Physics 106a/196a – Problem Set 3 – Due Oct 21, 2005

These problems cover the material on Lagrangian mechanics in Sections 2.1.1 through 2.1.8 of the lecture notes and Chapter 1 of Hand and Finch. Problems 1 and 2 are for 106a students only, 3 and 4 for 106a and 196a students, and 5 and 6 for 196a students only. The rewriting of the equation of motion in the simple harmonic oscillator form via Taylor expansions is a very general technique that we touched on during our discussion of equilibria earlier and will come back to when we discuss the SHO in detail later this term.

1. (106) A particle of mass m is constrained to move on a hyperboloid of revolution, which is the $z > 0$ half of a surface satisfying

$$\frac{z^2}{a^2} - \frac{x^2}{b^2} - \frac{y^2}{b^2} = 1 \qquad \text{with } a > b$$

   The lowest point of the hyperbola is at $z = a$. How many degrees of freedom are there? Define an appropriate set of generalized coordinates and calculate the kinetic energy. Hint: it's a hyperboloid, so think about hyperbolic functions.

2. (106) Two point masses m_1 and m_2 are connected by a massless rigid rod of length $2\,d$ to form a dumbbell. The rod is fixed to a frictionless pivot at its midpoint. From m_1 is hung a mass m_3 by a massless, inextensible string of length l, and a mass m_4 is similarly hung from m_2. All motion is constrained to be in a single vertical plane.

   (a) If $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$ and $\vec{r}_4$ are the Cartesian coordinates of the four masses, write down the constraints in terms of these Cartesian coordinates. (You don't need to write everything out in gory detail; statements of form $|\vec{r}_a - \vec{r}_b| = X$ are sufficient.) How many degrees of freedom are left when the constraints are applied?

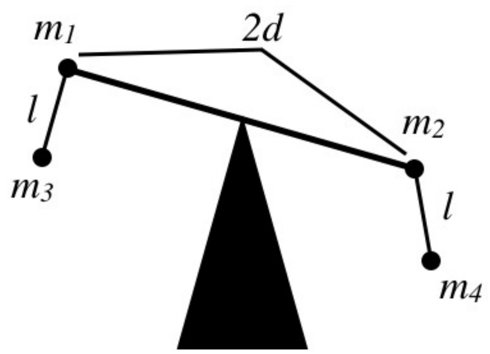
   (b) Write down a set of generalized coordinates that accomodate the constraints.

   (c) Write down the kinetic energy in terms of the generalized coordinates.

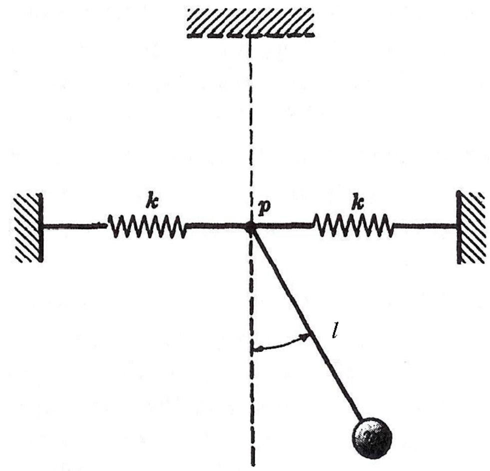
3. (106/196) The pendulum bob of mass m shown in the figure below is suspended by an inextensible string of length l from the point p. This point is free to move along a straight horizontal line under the action of the massless springs, each having spring constant k. Assume the mass is displaced only slightly from its equilibrium position, so appropriate approximations and Taylor expansions can be made.

   (a) Write down a set of generalized coordinates for the problem.

   (b) Write down the Lagrangian and obtain the Euler-Lagrange equations. How are the motion of p and the mass related (give an algebraic expression)?



Problem 2



Problem 3

(c) Show that the mass obeys a simple harmonic oscillator equation of motion of the form

$$\ddot{q} + \omega^2 q = 0 \qquad \text{with} \qquad \omega = \sqrt{\frac{2\,k\,g}{2\,k\,l + m\,g}}$$

(q is just an arbitrary symbol representing a generalized coordinate – which particular coordinate you use in the equation will depend on your choice of coordinate system.)

4. (106/196) A train is moving on a circular track of radius R at constant speed v. A pendulum consisting of a bob of mass m on a massless, inextensible string of length l is suspended in the train. The pendulum is only allowed to swing in a plane perpendicular to the direction of the train (this plane coincides with the plane formed by the radius of the track and the z axis). The pendulum makes an angle θ with the vertical.

