Ay 122 - Fall 2004 - Lecture 7

How Stars Work:

- **Basic Principles of Stellar Structure**
- **Energy Production**
- **The H-R Diagram**

(Many slides today c/o P. Armitage)

The Basic Principles:

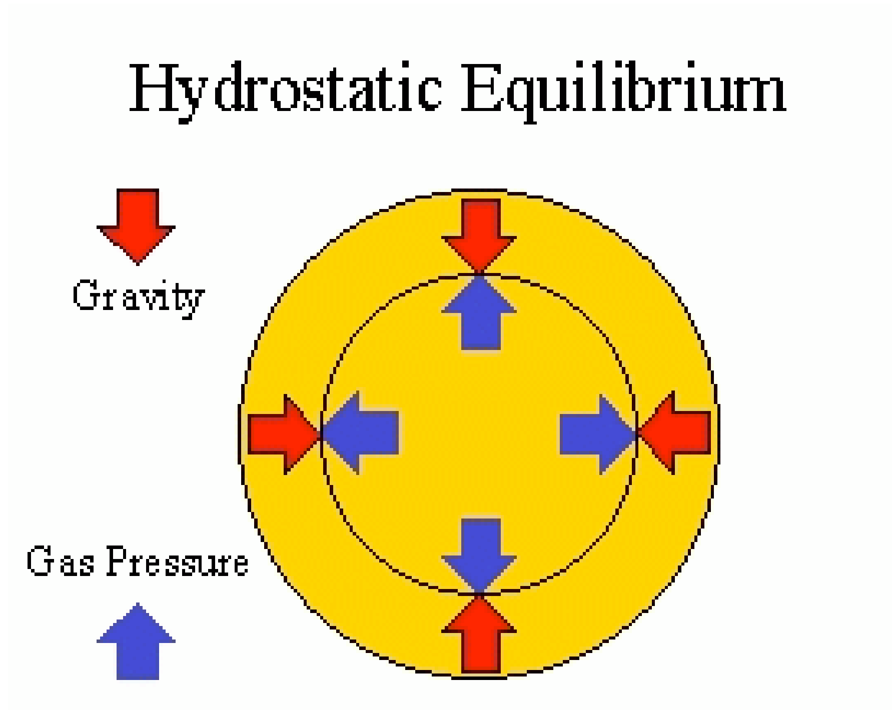
- **Hydrostatic equilibrium:** thermal pressure vs. gravity
  - Basics of stellar structure
- **Energy conservation:** $dE_{prod} / dt = L$
  - Possible energy sources and characteristic timescales
  - Thermonuclear reactions
- **Energy transfer:** from core to surface
  - Radiative or convective
  - The role of opacity

The H-R Diagram: a basic framework for stellar physics and evolution

- The Main Sequence and other branches
- Scaling laws for stars

Hydrostatic Equilibrium: Stars as Self-Regulating Systems

- Energy is generated in the star's hot core, then carried outward to the cooler surface.
- Inside a star, the inward force of gravity is balanced by the outward force of pressure.



- The star is stabilized (i.e., nuclear reactions are kept under control) by a pressure-temperature thermostat.

Self-Regulation in Stars

Suppose the fusion rate increases slightly. Then,

- Temperature increases.

(2) Pressure increases.

(3) Core expands.

(4) Density and temperature decrease.

(5) Fusion rate decreases.

So there's a feedback mechanism which prevents the fusion rate from skyrocketing upward.

We can reverse this argument as well …

Now suppose that there was no source of energy in stars (e.g., no nuclear reactions)

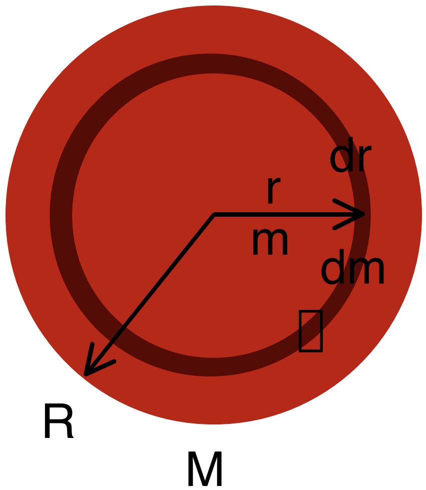
Core Collapse in a Self-Gravitating System

- Suppose that there was no energy generation in the core. The pressure would still be high, so the core would be hotter than the envelope.
- Energy would escape (via radiation, convection…) and so the core would shrink a bit under the gravity
- That would make it even hotter, and then even more energy would escape; and so on, in a feedback loop

➡ **Core collapse!** Unless an energy source is present to compensate for the escaping energy.

- In stars, nuclear reactions play this role. In star clusters, hard binaries do.

Description of a star in spherical symmetry



Let r be the distance from the center
Density as function of radius is $\rho(r)$

If m is the mass *interior* to r, then:

$$m(r) = \int_0^r 4\pi r^2 \rho(r)\,dr$$

Differential form of this equation is: $dm = 4\pi r^2 \rho\, dr$

Two **equivalent** ways of describing the star:

- Properties as f(r): e.g. temperature T(r)
- Properties as f(m): e.g. T(m)

Second way often more convenient, because (ignoring mass loss) total mass M of the star is fixed, while radius R evolves with time.

Stellar Structure

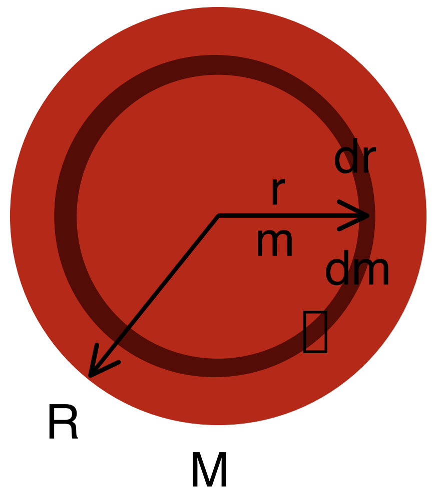
For an isolated, static, spherically symmetric star, four basic laws / equations needed to describe structure:

- **Conservation of mass**
- **Conservation of energy** (at each radius, the change in the energy flux equals the local rate of energy release)
- **Equation of hydrostatic equilibrium** (at each radius, forces due to pressure differences balance gravity)
- **Equation of energy transport** (relation between the energy flux and the local gradient of temperature)

Basic equations are supplemented by:

- **Equation of state** (pressure of a gas as a function of its density and temperature)
- **Opacity** (how transparent it is to radiation)
- **Nuclear energy generation rate** as $f(\rho,T)$

Conservation of Mass



Let r be the distance from the center
Density as function of radius is $\rho(r)$

Let m be the mass *interior* to r, then conservation of mass implies that:

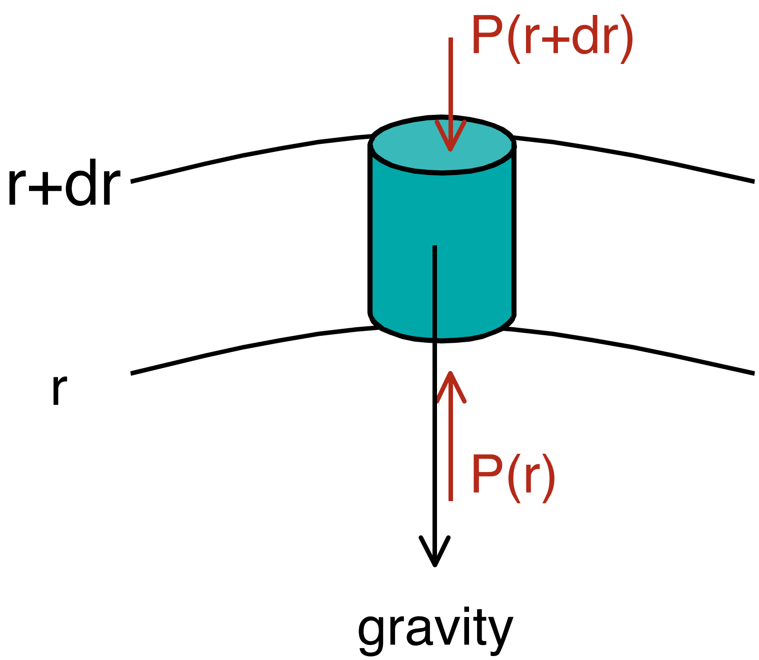
$$dm = 4\pi r^2 \rho\, dr$$

Write this as a differential equation:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

1st stellar structure equation

Equation of Hydrostatic Equilibrium



Consider small cylindrical element between radius r and radius r + dr in the star.

Surface area = dS
Mass = Δm

Mass of gas in the star at Smaller radii = m = m(r)

Radial forces acting on the element:

Gravity (inward):

$$F_g = -\frac{Gm\Delta m}{r^2}$$

Pressure (net force due to difference in pressure between upper and lower faces):

$$F_p = P(r)dS - P(r+dr)dS$$

$$= P(r)dS - \left[P(r) + \frac{dP}{dr}dr\right]dS$$

$$= -\frac{dP}{dr}drdS$$

Mass of element: $\Delta m = \rho drdS$

Applying Newton’s second law (`F=ma’) to the cylinder:

$$\Delta m\ddot{r} = F_g + F_p = -\frac{Gm\Delta m}{r^2} - \frac{dP}{dr}drdS$$

acceleration = 0 everywhere if star static

Setting acceleration to zero, and substituting for δm:

$$0 = -\frac{Gm\rho\, dr\, dS}{r^2} - \frac{dP}{dr}\, dr\, dS$$

<u>Equation of hydrostatic equilibrium</u>:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

2nd stellar structure equation

If we use enclosed mass as the dependent variable, can combine these two equations into one:

$$\frac{dP}{dm} = \frac{dP}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

⟵ alternate form of hydrostatic equilibrium equation

Equation of State in Stars

Interior of a star contains a mixture of ions, electrons, and radiation (photons). For most stars (exception very low mass stars and stellar remnants) the ions and electrons can be treated as an ideal gas and quantum effects can be neglected.

