Ay 20
Fall 2004

Kepler's Laws, Binaries, and Stellar Masses

Kepler's Laws:

1. The orbits of planets are elliptical, with the Sun at a focus
2. Radius vectors of planets sweep out equal areas per unit time
3. Squares of orbital periods are proportional to cubes of semimajor axes:

$$P^2\,[\mathrm{yr}] = a_{pl}^3\,[\mathrm{au}]$$

- Derived empirically from Tycho de Brahe's data
- Explained by the Newton's theory of gravity

Orbits in a Gravitational Potential

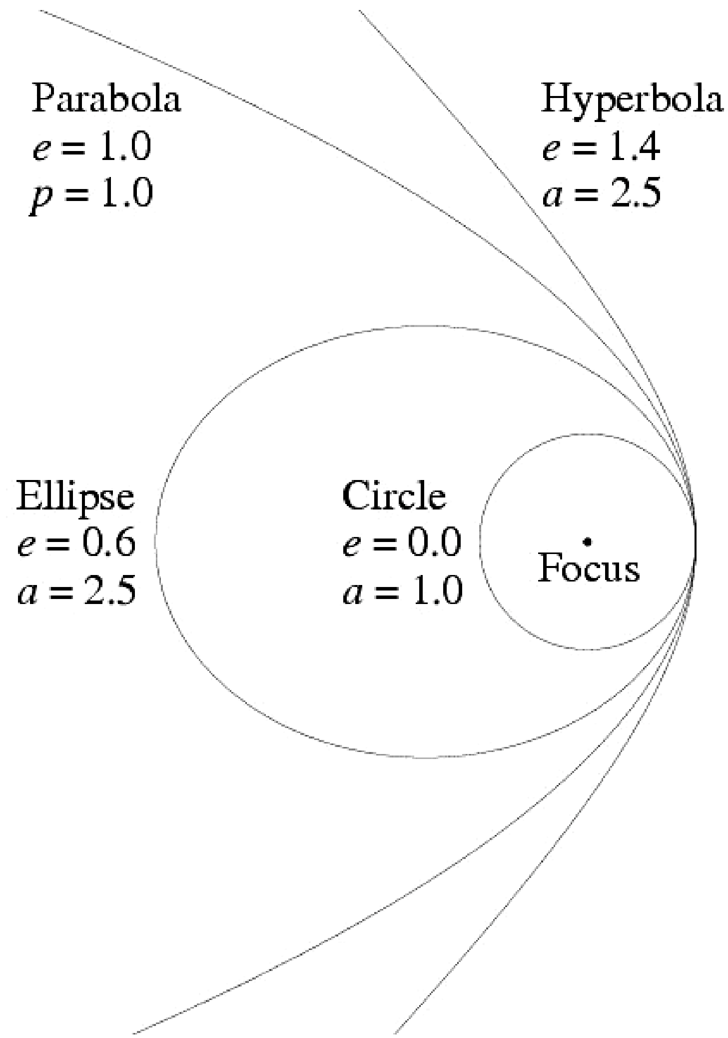
For a point mass (or spherically symmetric one), they are always the conic sections:

The shape depends on the sign of the total energy, $E_{tot} = E_{kin} - E_{pot}$:

$E_{tot} < 0 \rightarrow$ Ellipse

$E_{tot} = 0 \rightarrow$ Parabola

$E_{tot} > 0 \rightarrow$ Hyperbola



For the elliptical orbits, the eccentricity depends on the angular momentum: circular orbits have the maximum ang. mom. for a given energy

Kepler's 2nd Law: A quick and simple derivation

Angular momentum, at any time: $J = M_{pl}\, V\, r = const.$

Thus: $V\, r = const.$ (this is also an "adiabattic invariant")

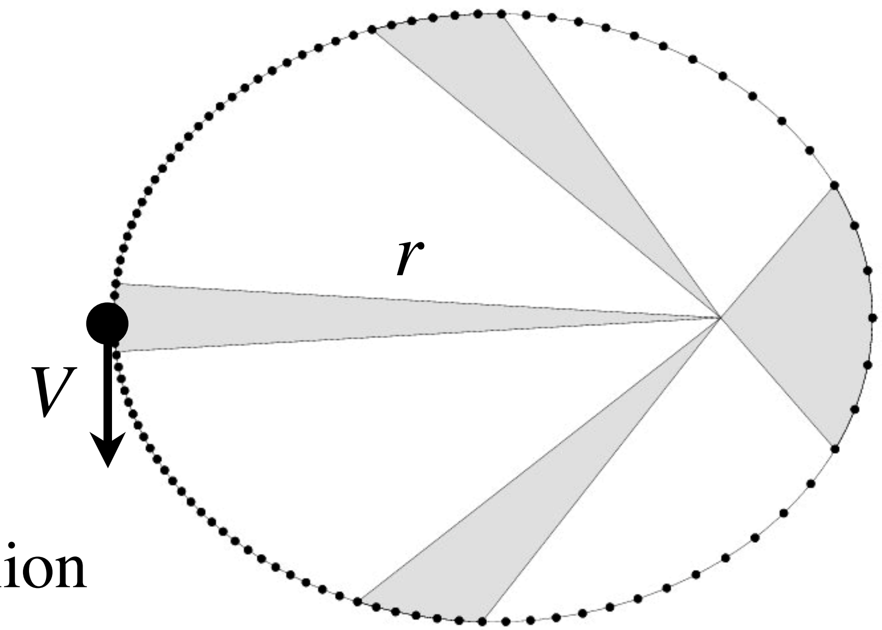
Element of area swept: $dA = V\, r\, dt$