   (a) Write down the position of the train in a coordinate system at rest with respect to the track (in terms of v, R, and t), the position of the bob in a coordinate system at rest with respect to the train (in terms of θ and l), and the position of the bob in the coordinate system at rest with respect to the track (in terms of v, R, t, θ, and l).

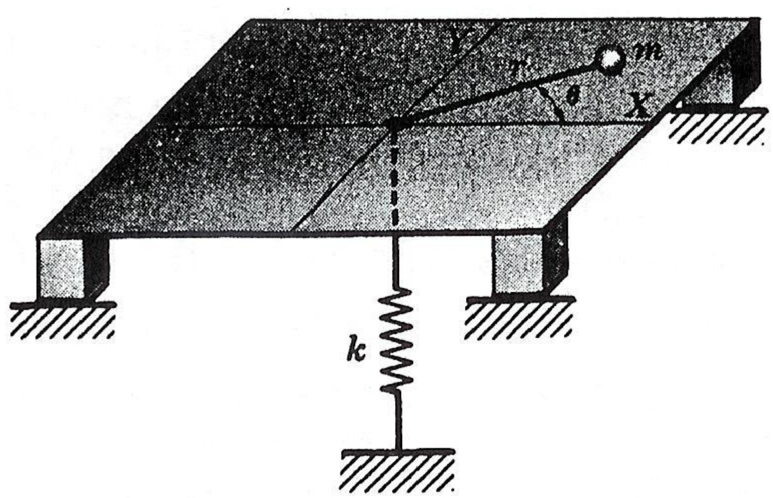
   (b) Show that the kinetic energy of the bob is

$$T = \frac{1}{2}\,m\left[v^2\left(1 + \frac{l}{R}\,\sin\theta\right)^2 + l^2\,\dot{\theta}^2\right]$$

   (c) Obtain the Lagrangian and use the Euler-Lagrange equation to show that the equation of motion of the bob is

$$\ddot{\theta} - \frac{v^2}{l\,R}\left(1 + \frac{l}{R}\,\sin\theta\right)\cos\theta + \frac{g}{l}\,\sin\theta = 0$$

   (d) Assume $\theta \ll 1$ so that you may Taylor expand the trigonometric functions around $\theta = 0$. Find the equilibrium position of the bob, θ_{eq} such that $\ddot{\theta} = 0$. Let $\eta(t) = \theta(t) - \theta_{eq}$



Problem 5

denote deviations from this equilibrium angle. Show that the equation of motion can be rewritten in simple harmonic oscillator form

$$\ddot{\eta} + \omega^2\,\eta = 0 \qquad \text{with} \qquad \omega^2 = \frac{g}{l} - \frac{v^2}{R^2}$$

This problem is similar to Example 7-6 in Thornton. In that case, though, the equation of motion is simpler and θ_{eq} can be found explicitly without making the small-angle approximation. It is useful to review that example to see how to Taylor expand correctly about θ_{eq} when θ_{eq} is not small.

5. (196) A particle of mass m is attached to a massless, inextensible string that passes through a hole in a smooth horizontal table and is fastened to a massless spring that is connected rigidly to the floor directly under the hole. The mass resides at the hole when the spring is at its rest length. The spring has constant k.

   (a) Write down a set of appropriate generalized coordinates for the problem and the generalized force for each generalized coordinate.

   (b) Write down the Lagrangian for the problem in terms of the generalized coordinates and obtain the Euler-Lagrange equations for the problem. (Notice how the generalized force appears in the E-L equations.)

   (c) What is the conserved canonical momentum? Use this conserved momentum to rewrite the Euler-Lagrange equation for the non-ignorable coordinate as an effective one-dimensional equation of motion.

   (d) Write down an effective one-dimensional potential energy and Lagrangian that would yield the same equation of motion. Sketch the potential energy function and describe qualitatively the dynamics.

   (e) Your effective Lagrangian looks almost the same as the Lagrangian one would obtain if one substituted the conserved canonical momentum into the original Lagrangian. How does it differ? Why is simply substituting back in L not a valid thing to do?

6. Nonholonomic constraints. Each of these individually is not very difficult, which is why they are grouped together to make one 10-point problem.

   (a) Hand and Finch 1-26b. The solution to 1-26a is in Appendix A of Chapter 1 (though you might find it interesting to spend 10 minutes trying to figure out for yourself how to determine when the particle leaves the surface of the ball).

   (b) Hand and Finch 1-27.