Physics 106a/196a – Midterm Exam – Due Nov 3, 2006

Instructions

Material: All lectures through Oct 19 and material on Lagrange multipliers on Oct 24, Sections 1, 2.1, and 2.2 of lecture notes, Hand and Finch Chapter 1, 2, and Sections 5.1 and 5.2. Only Ph196 students are responsible for the following topics:

- virtual work and generalized forces
- derivation of Euler-Lagrange equations via virtual work
- nonholonomic constraints
- Lagrangians for nonconservative forces
- incorporating nonholonomic constraints via Lagrange multipliers

Review the material ahead of time, consult me, the TAs, your fellow students, or other texts if there is material you are having trouble with.

Logistics: The exam consists of this page plus 2 pages of exam questions. Do not look at the exam until you are ready to start it. Please use a blue book if possible (makes grading easier), but there will be no penalty if you don't have one. Problems 1 and 2 are for 106 students only, 3 and 4 for 106 and 196 students, and 5 and 6 for 196 students only.

Time: 4 hrs, fixed time. You may take as many breaks as you like, but they may add up to no more than 30 minutes (2 x 15 minutes, 3 x 10 minutes, etc.).

Reference policy: Hand and Finch, official class lecture notes and errata, problem sets and solutions, your own lecture notes or other notes you have taken to help yourself understand the material. No other textbooks, no web searches, no interaction with your fellow students. Calculators and symbolic manipulation programs are neither needed or allowed.

Due date: Friday, Nov 3, 5 pm, my office (311 Downs). 5 pm means 5 pm. Late exams will require extenuating circumstances; otherwise, no credit will be given.

Grading: Each problem is 25 points out of 100. The exam is 1/3 of the class grade.

Suggestions on taking the exam:

- Go through and figure out roughly how to do each problem first; make sure you've got the physical concept straight before you start writing down formulae.
- Don't fixate on a particular problem. They are not all of equal difficulty. Come back to ones you are having difficulty with.
- Don't get buried in algebra (this really should not be an issue on this exam). Get each problem to the point where you think you will get most of the points, then come back and worry about the algebra.

1. (106) Consider a spherically symmetric distribution of mass density $\rho(r)$. Suppose a small hole is drilled along a diameter of the distribution so that a particle may be dropped in and allowed to move under influence of the gravitational field. What must the density profile be to make the motion of the particle simple harmonic? You should neglect the hole in determining the mass distribution.

2. (106) A nuclear reaction whose Q is known occurs in a photographic plate in which the tracks of the incident particle m_1 and the two product particles m_3 and m_4 can be seen. (The target particle m_2 is at rest.) To be clear, $Q > 0$ indicates that there is energy available to the final state mechanical energy that was not present in the initial state mechanical energy (and vice versa for $Q < 0$) – be careful about your sign convention. Find the energy of the incident particle in terms of m_1, m_3, m_4, Q, and the measured angles ψ_3 and ψ_4 between the incident track and the two final tracks. What happens if $Q = 0$?

3. (106/196) The Lagrangian for a relativistic particle in an electrostatic field is

$$L = -m\,c^2\sqrt{1 - \frac{v^2}{c^2}} - q\phi(\vec{r})$$

where c is the speed of light, $\vec{r}$ is the particle's position, $v = |\dot{\vec{r}}|$ is its speed, and $\phi(\vec{r})$ is the static electrostatic potential. (Yes, the negative sign on the first term is correct!) Find the canonical momenta and the Euler-Lagrange eqations (don't explicitly calculate the $\frac{d}{dt}$ term, leave it in the form $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)$). Calculate the Hamiltonian and prove that it is conserved. Is the Hamiltonian equal to

$$E = \frac{m\,c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

the total relativistic energy of the particle? Explain.

4. (106/196) Recall our example of the block on the inclined plane, with no friction between the block and the plane or between the plane and the surface on which it sits. Answer the following:

   (a) (10 pts) Under what symmetry transformation is the system invariant? What is the associated conserved momentum? Considering the block and the plane as a system of particles in the sense of Section 1.3 of the lecture notes, give an interpretation of the conserved momentum and what its conservation says about the total force acting on the system.

   (b) (15 pts) Obtain the equations of motion using Lagrange multipliers and the inertial system Cartesian coordinates – that is, use x_p, y_p, x_b, and y_b, with the latter two referenced to the same inertial origin as x_p and y_p. A full result consists of solutions for the Lagrange multipliers and for the accelerations in all four coordinates written in terms of the parameters of the problem (m, M, g, α). Find the constraint forces and give their physical interpretation.

   You do not need to do this problem completely from scratch – use whatever information from Hand and Finch or the lecture notes that you find useful. But realize that you must execute the full Lagrange multiplier methodology to get full credit.

5. (196) The term *generalized mechanics* has come to designate a variety of classical mechanics in which the Lagrangian contains time derivatives of q_i higher than the first. Problems for which $\dddot{x} = f(x, \dot{x}, \ddot{x}, t)$ have been referred to as "jerky" mechanics. By applying the methods of the calculus of variations, and assuming that there is a Lagrangian of the form $L(q_i, \dot{q}_i, \ddot{q}_i, t)$ and that Hamilton's principle holds with the zero variation of both q_i and $\dot{q}_i$ at the end points, obtain the analogue of the Euler-Lagrange equations. Apply your result to the Lagrangian

$$L = -\frac{m}{2}\, q\, \ddot{q} - \frac{k}{2}\, q^2$$

What familiar system does the equation of motion describe?

6. (196) Two parts:

(a) (10 pts) If L is a Lagrangian for a system of n degrees of freedom satisfying the Euler-Lagrange equations, prove that

$$L' = L + \frac{d}{dt} F(\{q_k\}, t)$$

yields the same Euler-Lagrange equations, where F is an arbitrary differentiable function of its arguments.

(b) (15 pts) The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\psi(\vec{r}, t)$$
$$\phi \rightarrow \phi' = \phi - \frac{\partial \psi}{\partial t}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field (5 pts)? Is the motion affected (5 pts)? How does your result in part (a) reconcile your result for the change in the Lagrangian with your result for the effect on the motion (5 pts)? (*Note: You must use the EM units convention of the lecture notes for this to work out. See Section 2.1.9 of the lecture notes.*)

For your own amusement after the exam, consider how this problem would change if we instead used the classical field Lagrangian from Problem Set 4, Problem 3, from which we obtained the Schrödinger equation. What additional transformation is necessary to leave the equations of motion unaffected?