Physics 106a – Problem Set 5 – Due Nov 11, 2004

Version 3

November 9, 2004

These problems cover the material on analytical mechanics in Hand and Finch Chapters 5 and 6 and Sections 2.3 and 2.4 of the lecture notes – Hamiltonian dynamics, canonical transformations, generating functions, Poisson brackets, Hamilton-Jacobi theory, action-angle variables. Please again write down the rough amount of time you are spending on each problem.
Changes since v. 1: add integration hint for problem 4.
Changes since v. 2: Apparently, some copies of Hand and Finch are missing pages from Section 6.4 through Section 7.2. The problems missing from those copies of the text have been included here. The lecture notes should provide sufficient information to do the problems.

1. (Using Hamilton's equations) Hand and Finch Problem 5.8. In addition, also write down the Euler-Lagrange equations for the system.

2. (Generating functions, Poisson Brackets) Hand and Finch Problem 6.6. Use the relation

$$\det\left|\frac{\partial(P,Q)}{\partial(p,q)}\right| = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = 1$$

   (*i.e.*, the Poisson bracket $[Q,P]_{q,p} = 1$) to prove whether or not the two functions below can be used as generating functions:

$$F_1(q,Q) = q\,e^Q \qquad F_1(q,Q) = q^2 + Q^4$$

   If it is a possible generating function, determine the transformation $q, p \rightarrow Q, P$ explicitly.

3. (Hamilton-Jacobi equation) One end of uniform rod of length $2\,l$ and mass m rests against a smooth horizontal floor and the other against a smooth vertical surface. Assume that the rod is constrained to move under gravity with its ends always in contact with the surfaces. (Figure 1.5 of Hand and Finch has an analogous setup, Question 6 in Section 1.7 of Hand and Finch will help you calculate the kinetic energy.) Use the "restricted" Hamilton-Jacobi equation (the differential equation for W in Section 2.4.4 of the lecture notes) to reduce the solution of the problem to a set of integrals. That is, you will find that your solutions for $q(t)$ and $p(t)$ will contain integrals that you cannot do; that's fine, just leave them. Do not forget to indicate how the initial conditions $q(t=0)$ and $p(t=0)$ should be used to determine the constants $\alpha_1 = E$ and β_1 in the problem. Reviewing the example of the 1D harmonic oscillator in Section 2.2.4 of the lecture notes will help you understand how to do this problem.

4. Hand and Finch 6.18. A particle of unit mass moves in a potential of the form

$$V(q) = U\,\tan^2(aq)$$

where U and a are positive constants. Find the turning points of the motion. Prove that the action variable I obeys the relation

$$\frac{a\,I}{\sqrt{2}} = \sqrt{E+U} - \sqrt{U}$$

where E is the total energy, and thus prove that the frequency ω has the energy dependence

$$\frac{\omega}{a\sqrt{2}} = \sqrt{E+U}$$

The increase of the frequency with energy reflects the fact that the restoring force increases with displacement q faster than linearly.

There is a difficult integral in this problem, here are two hints on how to do it:

- Make the trigonometric substitution $\sin\theta = \sqrt{\frac{U}{E}}\tan a\,q$ (why is it ok to do this?)
- Use the following formula (from the *CRC Standard Math Tables*, a good investment for any physicist):

$$\int dx \frac{\cos^2 c\,x}{a^2 + b^2\,\sin^2\,cx} = \frac{\sqrt{a^2+b^2}}{a\,b^2\,c}\ \text{arctan}\ \frac{\sqrt{a^2+b^2}\ \tan c\,x}{a} - \frac{x}{b^2}$$

If you find an easier method, let us know!

5. (Adiabatic invariants – see Hand and Finch Section 6.5 and/or lecture notes Section 2.4.5 for explanation – there was not time to cover this in class, but it is fairly easy) A plane pendulum of **small** amplitude is constrained to move on an inclined plane, as shown in the accompanying figure. How does its amplitude change when the inclination angle α of the plane is changed slowly? (Note: you do not need to rederive simple harmonic oscillator results that are in Hand and Finch or the notes.)