t) \qquad \mathcal{R} = \text{comoving radius of curvature}$$

Then lengths vary according to:

$$dl^2 = a(t)^2 \left[dr^2 + \begin{Bmatrix} r^2 \\ \mathcal{R}^2 \sin^2 r/\mathcal{R} \\ \mathcal{R}^2 \sinh^2 r/\mathcal{R} \end{Bmatrix} \cdot \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right].$$

(flat, closed, open).

In relativity we more commonly discuss the proper time interval between two neighboring events, which is:

$$ds^2 = \underbrace{dt^2}_{\text{time}} - \frac{\overbrace{dl^2}^{\text{spatial distance}}}{c^2}$$

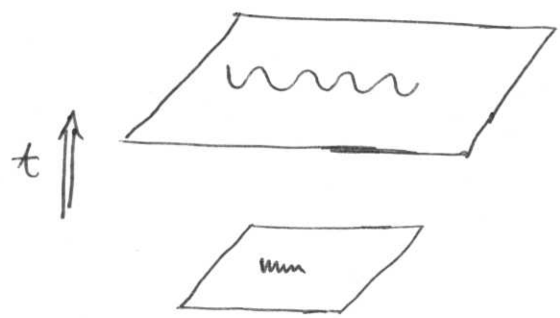
or:
$$ds^2 = dt^2 - \frac{a(t)^2}{c^2} \left[dr^2 + \begin{Bmatrix} r^2 \\ \mathcal{R}^2 \sin^2 \frac{r}{\mathcal{R}} \\ \mathcal{R}^2 \sinh^2 \frac{r}{\mathcal{R}} \end{Bmatrix} \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) \right].$$

Notes:
- RW metric has a global cosmic time t. Property of FRW universe - not all cosmologies have this.
- Spherical symmetry is manifest. Surface at constant t & constant radial coordinate r is a 2-sphere.
- Definition: $r_1 = \begin{Bmatrix} r \\ \mathcal{R} \sin r/\mathcal{R} \\ \mathcal{R} \sinh r/\mathcal{R} \end{Bmatrix}$ is comoving angular diameter distance.

3) Observational parameters.

Let's suppose we're in an RW universe and want to determine $a(t)$ and R (and if we are flat/closed/open). Need observations. Will develop theoretical tools here... actual data & techniques are in next lecture.

A) Redshift.



Wavelength of photons $\lambda \propto a(t)$ (stretched by expansion of universe).

Therefore if a photon is emitted at time t_1 and observed today @ t_0:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_0)}{a(t_1)}, \quad \text{or:} \quad \frac{\lambda_{obs}}{\lambda_{em}} = \frac{1}{a(t_1)}.$$

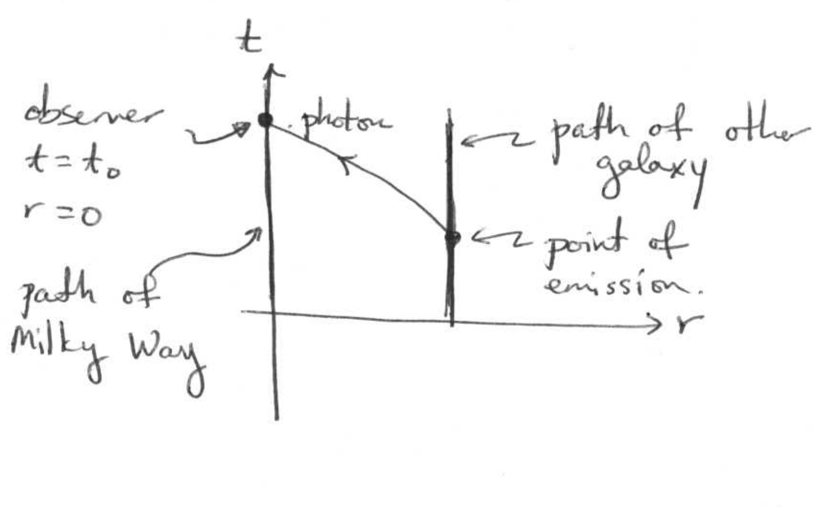
[By convention, $a(t_0) = 1$.]

Define redshift, $z \equiv \frac{\lambda_{obs}}{\lambda_{em}} - 1 = \frac{1}{a(t_1)} - 1$.

