Physics 106b/196b – Problem Set 8 – Due Jan 12, 2007

Version 3: Jan 12, 2007

This problem set focuses on the mathematics of rotations and some initial material on physics in rotating coordinate systems, Sections 5.1 and 5.2.1-5.2.2 of the lecture notes. The length of the problem set is misleading; much of it is in expository material. Problems 1 and 2 are for 106b students only, 3 though 5 for 106b an 196b students, and 6 and 7 are for 196b students only.

Version 2: Some clarifications on (1), (3), and (6):

- "why" part of Problem 1 unclear.
- In Problem 3, you should assume the orthogonal matrices are real.
- In Problem 6, "given by the angle between the two axes of rotation" is misleading.

Version 3: Further correction to Problem 6.

1. (106b) Show by a specific example that two finite rotations about different axes do not commute. On an algebraic level, why is it that finite rotations do not commute but infinitesimal ones do (try your example with infinitesimal rotations instead)? Specifically, what terms appear when you write out a finite rotation that ruin commutativity, and why are these terms not important for infinitesimal rotations?

2. (106b) Show that the product $\mathbf{C} = \mathbf{A}\,\mathbf{B}$ of two orthogonal matrices $\mathbf{A}$ and $\mathbf{B}$ is an orthogonal matrix. You will do this by using the orthogonality properties of $\mathbf{A}$ and $\mathbf{B}$ to prove that $\mathbf{C}$ satisfies the orthogonality relations given in Section 5.1.2 of the lecture notes.

3. (106b/196b) Show that, for arbitrary spatial dimension N, the eigenvalues of a *real* orthogonal matrix $\mathbf{R}$ all have unit magnitude (though they may be complex, even though the matrix is real). You can do this by allowing $\vec{c}$ to be a (possibly complex) eigenvector of $\mathbf{R}$ and considering the quantity $(\mathbf{R}\,\vec{c})^T\,(\mathbf{R}\,\vec{c}^*)$. You must of course use the orthogonality properties of $\mathbf{R}$. You should *not* assume that $\mathbf{R}$ preserves the norm of $\vec{c}$, which is given by $\vec{c}^T\,\vec{c}^*$ (or $(\vec{c}^*)^T\,\vec{c}$) – it is not necessary to make such an assumption. (Hint: you need to consider two different ways to reduce the expression $(\mathbf{R}\,\vec{c})^T\,(\mathbf{R}\,\vec{c}^*)$ in order to get a constraint on the magnitude of the eigenvalue.)

   Orthogonal matrices have determinant ± 1. We usually consider only the "special" subset of orthogonal matrices that have determinant $+1$ because these are the ones that correspond to rotations; including those with determinant -1 would include reflections or combinations of rotations and reflections. (By the way, you can see how to prove this fact about the determinant in Section 5.3.1 of the lecture notes, subsection **Relation of Euler Form to Single-Axis Rotation**.). Considering only rotations in $N = 3$ dimensions, explain by a qualitative argument why this implies that one eigenvalue must be 1. (Hint: what does it mean, physically or intuitively, for a vector to be the eigenvector of a rotation matrix with eigenvalue 1?) Then, using the facts that the determinant is 1 and one eigenvalue is 1 for $N = 3$, show that the other two eigenvalues are of the form $\exp(\pm i\alpha)$ for some angle α. (Hint: recall that the determinant of a matrix is independent of whether it has been diagonalized or not. Can you therefore write the determinant in terms of the eigenvalues?)

4. (106b/196b) Practice with indices and the summation convention. Though 106b students will not be tested on tensors, the use of indices and the summation convention will be something you will find useful later in E&M and future courses, so you should acquire some familiarity with them. We state in Appendix A of the notes, Equations A.25, A.26, and A.28, the following vector identities:

$$\begin{aligned}\vec{a} \times (\vec{b} \times \vec{c}) &= (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{a} \cdot \vec{b})\,\vec{c} \\ (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) &= \vec{a} \cdot \left[\vec{b} \times (\vec{c} \times \vec{d})\right] \\ &= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})\end{aligned}$$

Using the definitions of the dot and cross product in Equations A.13 and A.14 using index notation, the properties of ϵ_{ijk} under various permutations of the indices (Section A.1), and the identities given in Equations A.20-A.23 (which you may assume to be true, you don't need to prove them!), show that these identities are true.

Next, suppose, for the $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ identity, we have that $\vec{a} = \vec{\nabla}$, $\vec{c} = \vec{\nabla}$, and $\vec{b}$ and $\vec{d}$ are position-dependent. Why are the expressions on the right hand side given for that identity incorrect or misleading in some way? Look at this section of the lecture notes errata for a discussion of a similar problem with the first $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$ identity. This is one reason why indexed notation is a vast improvement over the vector notation you are used to.

5. (106b/196b) Modified version of Hand and Finch 7.1. We have given one proof in the lecture notes of addition of angular velocities; here we will develop another. This problem will also provide good practice with angular velocities and their coordinate representations, which is a subtle topic and one that you *must* understand to make sense of rotating coordinate systems and dynamics of rigid bodies.