Sectorial velocity: $dA/dt = V\, r = const.$

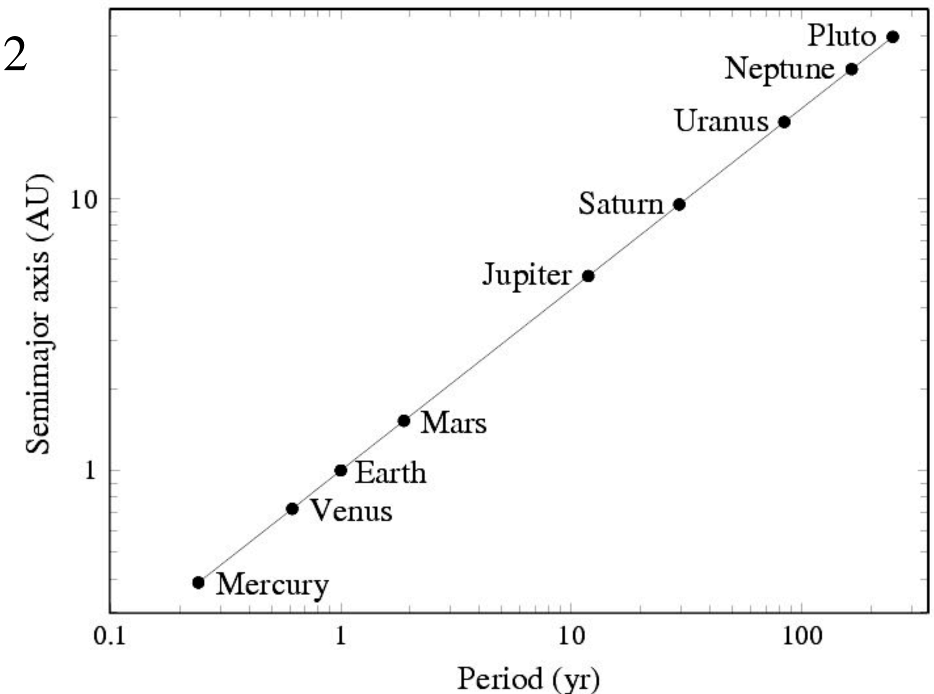
Independent of M_{pl} !

It is ***a consequence of the conservation of angular momentum.***

Planets move slower at the aphelion and faster at the perihelion



Kepler's 3rd Law: A quick and simple derivation



$F_{cp} = G\, M_{pl}\, M_\odot / (a_{pl} + a_\odot)^2$

$\approx G\, M_{pl}\, M_\odot / a_{pl}^{\ 2}$

(since $M_{pl} << M_\odot$, $a_{pl} >> a_\odot$)

$F_{cf} = M_{pl}\, V_{pl}^{\ 2} / a_{pl}$

$= 4\, \pi^2\, M_{pl}\, a_{pl} / P^2$

(since $V_{pl} = 2\, \pi\, a_{pl} / P$)

$F_{cp} = F_{cf}$ ➞ $\boxed{4\, \pi^2\, a_{pl}^{\ 3} = G\, M_\odot\, P^2}$ (independent of M_{pl} !)

Another way: $E_{kin} = M_{pl} V_{pl}^{\ 2} / 2 = E_{pot} \approx G\, M_{pl} M_\odot / a_{pl}$