Total pressure:

$$P = P_I + P_e + P_r$$
$$= P_{gas} + P_r$$

- P_I is the pressure of the ions
- P_e is the electron pressure
- P_r is the radiation pressure

Gas Pressure

The equation of state for an ideal gas is: $P_{gas} = nkT$

$\uparrow$

n is the number of particles per unit volume; $n = n_I + n_e$, where n_I and n_e are the number densities of ions and electrons

In terms of the mass density ρ: $P_{gas} = \frac{\rho}{\mu m_H} kT$

…where m_H is the mass of hydrogen and μ is the average mass of particles in units of m_H. Define the **ideal gas constant**:

$$R \equiv \frac{k}{m_H} \quad \Longrightarrow \quad P_{gas} = \frac{R}{\mu} \rho T$$

Determining Mean Molecular Weight μ

μ will depend upon the composition of the gas and the state of ionization. For example:

- Neutral hydrogen: $\mu = 1$
- Fully ionized hydrogen: $\mu = 0.5$

In the central regions of stars, OK to assume that all the elements are fully ionized.

Denote abundances of different elements per unit mass by:

- **X** hydrogen - mass m_H, one electron
- **Y** helium - mass $4m_H$, two electrons
- **Z** the rest, `metals', average mass Am_H, approximately (A / 2) electrons per nucleus

If the density of the plasma is ρ, then add up number densities of hydrogen, helium, and metal nuclei, plus electrons from each species:

	H	**He**	**metals**
Number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4m_H}$	$\frac{Z\rho}{Am_H}$
Number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2Y\rho}{4m_H}$	$\left(\frac{A}{2}\right)\frac{Z\rho}{Am_H}$

$$n = \frac{\rho}{m_H}\left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right) = \frac{\rho}{\mu m_H}$$

…assuming that A >> 1

$$\mu^{-1} = 2X + \frac{3}{4}Y + 2Z$$

Radiation Pressure

For blackbody radiation: $$P_r = \frac{1}{3} a T^4$$

…where a is the *radiation constant*:

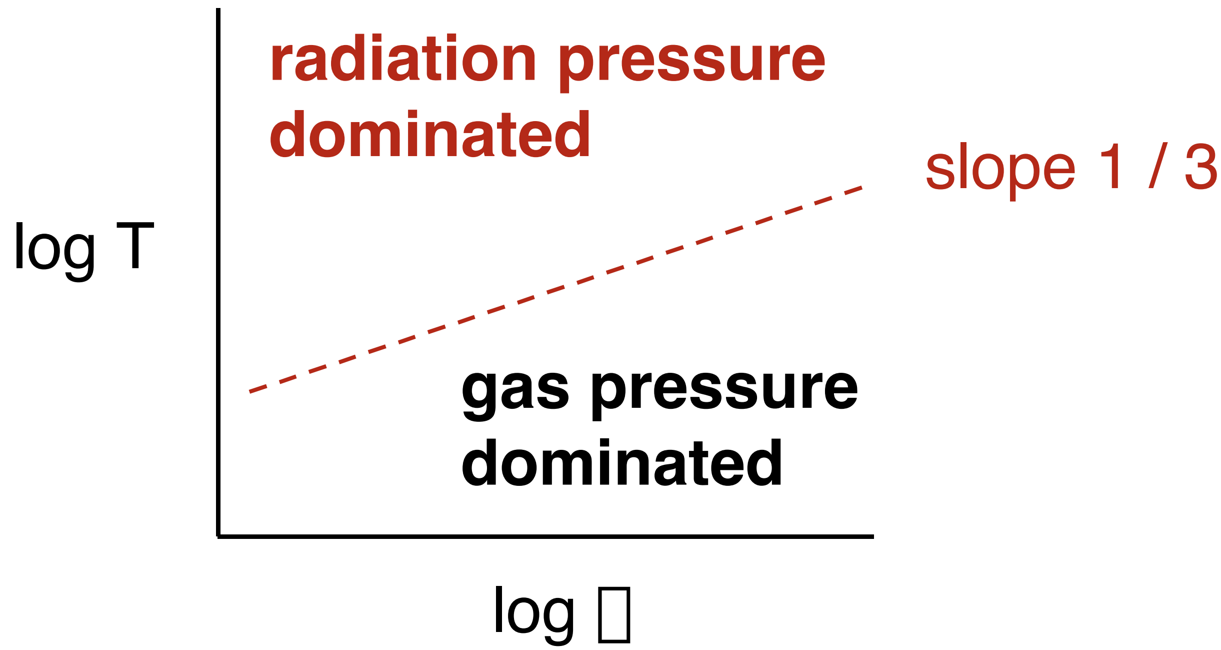
$$a = \frac{8\pi^5 k^4}{15 c^3 h^3} = \frac{4\sigma}{c}$$

$$= 7.565 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$$

$$= 7.565 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$$

In which stars are gas and radiation pressure important?

$$\left.\begin{aligned} P_{gas} &= \frac{R}{\mu}\rho T \\ P_r &= \frac{1}{3}aT^4 \end{aligned}\right\} \quad \text{equal when:} \quad T^3 = \frac{3R}{a\mu}\rho$$



Gas pressure is most important in **low mass stars**
Radiation pressure is most important in **high mass stars**

Conditions in the Solar core

A detailed model of the Sun gives core conditions of:

- $T = 1.6 \times 10^7$ K
- $\rho = 150$ g cm^{-3}
- $X = 0.34$, $Y = 0.64$, $Z = 0.02$ (note: hydrogen is almost half gone compared to initial or surface composition!)

$$\mu^{-1} = 2X + \frac{3}{4}Y + 2Z \quad \Rightarrow \quad \mu = 0.83$$

Ideal gas constant is $R = 8.3 \times 10^7$ erg g^{-1} K^{-1}

Ratio of radiation pressure to gas pressure is therefore:

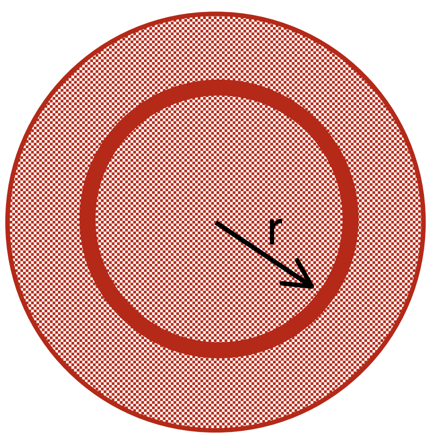
$$\frac{P_r}{P_{gas}} = \frac{\frac{1}{3}aT^4}{\frac{R}{\mu}\rho T} = \frac{\mu a}{3R} \times \frac{T^3}{\rho} = 7 \times 10^{-4}$$

Radiation pressure is not at all important in the center of the Sun under these conditions

Equation of energy generation

Assume that the star is in thermal equilibrium - i.e. at each radius the gas is neither heating up nor cooling down with time.

Let the **rate of energy generation per unit mass** be q (with units erg s^{-1} g^{-1}). Then:



Shell, mass $dm = 4\pi r^2 \rho dr$
Luminosity at r: L(r)
Luminosity at r+dr: L(r)+dL

$$dL = 4\pi r^2 \rho dr \times q$$

$$\frac{dL}{dr} = 4\pi r^2 \rho q$$

4th stellar structure equation

Summary: equations of stellar structure

At radius r in a static, spherically symmetric star:

- density ρ
- enclosed mass m (mass at *smaller radii)*
- temperature T
- luminosity L

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

Mass conservation

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

Hydrostatic equilibrium

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$

Energy transport due to radiation (only)

$$\frac{dL}{dr} = 4\pi r^2 \rho q$$

Energy generation

4 equations in 4 unknowns - enough for a solution once we know $P(\rho,T)$, κ and q.

Simple stellar models

In general, since κ and (especially) q are strong functions of density and temperature, these equations need to be solved numerically.

General behavior can be determined from simpler models.

$$\left.\begin{aligned}\frac{dm}{dr} &= 4\pi r^2 \rho \\ \frac{dP}{dr} &= -\frac{Gm}{r^2}\rho\end{aligned}\right\}$$

Most useful approximation: if the pressure is **only** a function of the density then can solve these two equations independently from the equations involving temperature.

e.g. for degeneracy pressure:

$$P \propto \rho^{5/3}$$ …in non-relativistic case

So at least one important equation of state falls into this category.

Some Order-of-Magnitude Estimates

Let's see if we can estimate roughly the conditions in the Solar core. **Pressure** $P = F / A$:

$$F \approx G\, M_\odot^2 / R_\odot^2$$

$$A \approx 4 \pi R_\odot^2$$

$$P \approx G\, M_\odot^2 / 4 \pi R_\odot^4$$

($M_\odot \approx 2 \times 10^{33}$ g, $R_\odot \approx 7 \times 10^{10}$ cm, $G \approx 6.7 \times 10^{-8}$ cgs)

