Physics 106a/196a – Problem Set 5 – Due Nov 10, 2006

For 106a students, this problem set reviews material in Section 2.2 of the lecture notes that was already covered in Problem Set 4 and on the exam, and adds new material on Section 2.3 of the lecture notes – Hamiltonian dynamics, Liouville's Theorem, Virial Theorem. For 196a students, the problem set focuses on Section 2.4 of the lecture notes (though remember you are responsible for Section 2.3 also!). Problems 1-4 are for 106a students only, and 5-8 for 196a students only. 196a students are encouraged to take a look at 1-4 to get some practice on the Section 2.3 material.
The Virial Theorem was covered in class on Oct 26 but is not covered in the lecture notes. Refer to Section 7.13 of Thornton for it.

1. (106a) The midterm indicated a need for more practice on Lagrange multipliers, so: do Hand and Finch 1-22 using an inertial set of coordinates for the box and the pendulum and using Lagrange multipliers. Specifically: the box position and the pendulum position should use coordinates x_b, y_b, x_p, y_p that are all referenced to the same, inertial origin (don't confuse these coordinates with the symbols used in Midterm Exam Problem 4). Be sure you write down *all* the constraints in these four coordinates, and you treat *all* the coordinates as completely free variables to start with. You should stop after you have obtained the constraints and equations of motion – it is difficult to solve the system because the constraints are nonlinear in the coordinates.

2. (106a) Discuss the implications of Liouville's theorem on the focusing of beams of charged particles by considering the following simple case. An electron beam of circular cross section (radius R_0) is directed along the z axis. The density of electrons across the beam is constant, but the momentum components transverse to the beam (p_x and p_y) are distributed uniformly over a circle of radius p_0 in momentum space. If some focusing system reduces the beam radius from R_0 to R_1, find the resulting distribution of the transverse momentum components. What is the physical meaning of the result? (Consider the angular divergence of the beam.)

3. (106a) High-temperature heat capacity of a crystal via the Virial Theorem.

   Treat a crystal lattice as an ensemble of N harmonic oscillators. The equipartition theorem from statistical mechanics tells us that, for a three-dimensional crystal, the mean kinetic energy and the temperature are related by $\langle T \rangle = \frac{3}{2}\, k\, \Theta$ where T is kinetic energy, Θ is absolute temperature in Kelvins, and k is Boltzmann's constant. First, use the virial theorem to relate the temperature Θ of the system to the total energy, including the potential energy in the harmonic oscillators. From this, calculate the heat capacity, $dE/d\Theta$, where E is the total energy. (Remember, there are N harmonic oscillators!) The result is known as the Law of Dulong and Petit.

   The Dulong-Petit Law turns out in practice to only be valid at high temperatures. To obtain a correct result at all temperatures, Einstein showed that one needs to take into account Bose-Einstein statistics for the vibrations (when quantized as *phonons*) and Debye showed that one must take into account the fact that there is a maximum spatial wavelength of these vibrations due to the fact that a crystal is not continuous but is made of atoms at regularly spaced sites. The temperature above which Dulong and Petit is correct is given by the Debye temperature Θ_D, which is simply the temperature corresponding to the energy of the most energetic (shortest wavelength) vibrational mode, $\Theta_D = h\,\nu_D/k$. Debye temperatures vary from material to material, but tend to be in the hundreds of Kelvins.

4. (106a) A particle moves in a spherically symmetric force field with potential energy given by $U(r) = -k/r$. Calculate the Hamiltonian function in spherical coordinates, and obtain the canonical equations of motion. Sketch the path that a representative point for the system would follow on a surface H = constant in phase space. Begin by showing that the motion must lie in a plane so that the phase space is four-dimensional (r, θ, p_r, p_θ, but only the first three are nontrivial). Calculate the projection of the phase space path on the $r - p_r$ plane, then take into account the variation with θ.

5. (196a) One of the attempts at combining the two sets of Hamilton's equations into one tries to take q and p as forming a complex quantity. Show directly from Hamilton's equations of motion that, for a system with one degree of freedom, the transformation

$$Q = q + i\,p \qquad\qquad P = Q^*$$

is not canonical if the Hamiltonian is left unaltered. Can you find another set of coordinates Q', P' that are related to Q, P by a change of scale (possibly different for q and p and for Q and P), and that are canonical? Can you construct the generating function for the transformation?

6. (196a) Using symplectic notation, show that the "product" of two canonical transformations is also a canonical transformation. By "product", we mean functional composition:

$$\left[f \cdot g\right](x) = f(g(x))$$

Do not use just use the fact that the product of the Jacobian determinants is unity if each individual transformation has unity Jacobian determinant – make a proof along the lines of the proof of the symplectic condition in Section 2.4.2 of the lecture notes.

Also, why is any canonical transformation invertible?

Because a product of two canonical transformations is a canonical transformation, and because any canonical transformation has an inverse that is also a canonical transformation, canonical transformations form a mathematical *group*.

7. (196a) A particle of mass m moves in one dimension under a potential $V = -k/|x|$. For energies that are negative, the motion is bounded and oscillatory. By the method of action-angle variables, find an expression for the period of motion as a function of the particle's energy.

8. (196a) Show that the Hamilton-Jacobi equation is separable for a particle moving in a potential of the form

$$V(r, \theta, \phi) = V_r(r) + \frac{V_\theta(\theta)}{r^2} + \frac{V_\phi(\phi)}{r^2 \sin^2\theta}$$

where r, θ, ϕ are spherical coordinates. Find the general solution for Hamilton's Principal Funcion $S(q, P, t)$ and from it derive the general solution to the equations of motion.