   In an inertial reference frame F'', a locomotive is rounding a curve of radius R at a speed v in the counterclockwise direction. Let $\vec{\Omega}_t$ be the angular velocity of the train as measured in the inertial frame F''. The wheels of the locomotive are turning at angular speed Ω_w with respect to the train (whose rest frame we call F'). Define $\vec{\Omega}_w$ as the angular velocity of the wheels (whose rest frame we call F) as measured in the train frame F'. Denote by $\vec{\omega}$ the angular velocity of the spinning wheels relative to the inertial (space) frame F''. Denote by $\mathbf{R}_1$ the rotation matrix that transforms from F to F' and by $\mathbf{R}_2$ the rotation matrix that transforms from F' to F''.

   (a) Addition of angular velocities: First, write down the natural coordinate representations $\underline{\vec{\Omega}}_w'$ and $\underline{\vec{\Omega}}_t''$. Why are these the "natural" representations? Next, because these angular velocities are the angular velocities of F relative to (as measured in) F' and of F' relative to (as measured in) F'', it holds that $\dot{\mathbf{R}}_1\mathbf{R}_1^T = \underline{\vec{\Omega}}_w' \cdot \vec{\mathbf{M}}$ and $\dot{\mathbf{R}}_2\mathbf{R}_2^T = \underline{\vec{\Omega}}_t'' \cdot \vec{\mathbf{M}}$. (Why do we not have to worry about the difference between $\vec{\mathbf{M}}$, $\vec{\mathbf{M}}'$ and $\vec{\mathbf{M}}''$?) It holds that the rotation matrix to go from F to F'' is $\mathbf{R} = \mathbf{R}_2\mathbf{R}_1$. It also holds that $\dot{\mathbf{R}}\,\mathbf{R}^T = \underline{\vec{\omega}}'' \cdot \vec{\mathbf{M}}$ because $\underline{\vec{\omega}}''$ is the coordinate representation in F'' of the angular velocity of F relative to (as measured in) F''. Use these various equalities to demonstrate that angular velocities add by showing that $\underline{\vec{\omega}}'' = \mathbf{R}_2\,\underline{\vec{\Omega}}_w' + \underline{\vec{\Omega}}_t''$.

(b) Practice with notation and coordinate representations of angular velocities: Write down explicit forms (*i.e.*, in terms of R, v, Ω_w, and t) for all three coordinate representations of all three velocities: $\mathbf{R}_1^T\underline{\vec{\Omega}}_w'$, $\mathbf{R}^T\underline{\vec{\Omega}}_t''$, $\mathbf{R}^T\underline{\vec{\omega}}''$, $\underline{\vec{\Omega}}_w'$, $\mathbf{R}_2^T\underline{\vec{\Omega}}_t''$, $\mathbf{R}_2^T\underline{\vec{\omega}}''$, $\mathbf{R}_2\underline{\vec{\Omega}}_w'$, $\underline{\vec{\Omega}}_t''$, and $\underline{\vec{\omega}}''$. You may do this completely algebraically by writing down the "natural" coordinate representation of each velocity and using the rotation matrices, or you may do it more intuitively by just writing down the results with some written justification.

6. (196b) More difficult version of (1): Suppose two successive finite rotations defined by vectors $\vec{\phi}_1$ and $\vec{\phi}_2$ are carried out, equivalent to a single rotation $\vec{\phi}$. (This is shown in Section 5.3.1 of the lecture notes, the subsection titled **Relation of Euler Form to Single-Axis Rotation**.) Show that $\frac{1}{2}\,\vec{\phi}_1$, $\frac{1}{2}\,\vec{\phi}_2$, and $\frac{1}{2}\,\vec{\phi}$ form the sides of a spherical triangle, with the angle opposite to $\frac{1}{2}\,\vec{\phi}$ determined by the angle between the two axes of rotation $\vec{\phi}_1$ and $\vec{\phi}_2$. You will have to do some research on your own in spherical geometry to know what relations $\vec{\phi}_1$, $\vec{\phi}_2$, and $\vec{\phi}$ must satisfy to form the sides of a spherical triangle and what the angle opposite to $\vec{\phi}$ is in terms of $\vec{\phi}_1$ and $\vec{\phi}_2$.

7. (196b) Show that the contraction of a tensor of rank m and a tensor of rank n is a tensor of rank $m+n-2$. Specifically, show that if the object $\mathcal{Z}$ is defined in terms of the tensors $\mathcal{X}$ and $\mathcal{Y}$ by, in any given coordinate representation

$$Z_{i_1\cdots i_{m-1}j_1\cdots j_{n-1}} = X_{i_1\cdots i_{m-1}k}Y_{j_1\cdots j_{n-1}k}$$

(summation convention used), then $\mathcal{Z}$ satisfies the transformation properties of a rank $m+n-2$ tensor.