Thus: $P_{est} \sim 10^{15}$ dyn / cm^2 -- and surely an underestimate

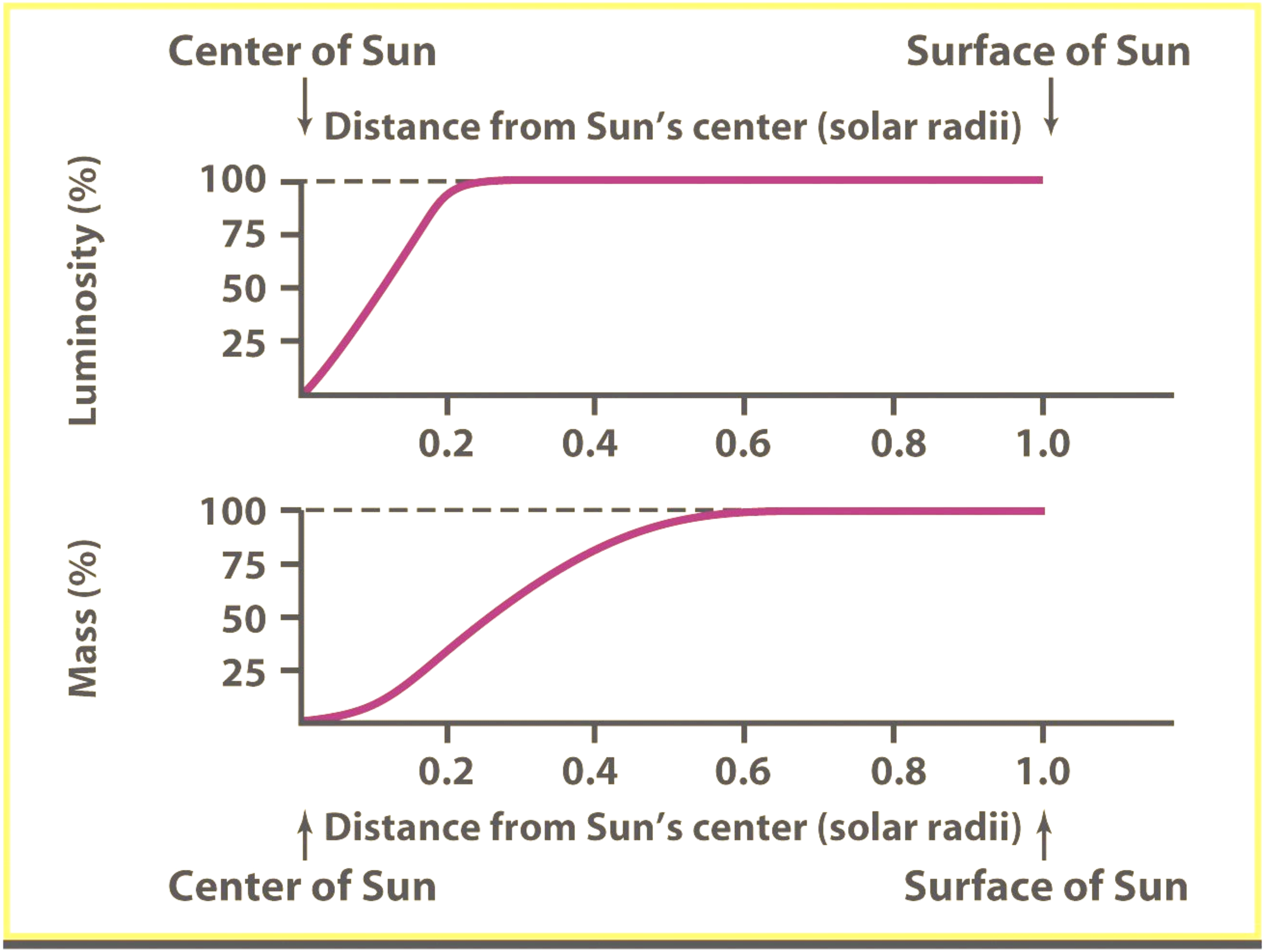
True value: $P_c \approx 2 \times 10^{17}$ dyn / cm^2

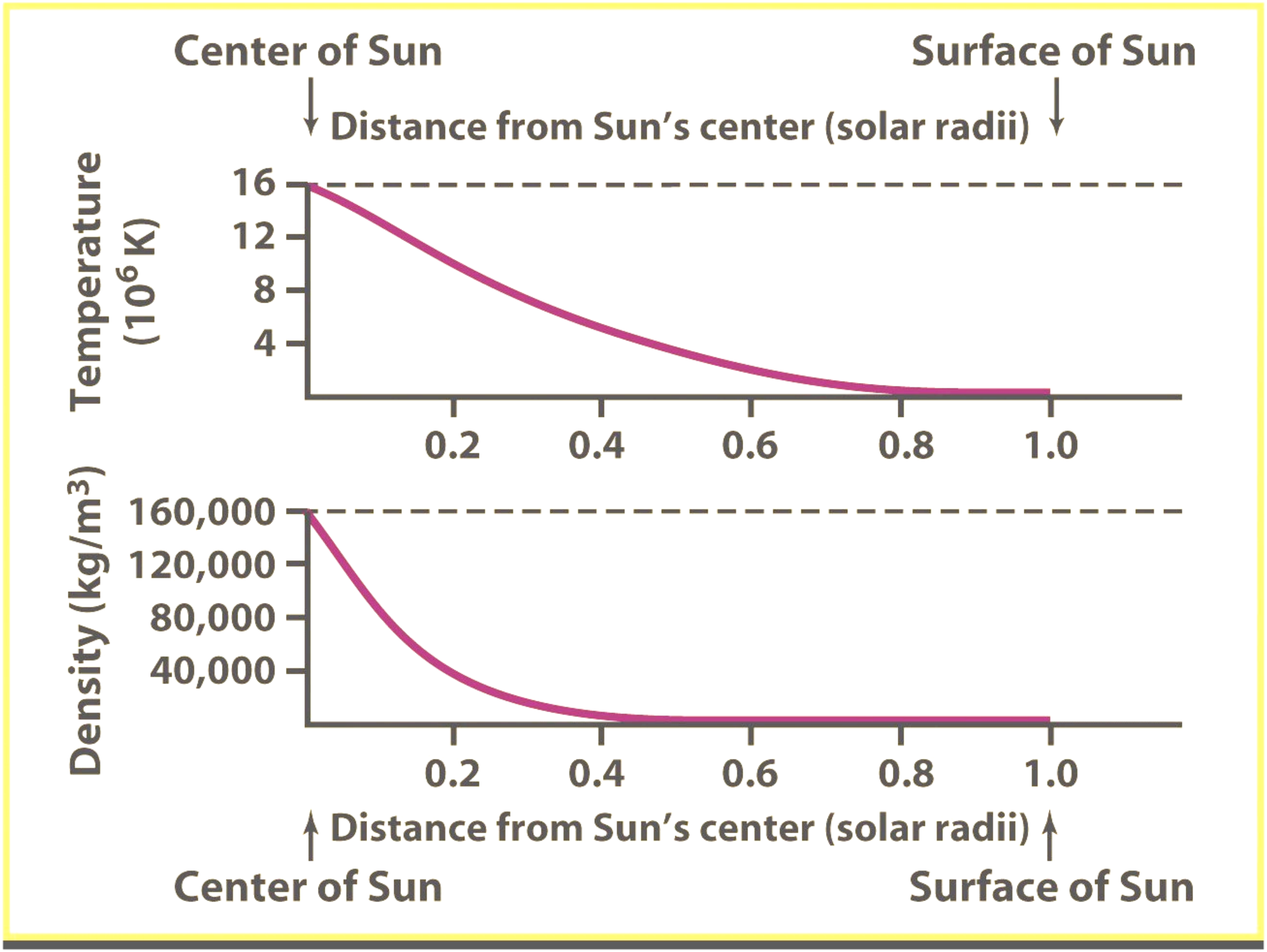
Now the **temperature:** $3/2\, k\, T \approx G\, m_p M_\odot / R$

($k \approx 1.4 \times 10^{-16}$ erg/K, $m_p \approx 1.7 \times 10^{-24}$ g)

Thus: $T_{est} \approx 1.6 \times 10^7$ K

True value: $T_c \approx 1.57 \times 10^7$ K -- not bad!


Center of Sun
Surface of Sun
Distance from Sun's center (solar radii)
Luminosity (%)
100
75
50
25
0.2
0.4
0.6
0.8
1.0
Mass (%)
100
75
50
25
0.2
0.4
0.6
0.8
1.0
Distance from Sun's center (solar radii)
Center of Sun
Surface of Sun


Center of Sun
Surface of Sun
Distance from Sun's center (solar radii)
Temperature (10^6 K)
16
12
8
4
0.2
0.4
0.6
0.8
1.0
Density (kg/m^3)
160,000
120,000
80,000
40,000
0.2
0.4
0.6
0.8
1.0
Distance from Sun's center (solar radii)
Center of Sun
Surface of Sun

We made some basic assumptions …

Problem of stellar structure is simplified by making several reasonable assumptions, which hold in most (not all) cases.

1) Spherical symmetry

An isolated, non-rotating star which does not contain strong magnetic fields will be spherically symmetric, i.e.:

All quantities (e.g. density, temperature, pressure) depend only on the distance from the center of the star - radius r.

Not a general property of all self-gravitating systems - e.g. an elliptical galaxy remains elliptical because interactions between stars are rare.

What About Rotation?

Gravitational potential at distance r from a point mass m is:

$$\Phi = -\frac{Gm}{r}$$

`Average' element of gas in a star is about distance R From the center, and has mass M interior to its radius, where R and M are the stellar radius and total mass. Typical potential is thus:

$$\Phi \sim -\frac{GM}{R}$$

⇒ gravitational binding energy $E_{grav} \sim M\Phi \sim -\frac{GM^2}{R}$

Solar rotation period is about P = 27 days. Angular velocity:

$$\omega = \frac{2\pi}{P} \approx 2.7 \times 10^{-6} \text{ s}^{-1}$$

Rotation energy is of the order of:

$$E_{rotation} \sim M\omega^2 R^2$$

Compare magnitude of gravitational and rotational energy:

$$\varepsilon = \frac{E_{rotation}}{\left|E_{grav}\right|} = \frac{M\omega^2 R^2}{GM^2/R} = \frac{\omega^2 R^3}{GM} \sim 2 \times 10^{-5}$$

↑ Depends upon **square** of rotation velocity

…even rotation rates much faster than the Sun ought to be negligibly small influence on structure.

What About Magnetic Fields?

Magnetic fields in sunspots are fairly strong, of the order of kG strength. Suppose same field fills Sun:

$$E_{magnetic} = \text{Volume} \times \text{Energy density}$$

$$= \frac{4}{3}\pi R^3 \times \frac{B^2}{8\pi} = \frac{B^2 R^3}{6}$$

Ratio to gravitational energy is:

$$\frac{E_{magnetic}}{\left|E_{grav}\right|} = \frac{B^2 R^3 / 6}{GM^2 / R} = \frac{B^2 R^4}{6GM^2} \sim 10^{-11}$$

Estimates suggest that unless something really weird is going on (e.g. Sun rotates super-fast on the inside but not at the surface) magnetic fields / rotation are too small to seriously affect assumption of spherical symmetry.

2) Isolation

In the Solar neighborhood, distances between stars are enormous: e.g. Sun's nearest stellar companion is Proxima Centauri at d = 1.3 pc. Ratio of Solar radius to this distance is:

$$\frac{R_{sun}}{d} \approx 2 \times 10^{-8}$$

Two important implications:

- Can ignore the gravitational field and radiation of other stars when considering stellar structure.
- Stars (almost) never collide with each other.

Once star has formed, initial conditions rather than interactions with other stars determine evolution.

3) Uniform initial composition

Suppose the star forms from a molecular cloud with a composition of:

- **Hydrogen**, fraction of gas by mass X
- **Helium**, fraction of gas by mass Y
- **All other elements (`metals')** Z

$X+Y+Z=1$

Reasonable to assume that initially, composition is constant throughout the star - i.e. X(r) = a constant.

Not true once fusion is underway - e.g. core of the Sun is enriched in helium relative to the surface (Y is larger in the core, smaller toward the surface).