Substitute for V_{pl} : $4\, \pi^2\, a_{pl}^{\ 3} = G\, M_\odot\, P^2$

➞ It is ***a consequence of the conservation of energy***

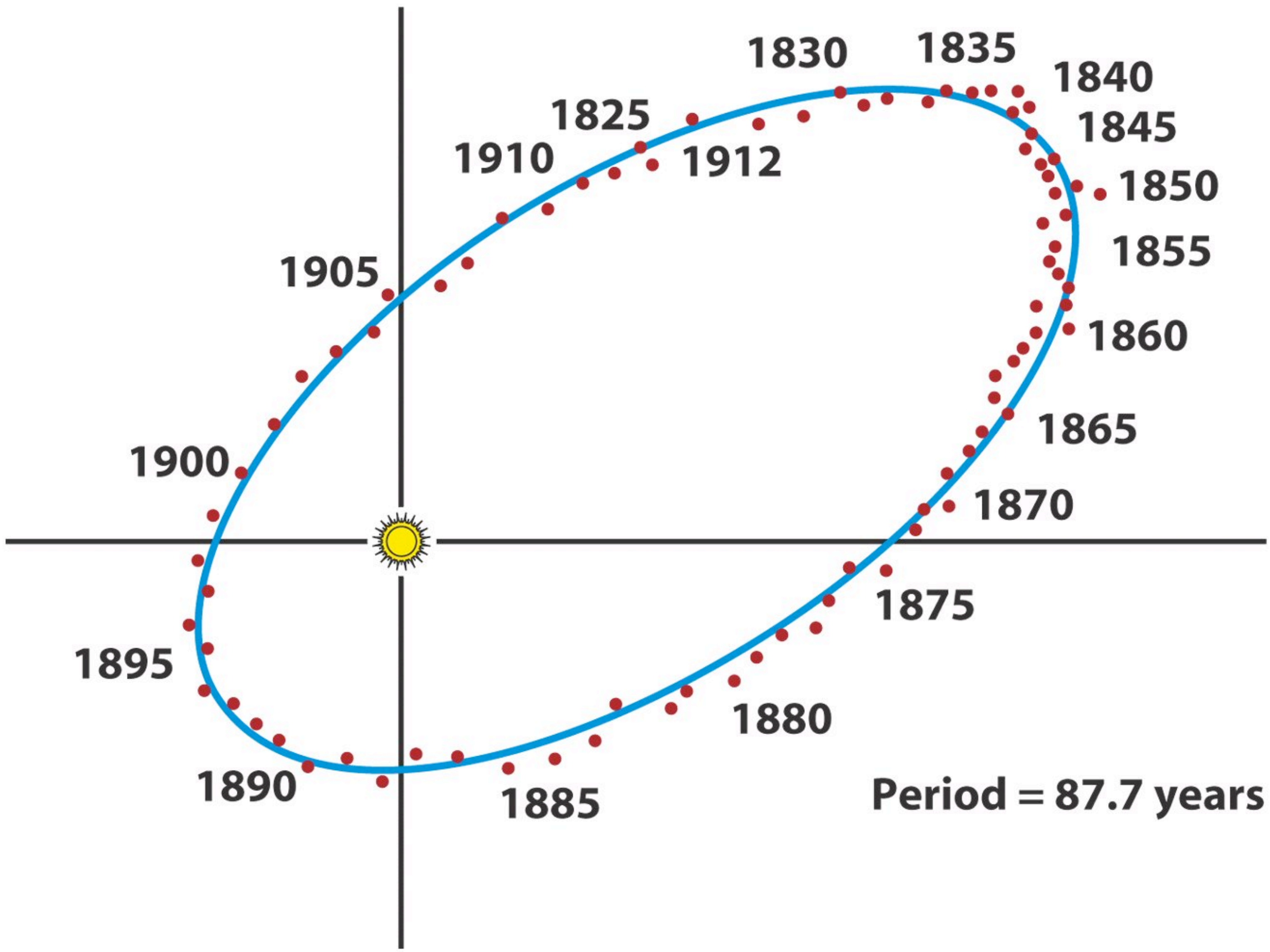
It Is Actually A Bit More Complex …

- Kepler’s laws are just an approximation: we are treating the whole system as a collection of isolated 2-body problems
- There are no analytical solutions for a general problem with > 2 bodies! But there is a good ***perturbation theory,*** which can produce very precise, but always approximate solutions
- Relativistic effects can be incorporated (➙ tests of GR)
- Dynamical resonances can develop (rotation/revolution periods; asteroids; Kirkwood gaps; etc.)
- If you wait long enough, more complex dynamics can occur, including ***chaos!***

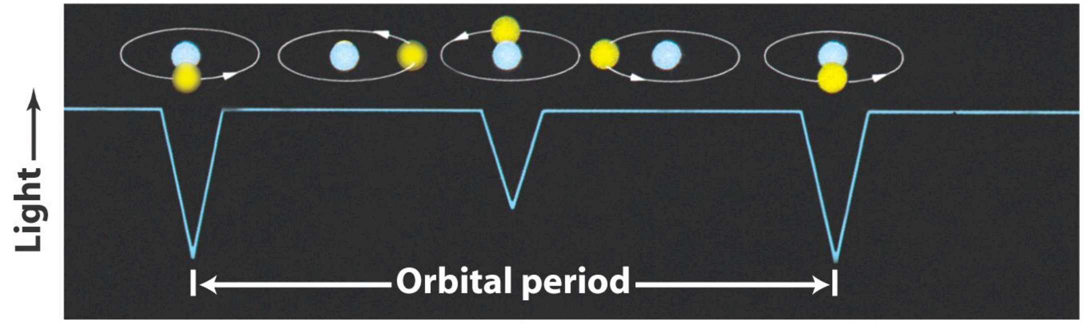
Binary Stars

1. **Optical/apparent:** not physical pairs, just a projection effect
2. **Physical binaries:** bound by gravity (Ekin = Epot), orbiting a common center of mass
   - ***Visual or astrometric:*** the binary is resolved, and the orbit can be mapped via precise astrometry
   - ***Eclipsing:*** orbital plane close to the line of sight, the stars occult each other, as seen in the light curve; generally unresolved
   - ***Spectroscopic:*** unresolved, but 2 line systems are seen in the combined spectrum, with periodic and opposite Doppler shifts

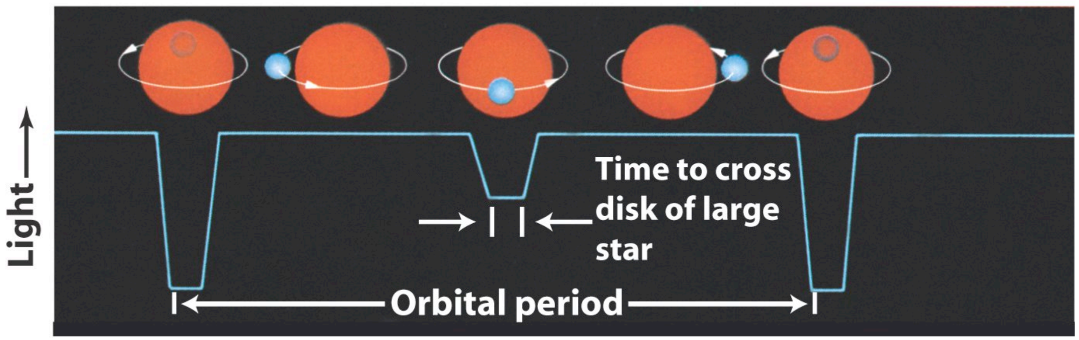
A Visual Binary Star System



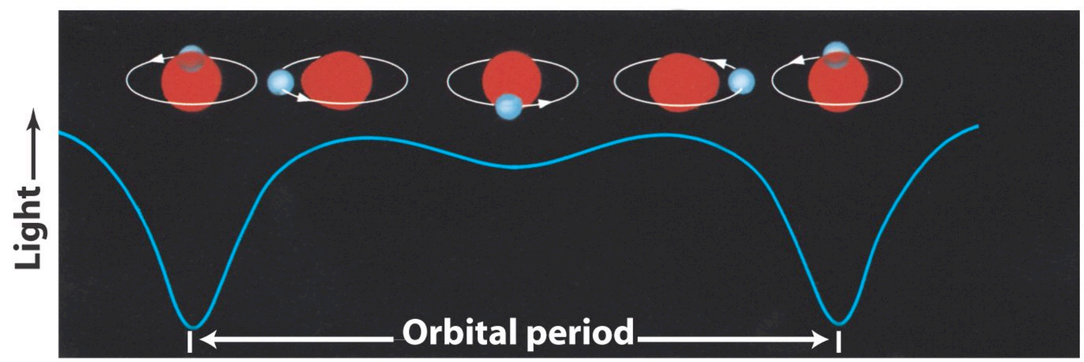
The shape of the light curve of an eclipsing binary depends on the types of stars involved (their L, R, T), and the inclination of the orbit:



Partial eclipse Time →



Total eclipse Time →

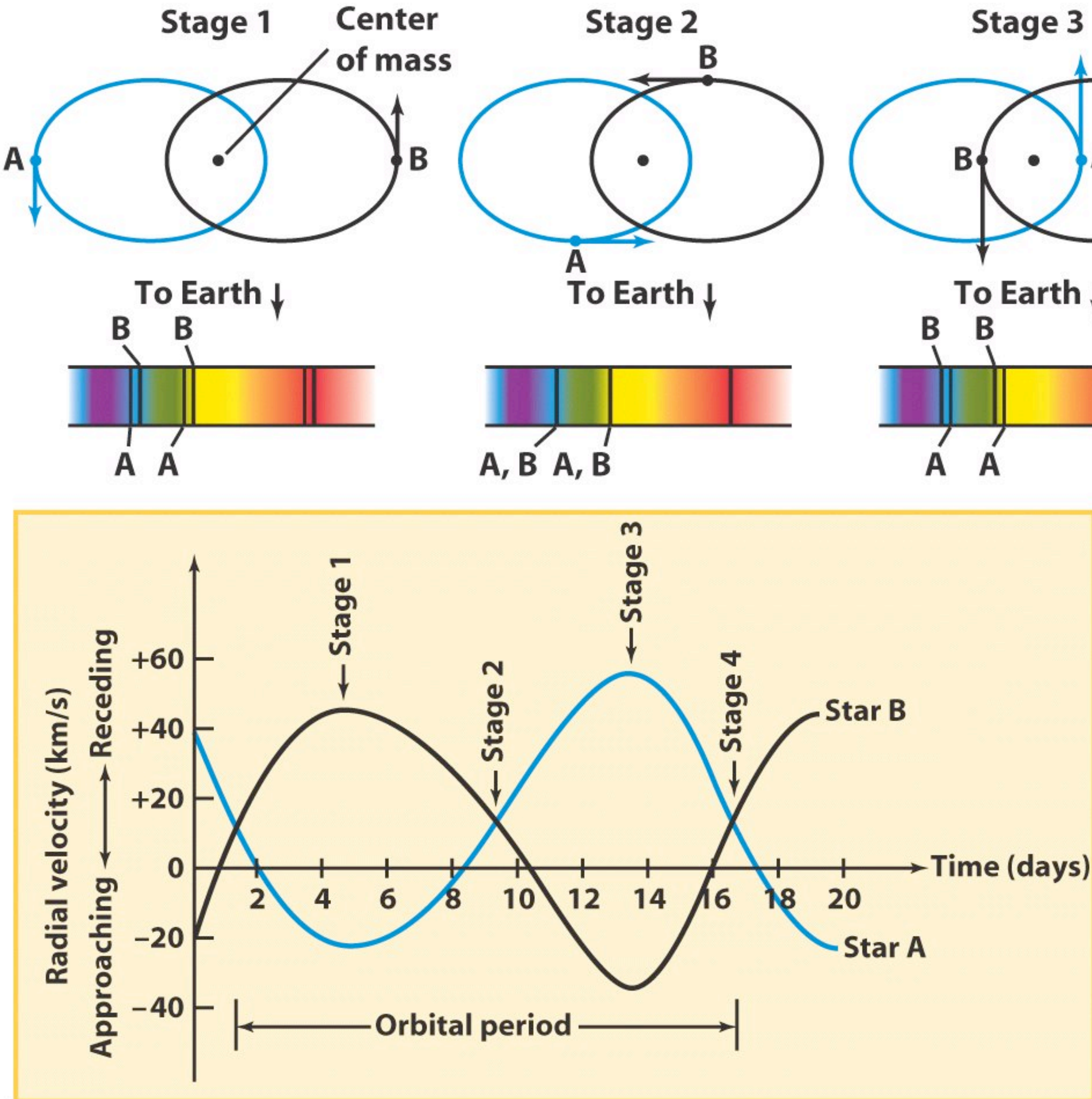


Tidal distortion Time →

Spectroscopic Binaries

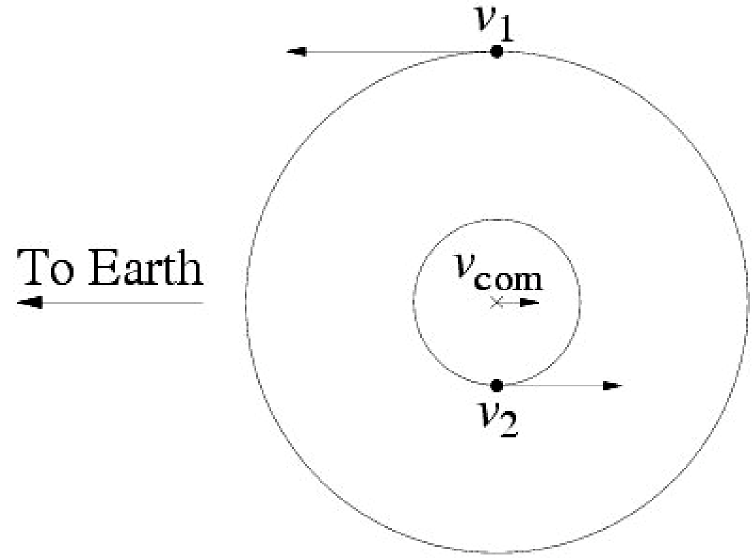
- Some unresolved binaries have spectra with the absorption lines for two distinctly different spectral types

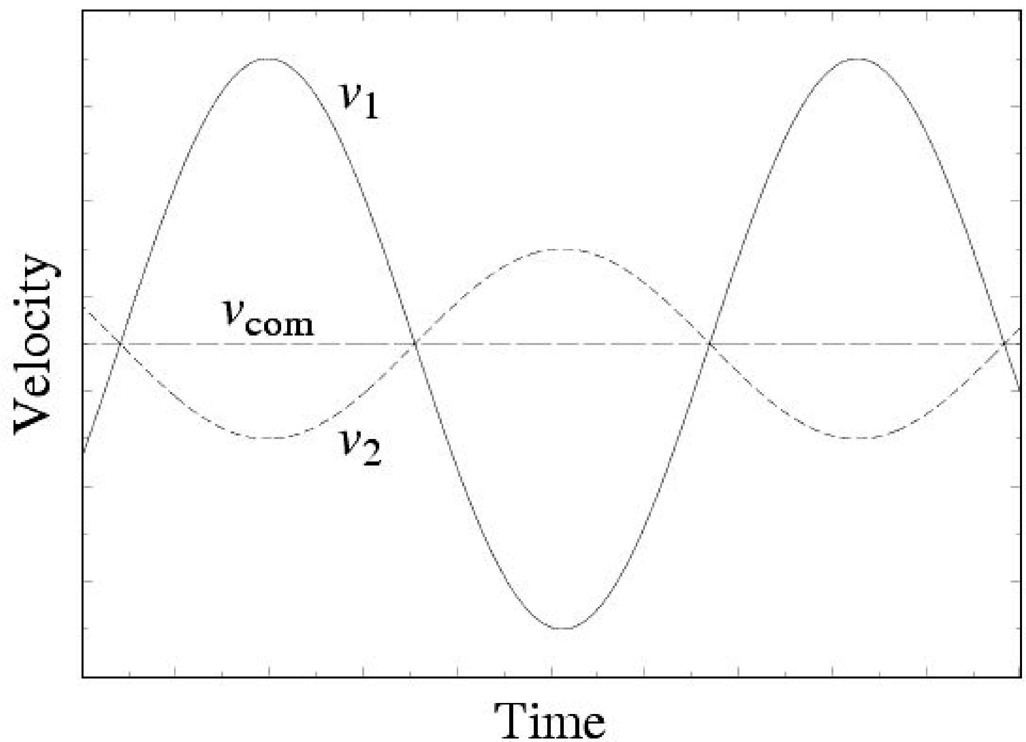
- A spectroscopic binary has spectral lines that shift back and forth in wavelength, due to the Doppler effect