Since $a(t_1) < 1$ (Universe smaller in past), $z > 0$.

The redshift of an object is related to radial comoving distance – more distant objects are more redshifted since we see them at an earlier stage in cosmic time.

Can determine $z - r$ relation by considering path of photon:



Photon travels at the speed of light so on trajectory:

$$dl = -c\,dt \qquad (-\text{ sign } = \text{ toward observer})$$

Here, for radial trajectory, $dl = a(t)\,dr$ so:

$$dr = -\frac{c\,dt}{a(t)}.$$

Integrate:

$$\int_{r_{em}}^{0} dr = -c \int_{t_{em}}^{t_0} \frac{dt}{a(t)} \qquad \text{or:}$$

$$r_{em} = c \int_{t_{em}}^{t_0} \frac{dt}{a(t)}.$$

Thus for given $a(t)$ we can find the coordinate distance to an object:

$$\boxed{r = c \int_{t_1}^{t_0} \frac{dt}{a(t)}}$$

Warnings:

- r is not directly measurable. Just a coordinate.
- Usually want inverse problem: data $\Rightarrow a(t)$, not $a(t) \Rightarrow$ data.

B) Hubble law.

Since separation of two infinitesimally seperated objects is $\dot{\ell} \propto \ell$:

$$\ell \propto a(t)$$

$$\dot{\ell} = \ell \frac{\dot{a}}{a}$$

we define Hubble rate: $H(t) = \dfrac{\dot{a}(t)}{a(t)}$ and

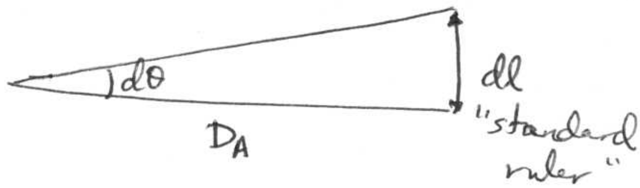
$$\dot{\ell}(t) = H(t)\,\ell(t).$$

$\uparrow$ relative velocity.

Current value $= H(t_0) = H_0$.

C) Angular diameter distances.

Suppose we observe an object of known length dl. Then we can determine its distance by finding the angle $d\theta$ it subtends.



$$D_A = \frac{dl}{d\theta} = \text{angular diameter distance.}$$

From metric: if from length of object only $d\theta$ is nonzero,

$$dl = a(t) \begin{Bmatrix} r \\ R \sin r/R \\ R \sinh r/R \end{Bmatrix} d\theta \quad \text{so}$$

$$D_A = a(t) \begin{Bmatrix} r \\ R \sin r/R \\ R \sinh r/R \end{Bmatrix} = \frac{1}{1+z} \begin{Bmatrix} r \\ R \sin r/R \\ R \sinh r/R \end{Bmatrix}.$$

Using relation for r, this allows us to relate D_A to t (and hence z) for any $a(t)$.

D) Luminosity distance.

Alternative way to measure distance is to use an object of known luminosity ("standard candle").

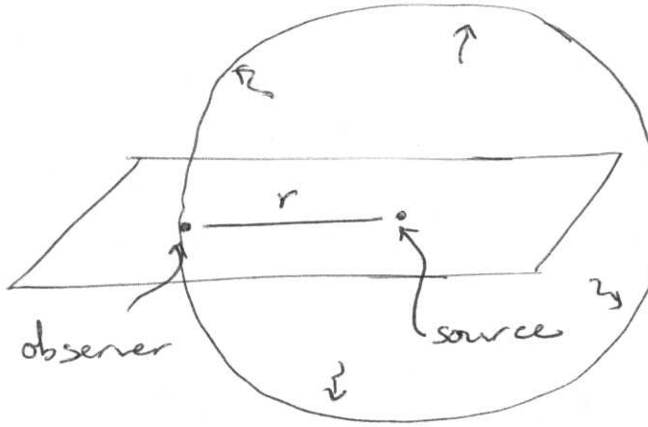
$$F = \frac{L}{4\pi D_L^2}$$

F → flux (erg/s/cm^2)

L → luminosity (erg/s)

D_L → luminosity distance (cm)

To get flux, note that photons from an object at angular diameter distance D_A are today spread over a sphere of radius ~~$(1+z)D_A$~~ r.