4) Newtonian gravity

Newtonian gravity is only an approximation to Einstein's theory of General Relativity. For the Sun:

$$\frac{1}{2}v_{esc}^{2} = \frac{GM}{R} \Rightarrow v_{esc} = \sqrt{\frac{2GM}{R}} = 620 \text{ km/s}$$

Ratio to speed of light: $$\frac{v}{c} \approx 2 \times 10^{-3}$$

Implies gravity in the Sun is very well approximated by ordinary Newtonian formulae.

Not true for neutron stars.

5) Static

Dynamical time scale for Sun: $t_{dyn} \approx \frac{R}{v_{esc}} \sim 20 \text{ minutes}$

Sun is obviously not collapsing / exploding on this time scale! Implies that pressure and gravitational forces are in very close balance within the Sun, i.e. Sun is very nearly static. Slow changes due to:

- Changing composition (time scales of Gyr)
- Mass loss due to Solar wind (even longer at current mass loss rates)

Note: important classes of stars pulsate - these can't be assumed to be static.

Timescales of stellar evolution

1. Dynamical time scale

Measure of the time scale on which a star would expand or contract if the balance between pressure gradients and gravity was suddenly disrupted (same as free-fall time scale):

$$\tau_{dyn} \equiv \frac{\text{characterististic radius}}{\text{characteristic velocity}} = \frac{R}{v_{esc}}$$

Escape velocity from the surface of the star: $v_{esc} = \sqrt{\frac{2GM}{R}}$

$$\tau_{dyn} = \sqrt{\frac{R^3}{2GM}} \sim \text{(for Sun) } 1100 \text{ s}$$

In terms of mean density: $\tau_{dyn} \sim \frac{1}{\sqrt{G\bar{\rho}}}$ ⟵ mean density of the star, molecular cloud, etc

2. Kelvin-Helmholtz time scale (or thermal time scale)

Suppose nuclear reaction were suddenly cut off in the Sun. Thermal time scale is the time required for the Sun to radiate all its reservoir of thermal energy:

$$\tau_{KH} \equiv \frac{U}{L}$$

← Virial theorem: the thermal energy U is roughly equal to the gravitational potential energy

$$\Rightarrow \quad \tau_{KH} \sim \frac{GM^2}{RL} = 3 \times 10^7 \text{ yr (for the Sun)}$$

Important time scale: determines how quickly a star contracts *before* nuclear fusion starts - i.e. sets roughly the pre-main sequence lifetime.

3. Nuclear time scale

Time scale on which the star will exhaust its supply of nuclear fuel if it keeps burning it at the current rate:

Energy release from fusing one gram of hydrogen to helium is 6×10^{18} erg, so:

$$\tau_{nuc} = \frac{qXM \times 6 \times 10^{18} \text{ erg g}^{-1}}{L}$$

…where:

- X is the mass fraction of hydrogen initially present (X=0.7)
- q is the fraction of fuel available to burn in the core (q=0.1)

$$\tau_{nuc} \approx 7 \times 10^{9} \text{ yr}$$

Reasonable estimate of the main-sequence lifetime of the Sun.

Ordering of time scales:

$$\tau_{dyn} << \tau_{KH} << \tau_{nuc}$$

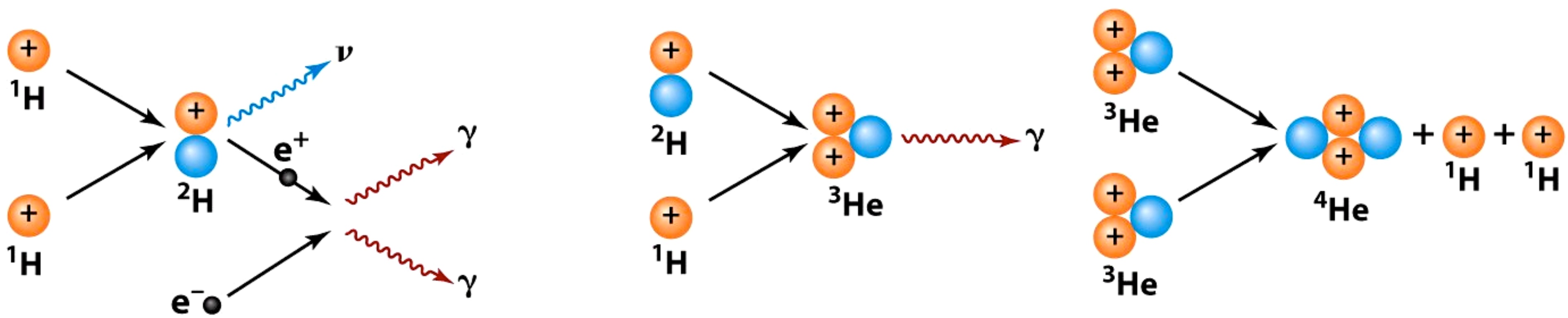
Most stars, most of the time, are in hydrostatic and thermal equilibrium, with slow changes in structure and composition occurring on the (long) time scale τ_{nuc} as fusion occurs.

Do observe evolution on the shorter time scales also:

- Dynamical - **stellar collapse / supernova**
- Thermal / Kelvin-Helmholtz - **pre-main-sequence**

Energy Production in Stars: Thermonuclear Reactions

The main process is hydrogen fusion into helium:



(a) Step 1:

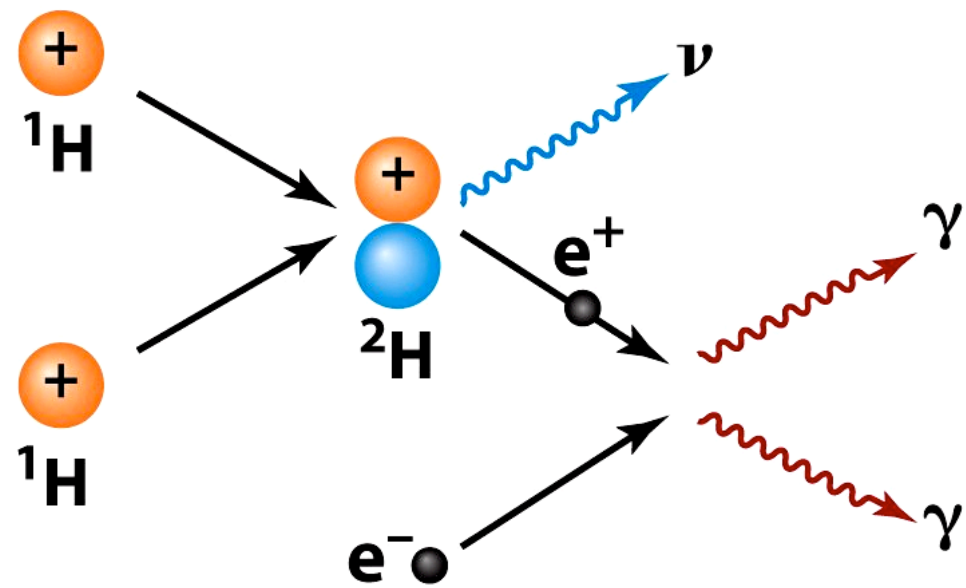
- Two protons (hydrogen nuclei, ^{1}H) collide.
- One of the protons changes into a neutron (shown in blue), a neutral, nearly massless neutrino (ν), and a positively charged electron, or positron (e^+).
- The proton and neutron form a hydrogen isotope (^{2}H).
- The positron encounters an ordinary electron (e^-), annihilating both particles and converting them into gamma-ray photons (γ).

(b) Step 2:

- The ^{2}H nucleus from the first step collides with a third proton.
- A helium isotope (^{3}He) is formed and another gamma-ray photon is released.

(c) Step 3:

- Two ^{3}He nuclei collide.
- A different helium isotope with two protons and two neutrons (^{4}He) is formed and two protons are released.



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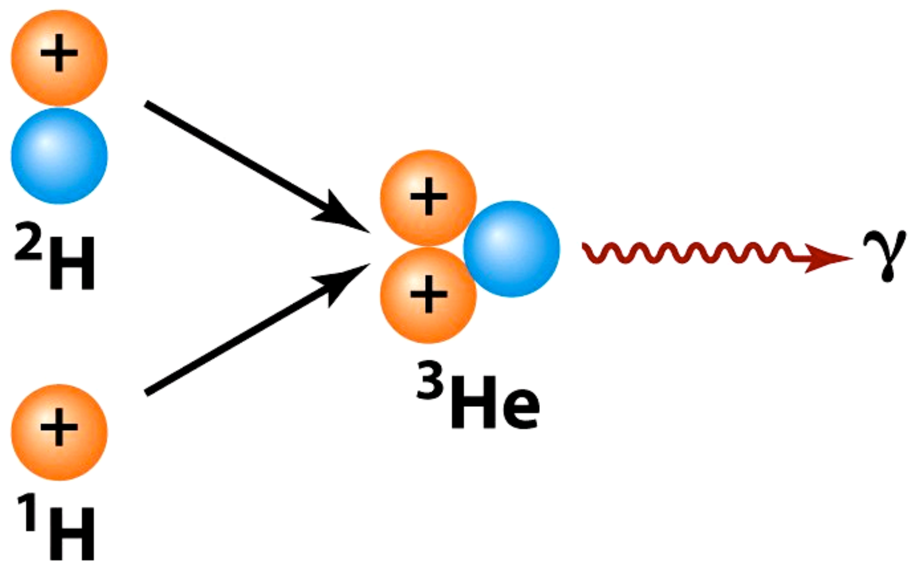