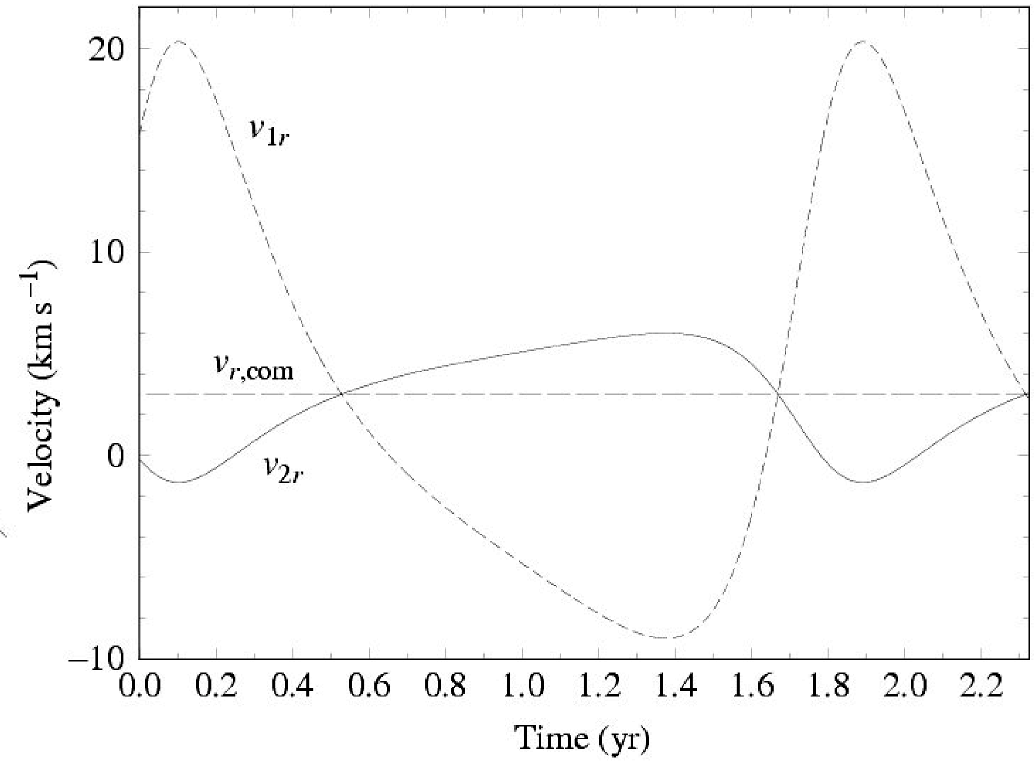
Velocity curves for a spectroscopic binary

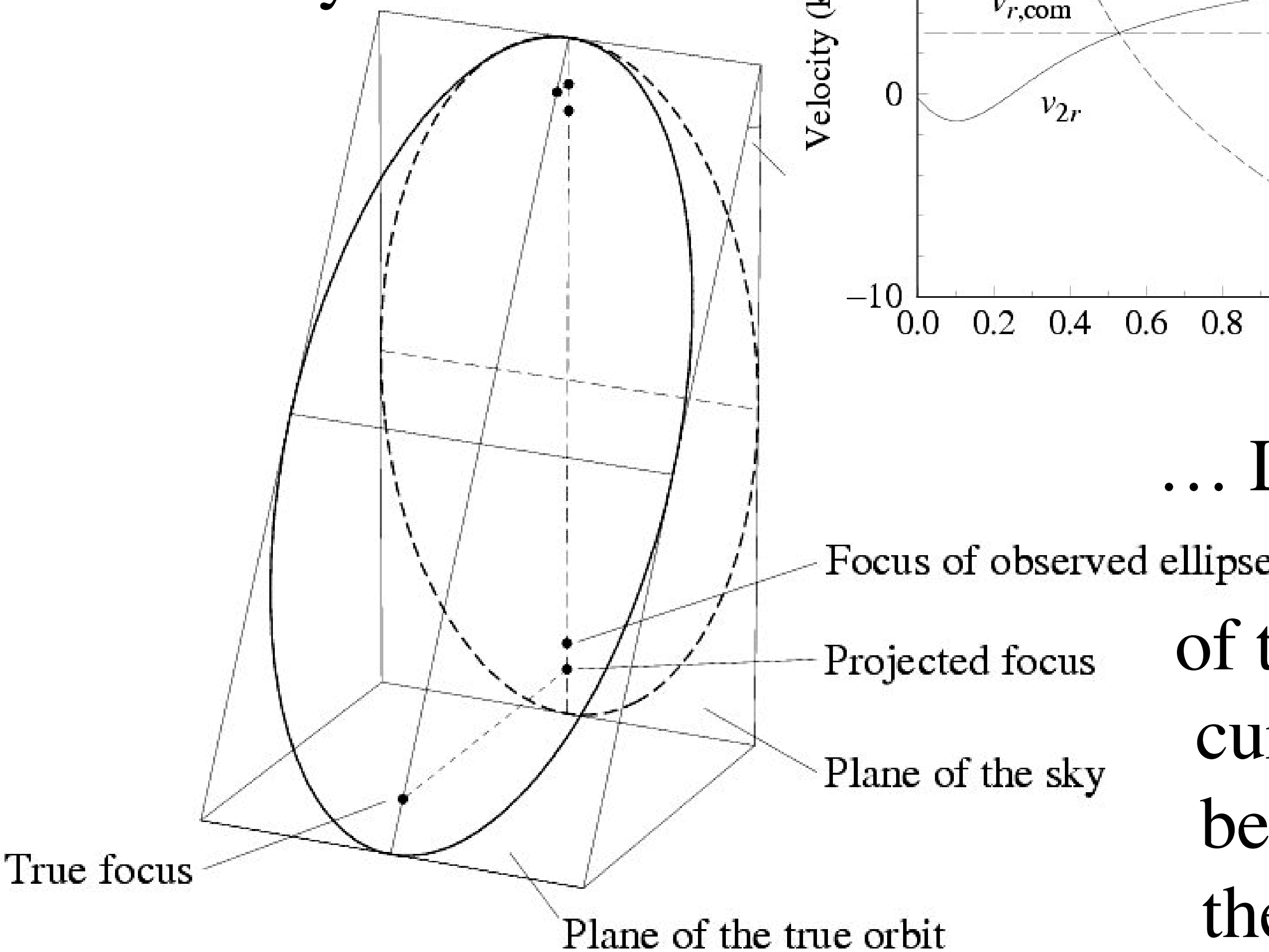
A simple orbit geometry and orientation leads to a simple and easy way to interpret the radial velocity curves