Therefore, the surface area of the sphere is:

$$A = 4\pi \begin{cases} r^2 \\ R^2 \sin^2 r/R \\ R^2 \sinh^2 r/R \end{cases}.$$

The flux is <u>not</u> L/A for two reasons:

- Each photon carries less energy since λ is longer (factor of $1+z$).
- Time between arrival of photons is stretched (factor of $1+z$).

$\Rightarrow$ $$F = \frac{L}{(1+z)^2 A} \quad \text{and} \quad D_L = (1+z)\sqrt{\frac{A}{4\pi}} \quad \text{or:}$$

$$D_L = (1+z) \begin{cases} r \\ R \sin r/R \\ R \sinh r/R \end{cases} = (1+z)^2 D_A.$$

general.

E) Comoving Volume, Number Counts.

Suppose we have objects (e.g. electrons) that are conserved as the Universe expands. Then their <u>comoving number density</u> n is fixed. (units: objects/Mpc^3)

We may count these objects in a given range of redshift & solid angle. The ~~comoving~~ volume is:

$$d\ell^2 = a(t)^2 \left[dr^2 + r_1^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right]$$

$\Rightarrow$ length element on the 3 axes is:

$$a(t)\,dr, \quad a(t) r_1 \, d\theta, \quad a(t) r_1 \sin\theta \, d\phi$$

so: $$dV_{phys} = a(t)^3 r_1^2 \sin\theta \, dr \, d\theta \, d\phi.$$

Comoving volume is then:

$$dV = D^2 \sin\theta \, dr \, d\theta \, d\phi \qquad (D \equiv r_1).$$

Count objects per unit redshift per unit solid angle:

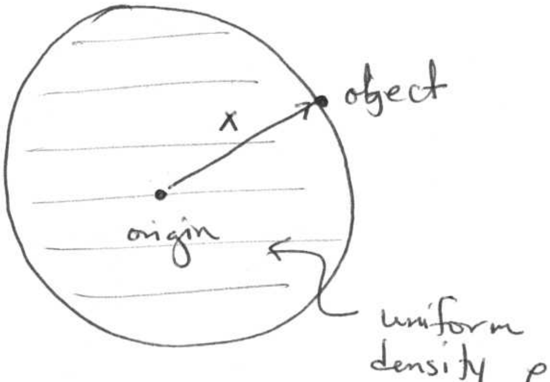
$$dz = d\left[\frac{1}{a(t)} - 1\right] = -\frac{\dot{a}\,dt}{a^2} = -\frac{\dot{a}}{a^2}\left(-\frac{a\,dr}{c}\right) = \frac{H\,dr}{c},$$

$$d\Omega = \sin\theta \, d\theta \, d\phi.$$

Then: $$\frac{dN}{dz\,d\Omega} = \frac{n\,D^2 \sin\theta \, dr \, d\theta \, d\phi}{\frac{H\,dr}{c} \sin\theta \, d\theta \, d\phi} = nc\,\frac{D^2}{H}.$$

4) Dynamics & the Friedmann Equations

Up until now we've only done geometry. Time for dynamics. Will do Newtonian calculation here. Consider dynamics of uniform sphere.



$$\ddot{x} = -\frac{GM}{x^2} = -\frac{4}{3}\pi G \rho x .$$

Since $x \propto a(t)$, scale factor in Newtonian model is:

$$\ddot{a} = -\frac{4}{3}\pi G \rho a .$$

Now the density $\rho = \rho_0 / a^3$ for matter, so:

$$\ddot{a} = -\frac{4}{3}\pi G \frac{\rho_0}{a^2} .$$

Can find first integral: (multiply by $2\dot{a}$, integrate):

$$\underbrace{\int 2\dot{a}\ddot{a}\, dt}_{\dot{a}^2} = -\frac{8}{3}\pi G \rho_0 \underbrace{\int \frac{dt\, \dot{a}}{a^2}}_{-\frac{1}{a}}$$

so:

$$\dot{a}^2 = \frac{8}{3}\pi G \rho a^2 + \text{constant}.$$

(Friedmann eq.) Argument is correct in GR because of Birkhoff's thm on spherically symmetric systems.

GR further tells us via constraint equation:

$$\boxed{\dot{a}^2 = \frac{8}{3}\pi G \rho a^2 \mp \frac{c^2}{R^2}}$$

(matter $\Leftrightarrow$ geometry).