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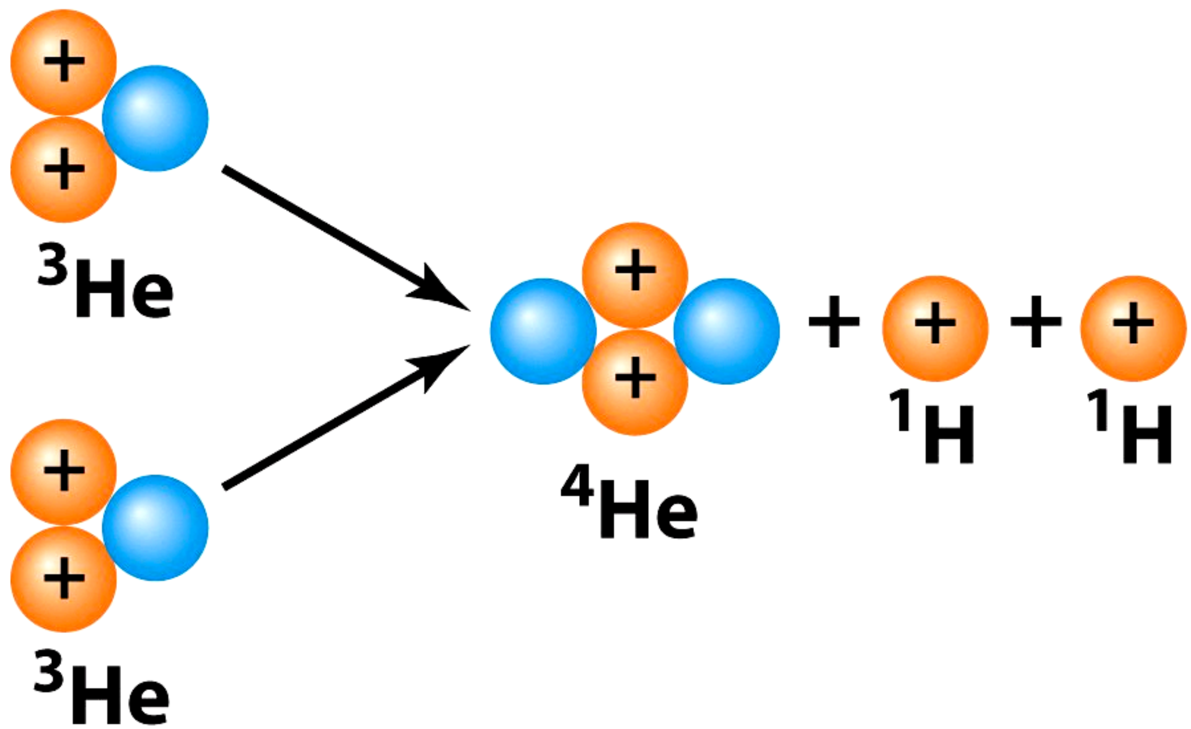

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Nuclear Processes in Stars

Mass of nuclei with several protons and / or neutrons does not exactly equal mass of the constituents - slightly smaller because of the **binding energy** of the nucleus.

Since binding energy differs for different nuclei, can release or absorb energy when nuclei either fuse or fission.

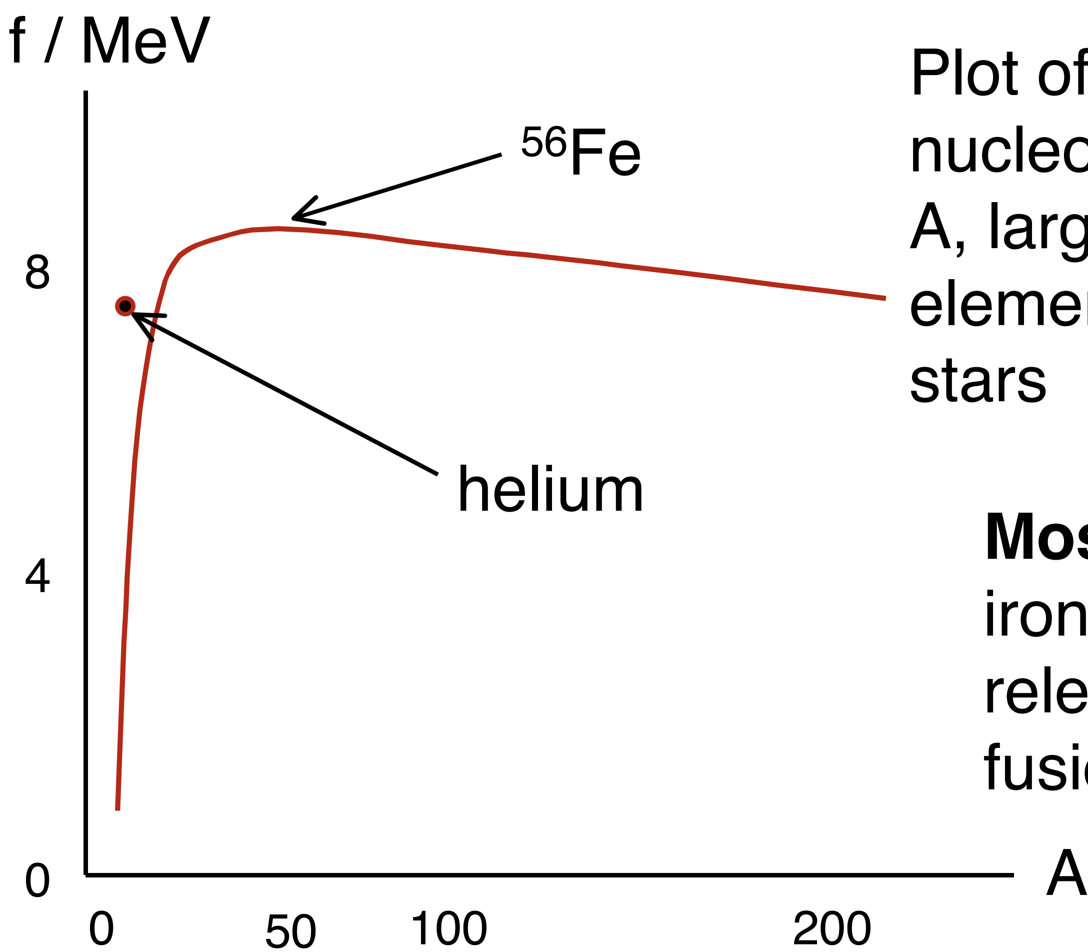
Example: $$4 \, {}^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He}$$

4 protons, each of mass 1.0081 atomic mass units: 4.0324 amu

mass of helium nucleus: 4.0039 amu

Mass difference: 0.0285 amu = 4.7 x 10^{-26} g

$$\Delta E = \Delta M c^2 = 4.3 \times 10^{-5} \text{ erg} = 27 \text{ MeV}$$



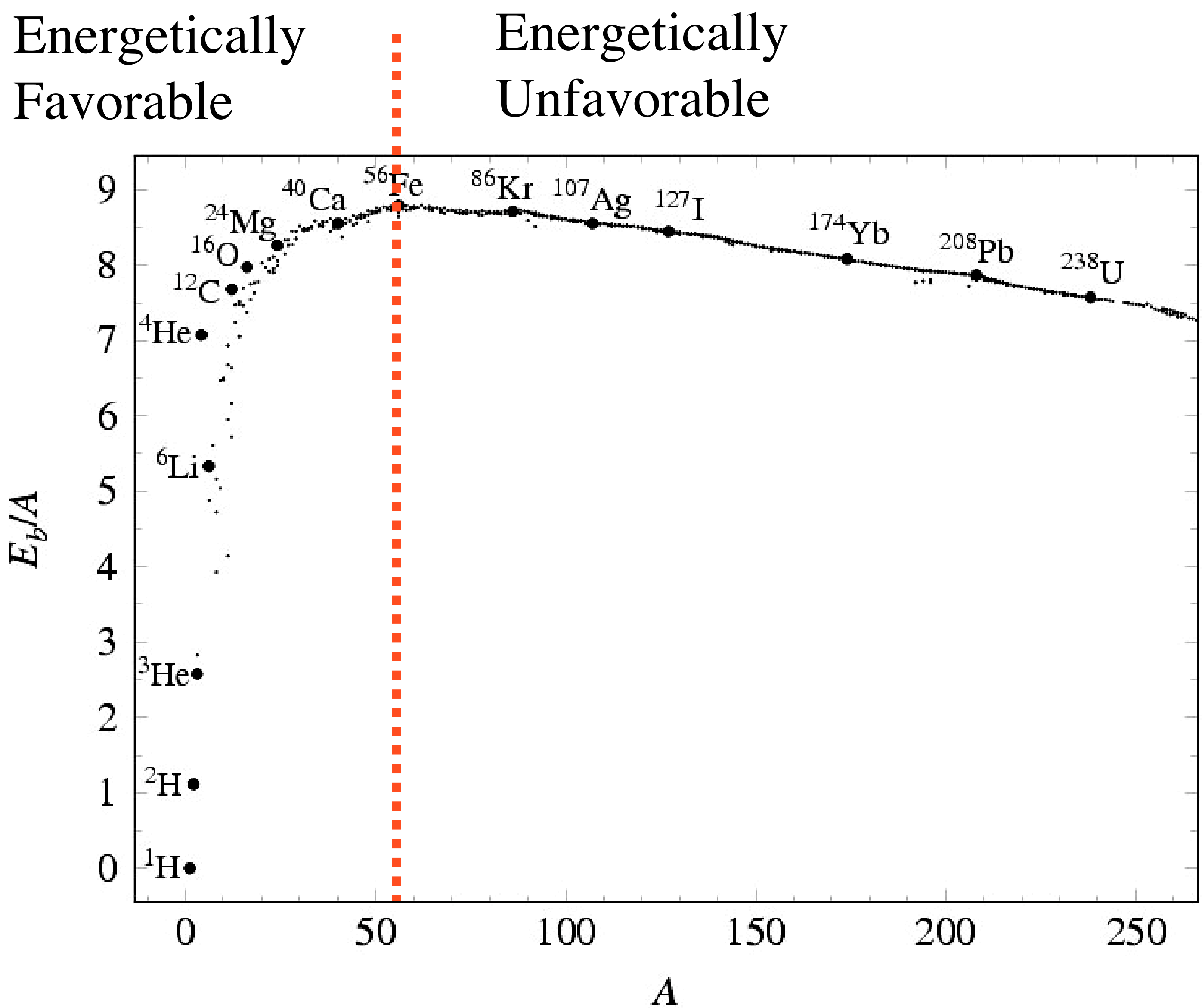
Plot of binding energy per nucleon, f, vs. atomic number A, largely determines which elements can be formed in stars

Most bound nucleus is iron 56: A < 56 fusion releases energy, A > 56 fusion requires energy.

Yield for fusion of hydrogen to ^{56}Fe: ~8.5 MeV per nucleon
Most of this is already obtained in forming helium (6.6 MeV)

Drawn curve as smooth - actually fluctuates for small A - He is more tightly bound than `expected'.