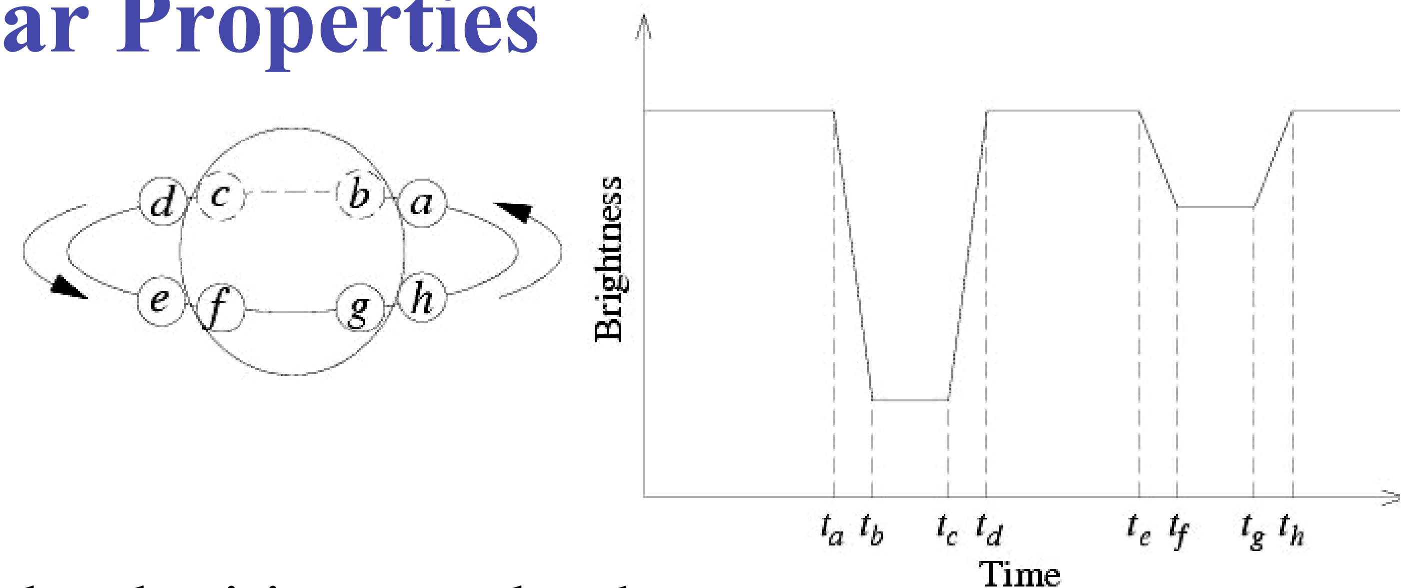
In general, the orbital plane will be tilted relative to the plane of the sky …





… Leading to a more complex shape of the radial velocity curves, which could be used to constrain the orbit orientation

Using Eclipsing Binaries to Measure Stellar Properties



If radial velocities are also known,
from the transit times we can compute ***stellar radii.***
If we measure the temperatures from stellar spectra, we can then estimate the ***luminosities***, and from the apparent brightness the ***distance*** to the system.
From the velocities and periods, we can get the orbit size, and then the ***masses***.

Binary Stars …are common:

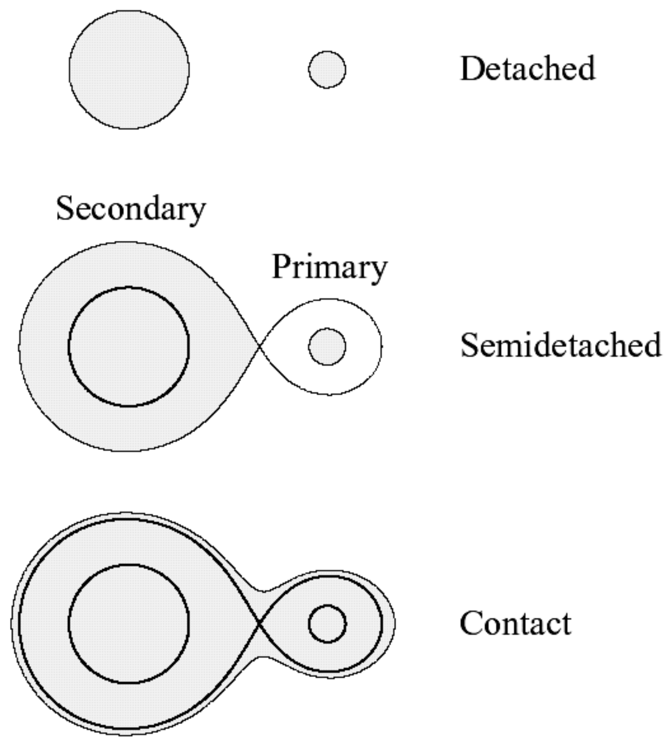
"3 out of every 2 stars are in binaries"