+ = open

− = closed.

We may also re-write as:

$$H^2 = \frac{8}{3}\pi G \rho \mp \frac{c^2}{a^2 R^2}.$$

Definitions: • The <u>critical density</u> $\rho_c \equiv \frac{3H^2}{8\pi G}$.

Universe is:
- flat if $\rho = \rho_c$ $(R \to \infty)$
- open if $\rho < \rho_c$
- closed if $\rho > \rho_c$.

• <u>Density parameter</u> $\frac{\rho}{\rho_c} = \Omega$.
- Can break up: $\Omega = \Omega_{baryon} + \Omega_{dm} + \Omega_{de}$.
- define $\Omega_K \equiv 1 - \Omega = \mp \frac{c^2}{a^2 R^2 H^2}$.
- Generally a function of time, but usually quote values at present epoch.

Phenomenology:

A) Einstein-de Sitter model. $\Omega = 1$, no Λ.

Always have $H = \sqrt{\frac{8}{3}\pi G \rho} = \sqrt{\frac{8}{3}\pi G \rho_0}\, a^{-3/2} = H_0 a^{-3/2}$.

$$\frac{da}{dt} = aH = H_0 a^{-1/2}.$$

Solve dif. eq. $\Rightarrow$ $a^{3/2} = \frac{3}{2} H_0 t + \text{constant}$.

Define origin of time as $t=0$ @ Big Bang, so:

$$a = (t/t_0)^{2/3}, \quad H_0 = \frac{2}{3t_0}.$$

B) Matter + Curvature model. $\Omega \neq 1$, no Λ.

Again we have a Friedmann equation:

$$H^2 = \underbrace{\frac{8}{3}\pi G\rho}_{\propto a^{-3}} \mp \underbrace{\frac{c^2}{a^2R^2}}_{\propto a^{-2}}.$$

In terms of critical density:

$$H^2 = \frac{8}{3}\pi G\rho_o\, a^{-3} \mp \frac{c^2}{a^2R^2}$$

$$H^2 = H_o^2\,\Omega\, a^{-3} + H_o^2(1-\Omega)\, a^{-2}.$$

$$\frac{H^2}{H_o^2} = \frac{\Omega}{a^3} + \frac{1-\Omega}{a^2}.$$

As the Universe becomes large:

- For $\Omega<1$, the 2nd term dominates and the long-term future is $H \propto 1/a$ or (since $H = \dot{a}/a$)

  $\dot{a} \to$ constant.

  Universe expands forever.

- For $\Omega>1$, the 2nd term is negative and H reaches zero when:

  $$a = a_{max} \equiv \frac{\Omega}{\Omega - 1}.$$

  Then the Universe turns around ($\ddot{a}<0$) and ends in the far future with a Big Crunch.

c) Matter + Λ, Flat.

$$\rho = \rho_m + \rho_\Lambda$$

$\rho_m \propto a^{-3}$ normal matter

$\rho_\Lambda = \text{const.}$ cosmological constant.

Density at other epochs:

$$\rho(a) = \rho_{c0}\left(\frac{\Omega_m}{a^3} + \underbrace{1-\Omega_m}_{\Omega_\Lambda}\right)$$

$$\frac{H^2}{H_0^2} = \frac{\Omega_m}{a^3} + 1 - \Omega_m .$$

For $\Omega_m < 1$, the far future is that $H \to$ constant; thus:

$$a \propto e^{H_{\text{future}}\, t} = \exp\left\{\sqrt{\tfrac{8}{3}\pi G \rho_\Lambda}\; t\right\}.$$

Universe will expand <u>exponentially</u>.

For open or Λ models, $D_A(z)$ etc. have no analytic solution but can be found numerically. But note the following property of Λ:

$$\frac{\dot a}{a} = H_0 \sqrt{\frac{\Omega_m}{a^3} + 1 - \Omega_m} \quad \text{or:}$$

$$\dot a = H_0 \sqrt{\frac{\Omega_m}{a} + (1-\Omega_m)a^2} .$$

This passes through a <u>minimum</u> at $a = \sqrt[3]{\dfrac{\Omega_m}{2(1-\Omega_m)}}$. At more recent times, the Universe is accelerating, $\ddot a > 0$, an unmistakable sign of something exotic (like Λ) if observed.