Binding Energy Per Nucleon vs. Atomic Number



Energetics of fusion reactions

Nuclei are positively charged - repel each other.
If charges on the nuclei are Z_1e and Z_2e, then at distance d the electrostatic energy is:

$$E = \frac{Z_1 Z_2 e^2}{d}$$

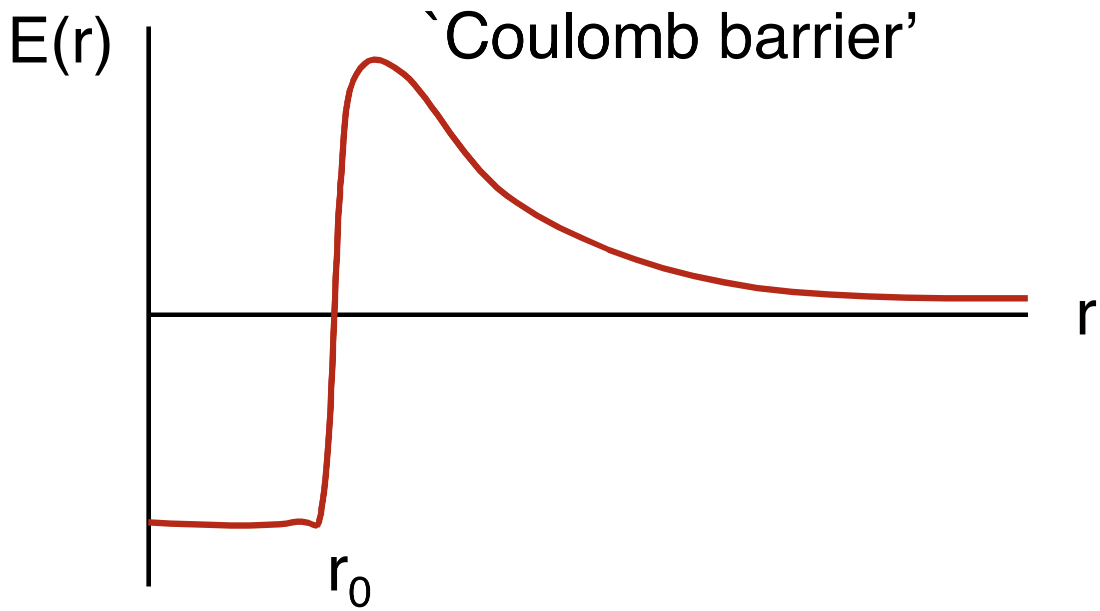
If the nuclei approach sufficiently closely, short range nuclear forces (attractive) dominate and allow fusion to take place.

Nuclear material has roughly constant density, so `close enough' means within a distance:

$$r_0 \approx 1.44 \times 10^{-13} A^{1/3} \text{ cm}$$

↑ atomic mass number

Schematically:



At $r = r_0$, height of the Coulomb barrier is:

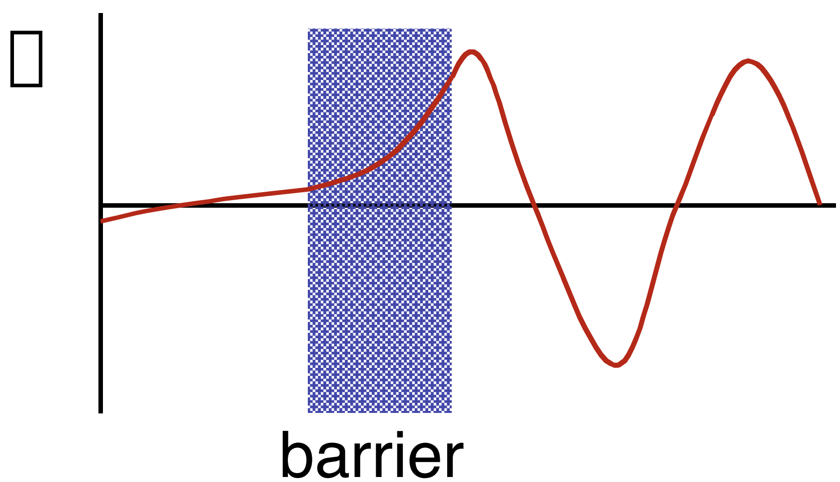
$$E = \frac{Z_1 Z_2 e^2}{r_0} \sim Z_1 Z_2 \text{ MeV}$$

i.e. of the order of 1 MeV for two protons…

For Solar core conditions $T = 1.5 \times 10^7$ K
Thermal energy of particles = kT = 1300 eV = 10^{-3} MeV

Classically, there are *zero* particles in a thermal distribution with enough energy to surmount the Coulomb barrier and fuse.

Quantum mechanically, lower energy particles have a very small but non-zero probability of tunnelling through the barrier:



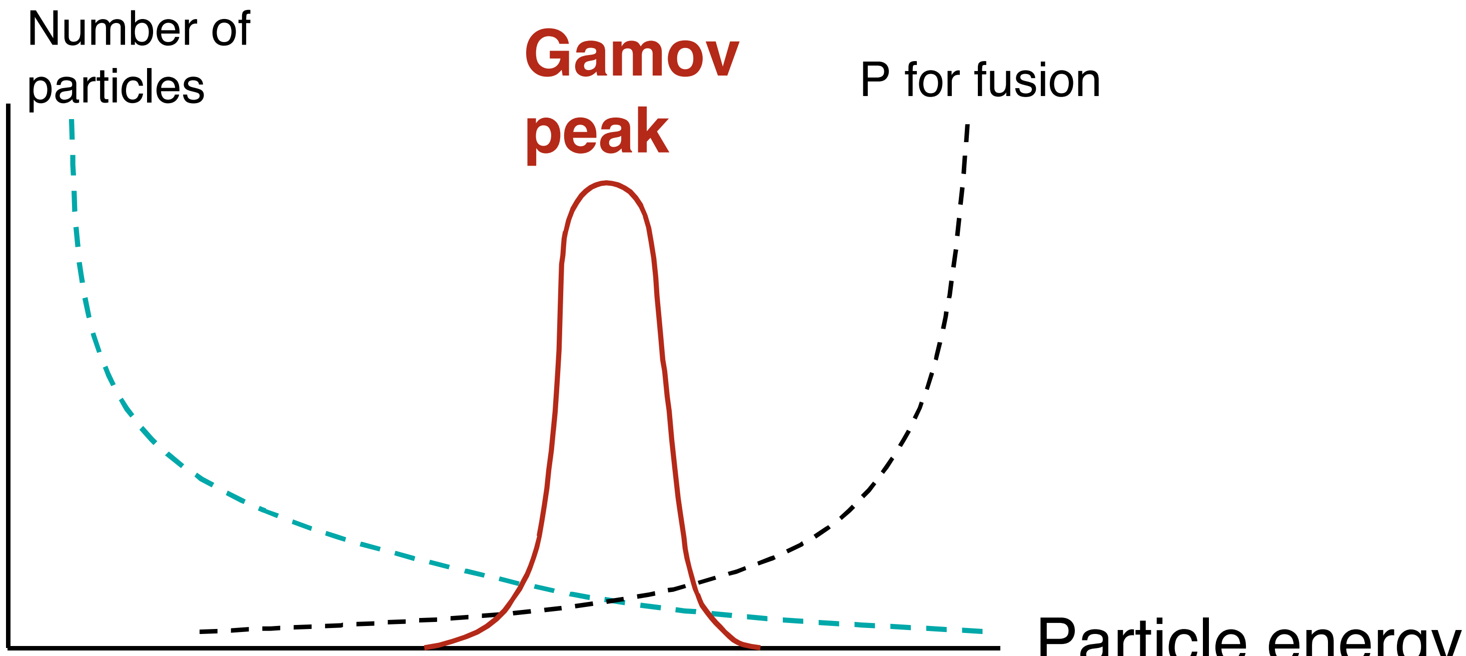
Probability of finding particle ~ $|\psi|^2$ - if barrier is not too wide then non-zero wavefunction allows some probability of tunnelling…

Probability of tunnelling depends upon the energy of the particles, their mass, and the charge:

$$P \propto E^{-1/2} e^{-2\pi\eta} \qquad \eta = \left(\frac{m}{2}\right)^{1/2} \frac{Z_1 Z_2 e^2}{\hbar E^{1/2}}$$

- P increases rapidly with E
- P decreases with $Z_1 Z_2$ - lightest nuclei can fuse more easily than heavy ones
- Higher energies / temperatures needed to fuse heavier nuclei, so different nuclei burn in well-separated phases during stellar evolution.

Competition: most energetic nuclei most likely to fuse, but very few of them in a thermal distribution of particle speeds:



Narrow range of energies around the Gamov peak where significant numbers of particles in the plasma are able to fuse. Energy is >> typical thermal energy, so fusion is slow.

Nuclear reactions in the Sun

Almost all reactions involve collisions of only two nuclei. So making helium from four protons involves a sequence of steps. In the Sun, this sequence is called the **proton-proton chain**:

Step 1

$$^{1}H + {}^{1}H \rightarrow {}^{2}H + e^{+} + \nu_{e}$$

deuteron - one proton + one neutron

positron

electron neutrino

This is the critical reaction in the proton-proton chain. It is slow because forming a deuteron from two protons requires transforming a proton into a neutron - this involves the weak nuclear force so it is slow…

Beyond this point, several possibilities. Simplest:

Step 2 **^{2}H + ^{1}H -> ^{3}He + □**

Step 3 **^{3}He + ^{3}He -> ^{4}He + ^{1}H + ^{1}H**

Results of this chain of reactions:

- Form one ^{4}He nucleus from 4 protons
- Inject energy into the gas via energetic particles: one positron, one photon, two protons
- Produce one electron neutrino, which will escape the star without being absorbed.

Energy yield is ~10^{-5} erg per proton, so ~4 x 10^{38} reactions per second needed to yield L_{sun}. About 0.65 billion tons of hydrogen fusing per second.