Possible origins:

1. ***Born that way:*** Fragmentation of a protostellar disk with an excess angular momentum; or formed close together and gravitationally bound via (2) or (3)
   - Similar story for the planet formation
2. ***Tidal capture:*** Nearby passage with energy tidal dissipation ➔ gravitationally bound pair
   - The impact parameter has to be just right: close enough for an effective interaction, but no merger
3. ***Three-body interaction:*** One star in a triple encounter takes away extra E_{kin}, and the other two get bound

Binaries: The Chief Physical Distinction

- Binaries can be ***detached*** (no physical contact or material exchange), or ***contact / interacting*** (some mass exchange)



- This can change as the stars evolve. If one star swells beyond its Roche lobe (the equipotential surface of the two stars), its material will usually flow onto its companion

- Stellar interactions can lead to all kinds of interesting variability (cataclysmic variables, novae, etc.) and even supernova explosions

Stellar Masses

- The most important physical property of stars: determines everything else (L, T, evolution …)
  - Mass-luminosity relation is a key concept
- It is only from binary stars that we can accurately determine the masses of individual stars
- Eclipsing binaries are the most useful in that the masses and radii of the individual stars can be determined from the light curve and radial velocity data, using the Kepler's laws:

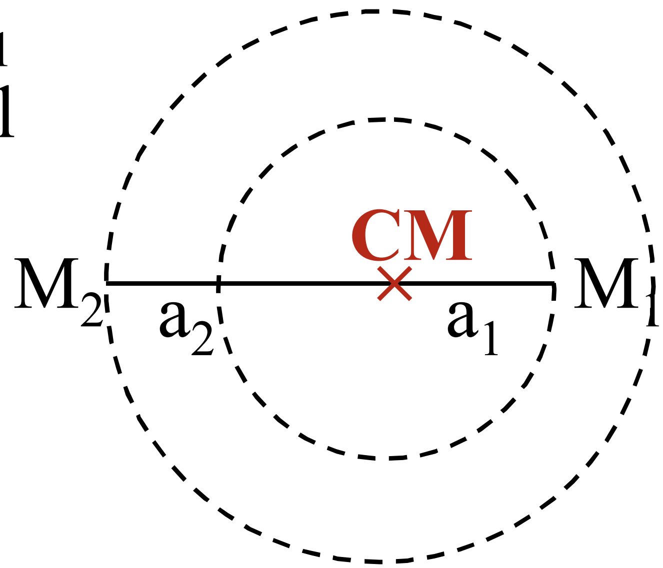
$$M_1 + M_2 = \text{const.} \times a^3 / P^2$$

(Note: exactly the same approach is used in radial velocity searches for extrasolar planets)

Stellar Masses From Binaries

Kepler's Law: consider for simplicity circular orbits

Stars mass: M_1 and M_2, orbital radii a_1 and a_2



In orbit around the center of mass (CM) of the system

From definition of center of mass: $M_1 a_1 = M_2 a_2$

Let total separation: $a = a_1 + a_2$

Then: $$a_2 = \frac{M_1}{M_1 + M_2} a$$

(From P. Armitage)

Apply Newton's law of gravity and condition for circular motion to M_2:

$$\frac{GM_1M_2}{a^2} = M_2 a_2 \omega^2$$

ω is angular velocity of the binary

Substitute for a_2: $$\omega = \sqrt{\frac{G(M_1 + M_2)}{a^3}}$$

$$P = \frac{2\pi}{\omega}$$

Visual binary: see each orbit so know immediately a_2 / a_1:

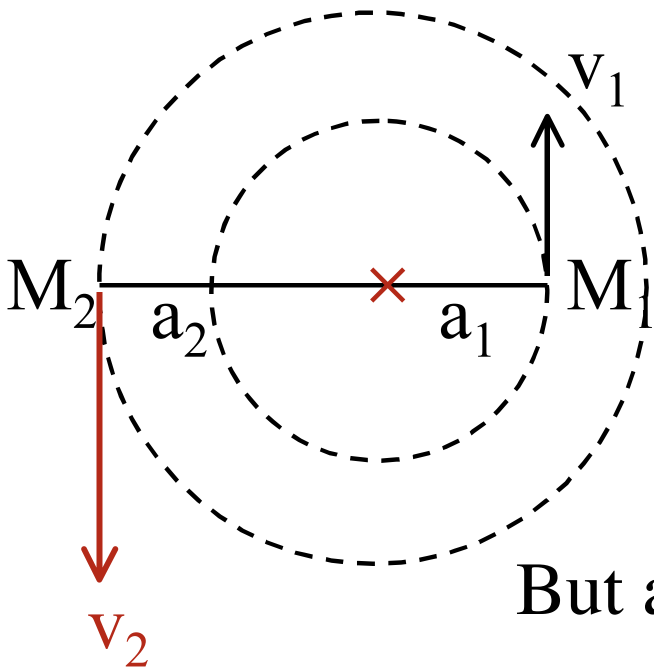
⇒ determines *ratio* of masses M_1 / M_2

If we know distance, then angular separation + d gives a, which with period P determines *sum* of masses $M_1 + M_2$

(From P. Armitage)

So this is enough information to get both M_1 and M_2…

Now consider spectroscopic binaries with circular orbits (often a good approximation because tides in close binaries tend to circularize the orbits)



Velocities are constant around the orbit:

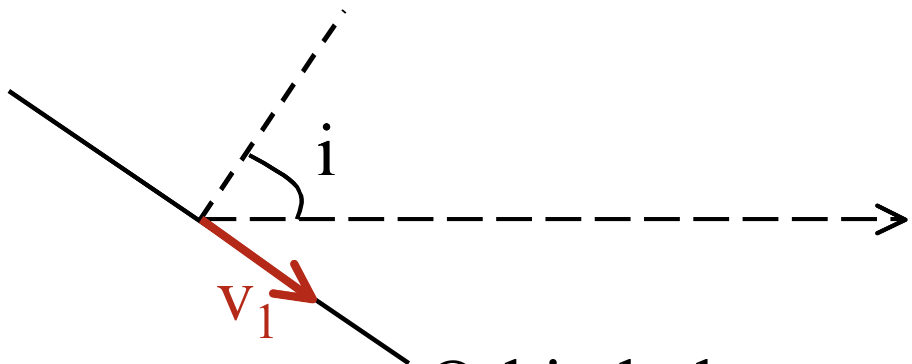
$$Pv_1 = 2\pi a_1$$

$$Pv_2 = 2\pi a_2$$

But alas, there are projection effects…

(From P. Armitage)

We don't observe v_1 and v_2 - only the component of those velocities along our line of sight:



Maximum component of velocity along the line of sight is:

$$v_{r1} = v_1 \cos(90 - i) = v_1 \sin i$$

$$v_{r2} = v_2 \sin i$$

Radial velocities are the observables

i is the inclination angle of the binary system

(From P. Armitage)

Ratio of maximum observed radial velocities is:

$$\frac{v_{r2}}{v_{r1}} = \frac{v_2 \sin i}{v_1 \sin i} = \frac{2\pi a_2 / P}{2\pi a_1 / P} = \frac{a_2}{a_1} = \frac{M_1}{M_2}$$

Ratio of masses can be found if we see spectral lines from both stars (a `double-lined' spectroscopic binary), without knowing the inclination. To find the sum of the masses, note:

$$a = a_1 + a_2 = \frac{P}{2\pi}(v_1 + v_2)$$

Use Kepler's law again:

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)} = \frac{P^3 (v_1 + v_2)^3}{2\pi G (M_1 + M_2)}$$

$$M_1 + M_2 = \frac{P}{2\pi G}(v_1 + v_2)^3$$

(From P. Armitage)

Replace v_1 and v_2 with the observable radial velocities:

$$M_1 + M_2 = \frac{P}{2\pi G} \frac{\left(v_{r1} + v_{r2}\right)^3}{\sin^3 i}$$

So… we can determine sum of masses (and hence the Individual masses M_1 and M_2) ***only*** if the inclination i can be determined.

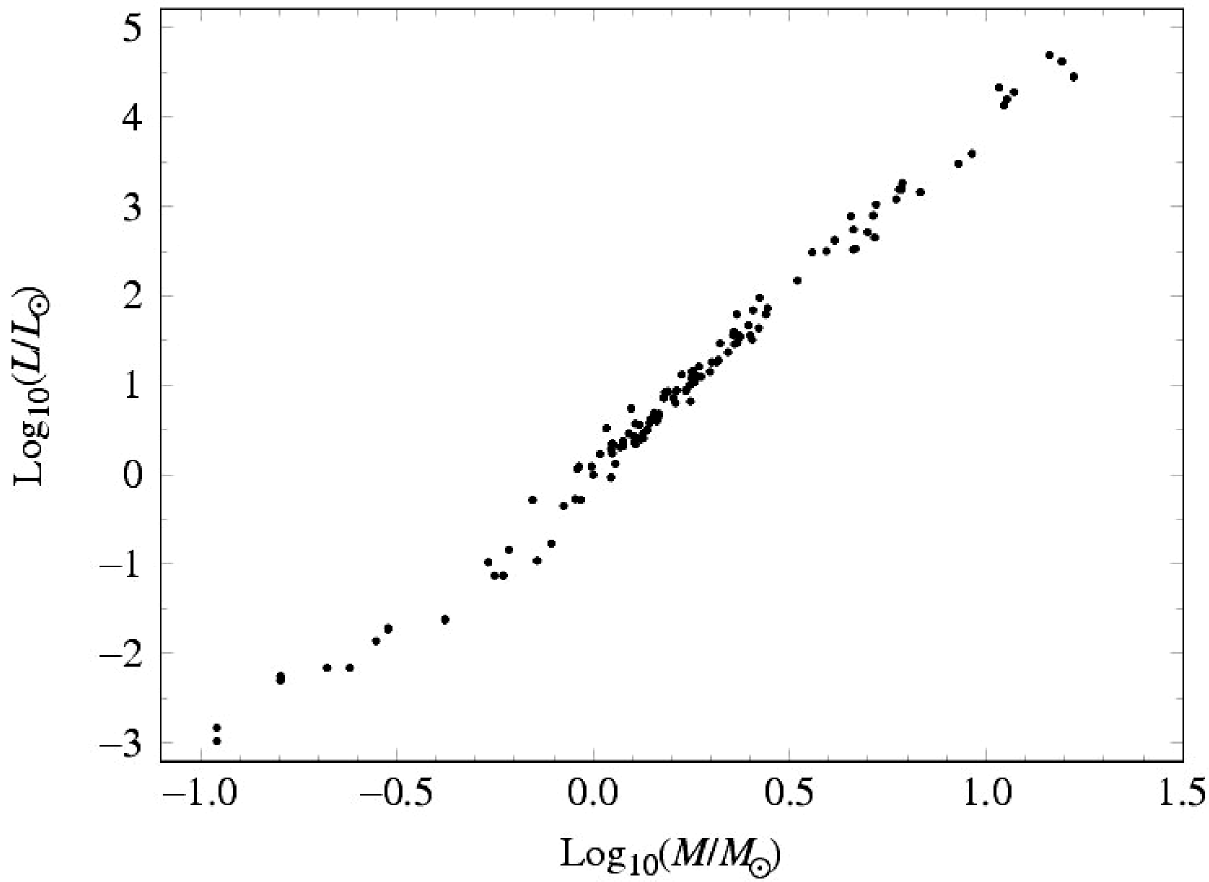
Requires that the stars are also eclipsing:

- Detailed shape of lightcurve gives i
- Obviously must be close to $i = 90^{\circ}$ to see eclipses!

Rare binaries are main source of information on stellar masses…

(From P. Armitage)

Stellar Mass-Luminosity Relation



$\boxed{L \sim M^4}$

- A key relation for understanding of stellar physics
- Main Sequence stars follow this relation, but giants, supergiants, & white dwarfs do not