The P-P Cycle

$$^{1}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \longrightarrow {}^{2}_{1}\mathrm{H} + e^{+} + \nu_e$$

$$^{2}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H} \longrightarrow {}^{3}_{2}\mathrm{He} + \gamma$$

69% | 31%

$$^{3}_{2}\mathrm{He} + {}^{3}_{2}\mathrm{He} \longrightarrow {}^{4}_{2}\mathrm{He} + 2\,{}^{1}_{1}\mathrm{H}$$

(PP I)

$$^{3}_{2}\mathrm{He} + {}^{4}_{2}\mathrm{He} \longrightarrow {}^{7}_{4}\mathrm{Be} + \gamma$$

99.7% | 0.3%

$$^{7}_{4}\mathrm{Be} + e^{-} \longrightarrow {}^{7}_{3}\mathrm{Li} + \nu_e$$

$$^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} \longrightarrow 2\,{}^{4}_{2}\mathrm{He}$$

(PP II)

$$^{7}_{4}\mathrm{Be} + {}^{1}_{1}\mathrm{H} \longrightarrow {}^{8}_{5}\mathrm{B} + \gamma$$

$$^{8}_{5}\mathrm{B} \longrightarrow {}^{8}_{4}\mathrm{Be} + e^{+} + \nu_e$$

$$^{8}_{4}\mathrm{Be} \longrightarrow 2\,{}^{4}_{2}\mathrm{He}$$

(PP III)

Thermonuclear Reactions (TNR)

- Burning of H into He is the only energy generation process on the Main Sequence; all others happen in post-MS evolutionary stages
- In addition to the **p-p cycle**, there is the **CNO Cycle**, in which the C, N, O, nuclei catalyze the burning of H into He
- The rates of TNR are usually very steep functions of temperature, due to high potential barriers
- Generally, more massive stars achieve higher T_c , and can synthesize elements up to Fe; beyond Fe, it happens in SN explosions

The Next Step: Burning Helium Into Carbon

- Requires much higher temperatures, $T \sim 10^9$ K
- Enabled by the “exact right” energy resonance for carbon

Energy Transport Mechanisms in Stars

How does the energy get out?

1. **Radiatively** (photon diffusion)
2. **Convectively**
3. Conduction (generally not important in stars)

… and the reality is fairly complex

Convection

When the radial flux of energy is carried by radiation, we derived an expression for the temperature gradient:

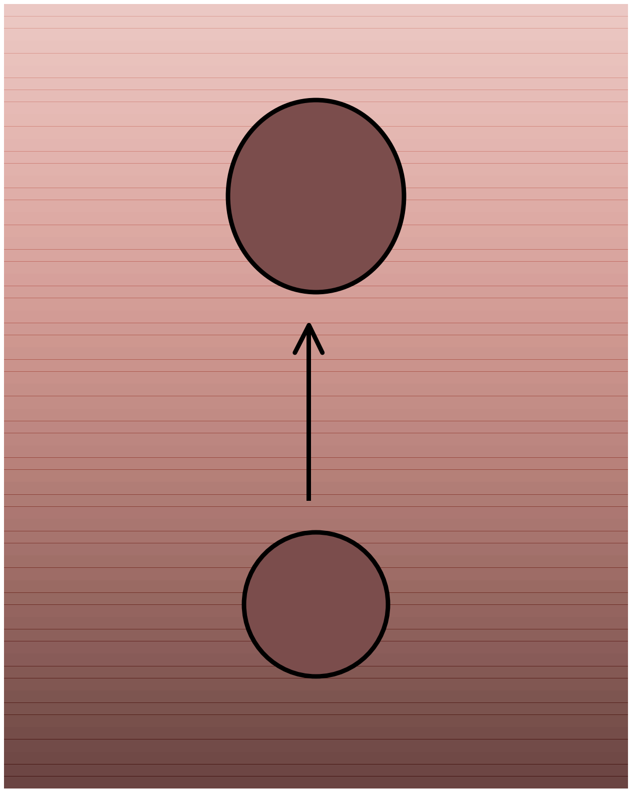
$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$

Large luminosity and / or a large opacity κ implies a large (negative) value of dT / dr.

For an ideal gas, the energy density (energy per unit volume) is given by: $\frac{3}{2}nkT$ …with n the number density of particles.

Hot gas near the center of the star has higher energy density than cooler gas above - if we could `swap' the gas over we could transport energy outward… especially if dT / dr is large.

Schwarzschild criteria for convective instability



Imagine displacing a small mass element vertically upward by a distance dr. Assume that **no heat** is exchanged with the surrounding, i.e. the process is **adiabatic**:

- Element expands to stay in pressure balance with the new environment
- New density will *not* generally equal the ambient density at the new location

If this mechanical energy transport is more efficient than the radiative case, the medium will be **convectively unstable**

Stability condition is:

Temperature gradient in the star

$$\left|\left(\frac{dT}{dr}\right)_{star}\right| < \left|\left(\frac{dT}{dr}\right)_{adiabatic}\right|$$

Temperature gradient when an element is moved adiabatically

The important physical point is:

Too steep a temperature gradient leads to the onset of convection in stars

Since a steep gradient is caused by a large luminosity, can convert this into an expression for the **maximum** luminosity that can be transported radiatively:

$$L_{max} \propto \left(\frac{1}{\kappa}\right)\left(1 - \frac{1}{\gamma}\right)$$

omitting lots of factors but keeping the important dependencies on opacity and adiabatic exponent

Larger luminosities lead to convection.

Which stars are convectively unstable?

Low mass stars

$$L_{max} \propto \frac{1}{\kappa}\left(1 - \frac{1}{\gamma}\right)$$

Near the surface, opacity is large (atomic processes) and $\gamma < 5 / 3$ due to ionization. Leads to **surface convection zones**.

High mass stars

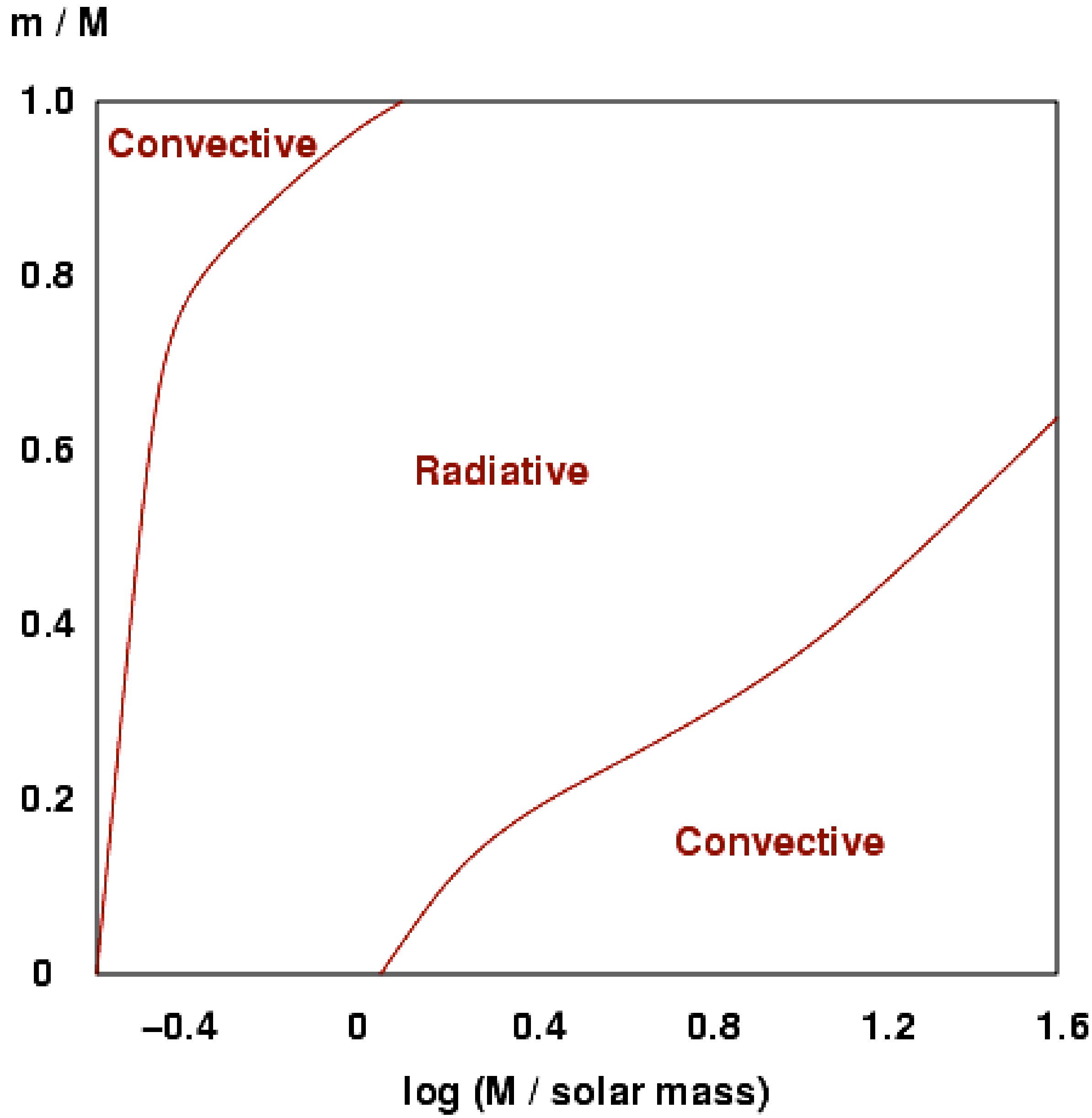
Luminosity of stars increases very rapidly with increasing stellar mass: $L \sim M^4$ for stars of around a Solar mass.

All this energy is generated very close to the core of the star. Can exceed the critical value - **core convection**.

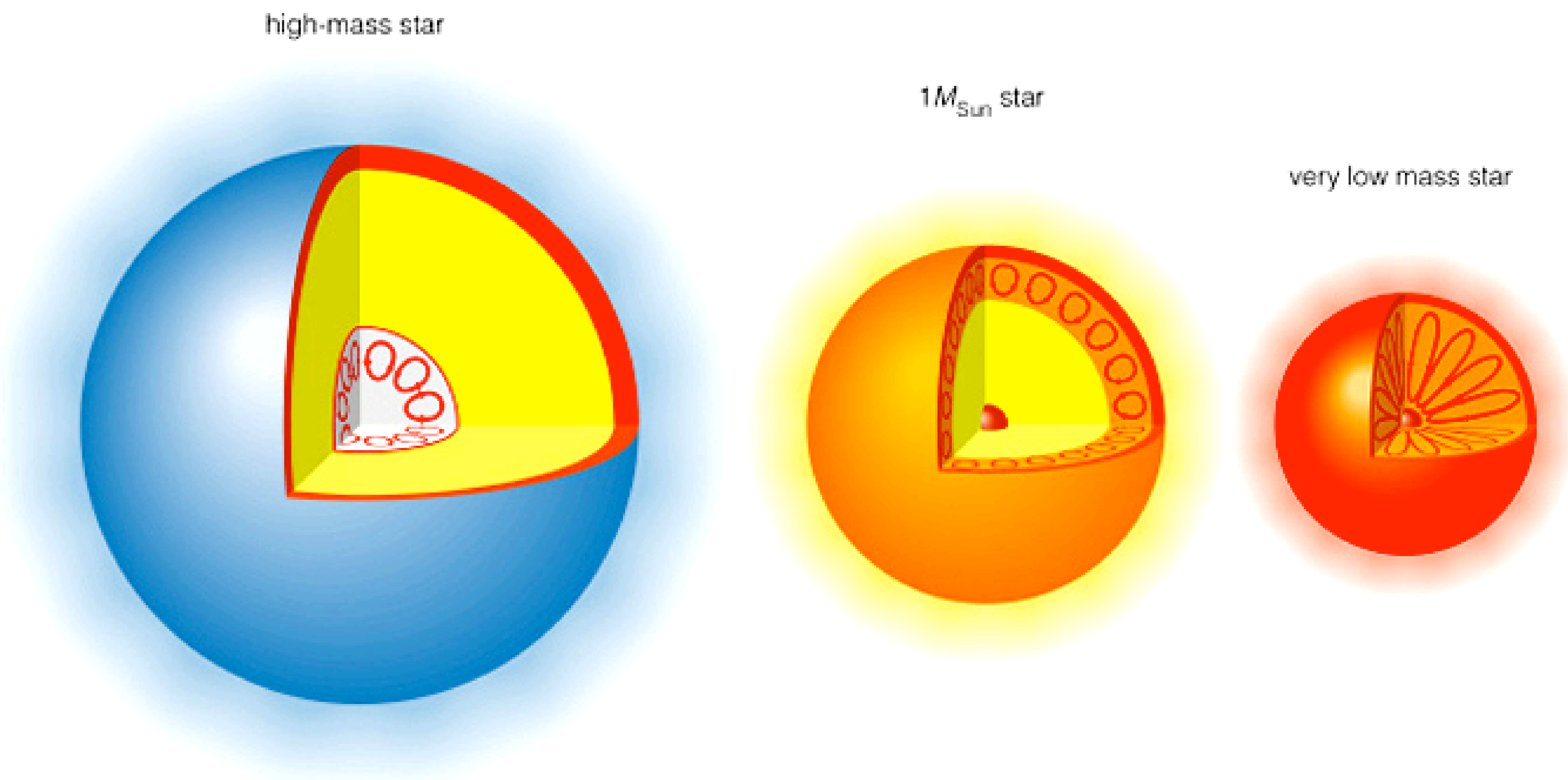
Pre-main-sequence stars

Fully convective due to the large dissipation of gravitational potential energy as they contract.

Regions of convection in main sequence stars



Differences in Stellar Structures



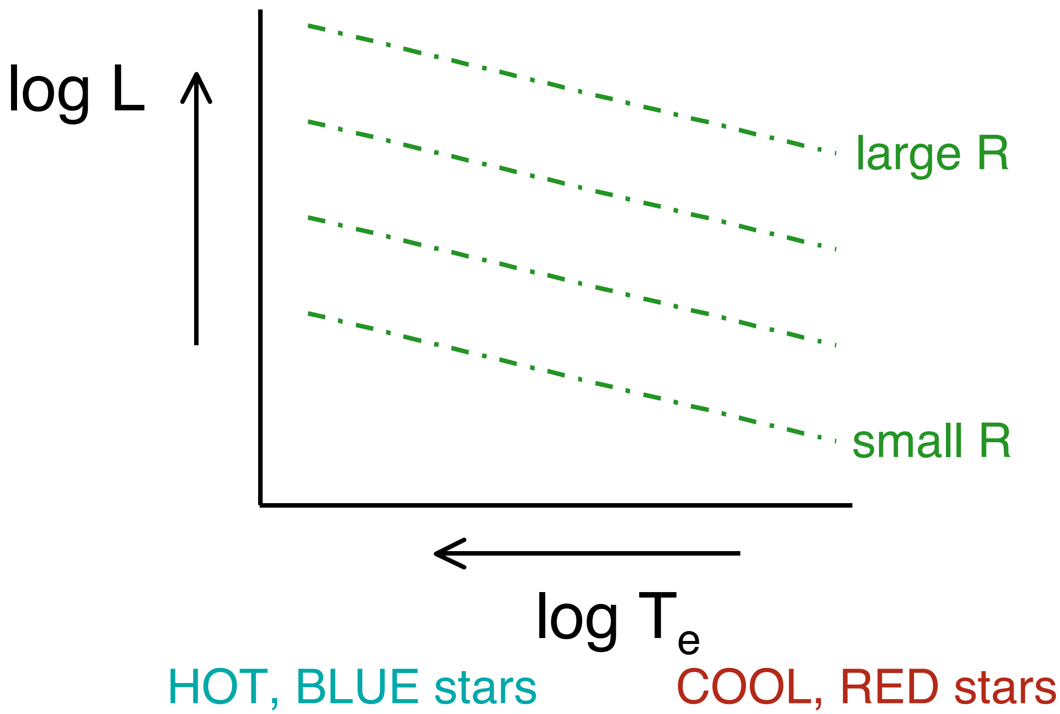
Hertzsprung-Russell Diagram

- The Fundamental Tool for understanding stars and their evolution.
- The Hertzsprung-Russell diagram (H-R diagram) classifies stars by their luminosity and temperature.
- Most stars fall on the Main Sequence of the H-R diagram, a sequence running from hot, luminous stars to cool, dim stars.
- Other stars, such as supergiants, giants, and white dwarfs, fall in different regions of the H-R diagram.
- ***Mass is the dominant parameter*** which determines where a star will fall on the HRD

The Hertzsprung-Russell Diagram

Plot T_e against L (theorists) or color (e.g. B-V) against Absolute magnitude (observers):

<u>H-R diagram</u>

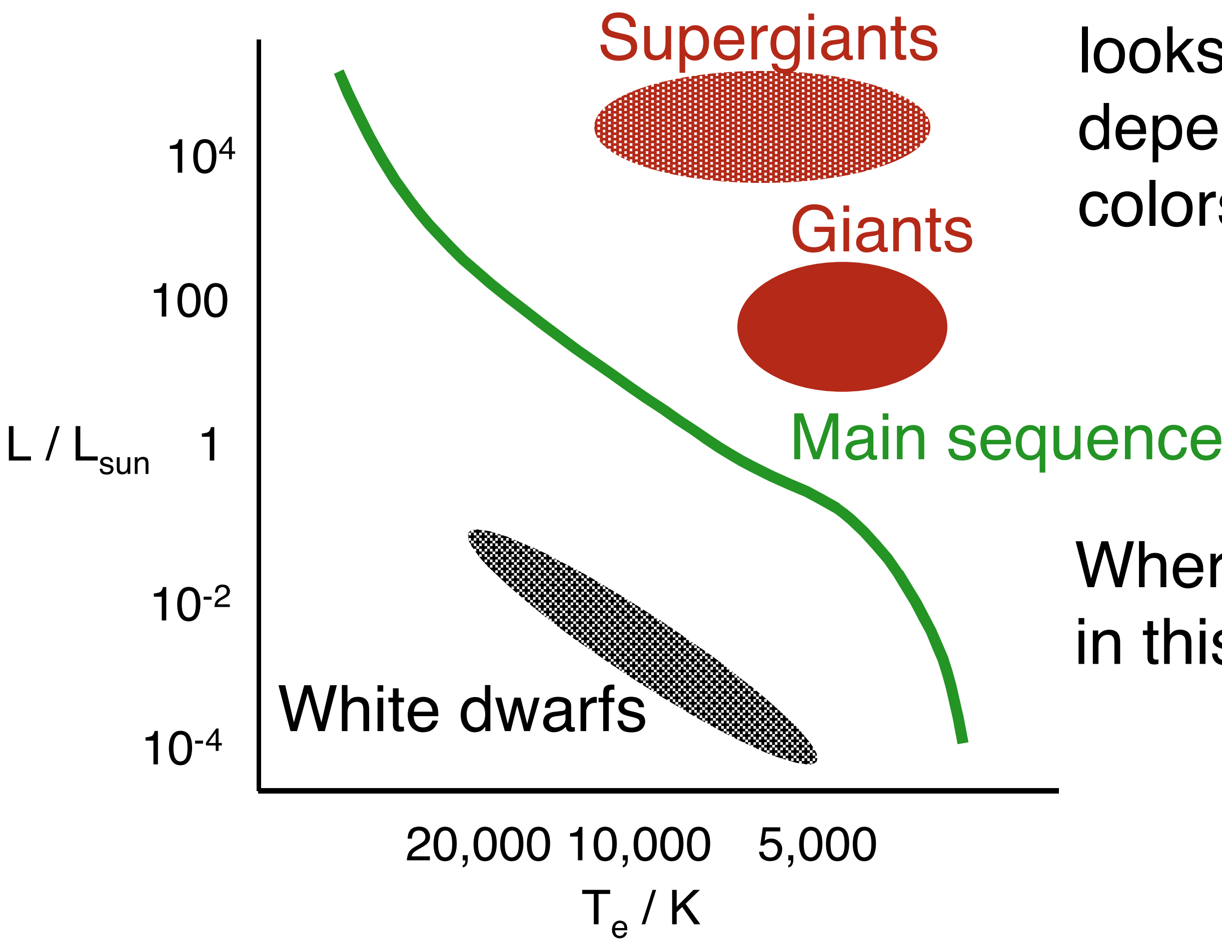


Plot lines of constant stellar radius on the H-R diagram using:

$$L = 4\pi R^2 \sigma T_e^4$$

Individual star is a single point in this plane.

Since stars with large radius fall in the upper right corner of the H-R diagram, can determine which stars lie in different areas of the plot:



Observational diagram looks a bit different depending on what colors / bands are plotted.

Where do binaries fall in this plot?

Life of Main Sequence Stars

- As a star burns H into He, it can’t provide the same amount of outer pressure due to the fact the the number of particles has decreased.
- This decrease in pressure causes the outer layers to contract, which then causes the core to heat up and burn H at a higher rate.
- This produces more energy, and thus the star must release this energy at a more efficient rate which causes the outer envelope to swell up.
- This swelling of the MS star means that the MS line is really a band in which stars move across during their lives.
- The MS stars begin on the lower edge of the MS band and as gradual changes in their T and L occur they move slightly right on the MS band.

Time on the Main Sequence Line

- The time a star spends on the main sequence line depends on its mass.
- Recall the nonlinearity of the M-L relation: the larger the star, the quicker it guzzles its own fuel, and therefore evolves faster
- Life expectancy:
  - T=fuel/rate of consumption or
    - $T=1/M^{2.5}$

Where M is expressed in solar masses, time is given in terms of solar lifetimes, which is 10×10^9 years