Lecture 42:
Rotations and Orbital Angular Momentum in Two Dimensions

Date Revised: 2009/02/04
Date Given: 2009/02/04

Plan of Attack

We will study the problem of rotations and orbital angular momentum in the following sequence:

- **Rotation Transformations in Two Dimensions**
  We will first review classical rotation transformations in two dimensions, derive the formula for the active rotation transformation of a quantum mechanical state, and show that the generator of the transformation is the quantum analogue of the classical z-axis angular momentum, L_z.
- **The L_z Eigenvector-Eigenvalue Problem**
  L_z will be a Hermitian, observable operator. For Hamiltonians for which $[H, L_z] = 0$ – *i.e.*, Hamiltonians with rotational symmetry in two dimensions – H and L_z are simultaneously diagonalizable. Therefore, eigenvectors of H must also be eigenvectors of L_z, and so the eigenvectors of L_z will be of interest. We calculate the eigenvectors and eigenvalues of L_z and see how the requirement that eigenvectors of H be eigenvectors of L_z reduces the Schrödinger Equation to a differential equation in the radial coordinate only.
- **Rotation Transformations in Three Dimensions**
  We then generalize classical rotation transformations to three dimensions and use correspondences to identify the three angular momentum operators L_x, L_y, and L_z, as well as the total angular momentum magnitude L^2.

Plan of Attack (cont.)

- **The L^2-L_z Eigenvalue Problem**
  In three dimensions, we shall see that L_x, L_y, L_z, and L^2 are all Hermitian, observable operators. But no two of L_x, L_y, and L_z commute, while each of them commutes with L^2, so it becomes clear that useful set of operators to work with for Hamiltonians that are rotationally invariant in three dimensions is H, L_z, and L^2. We therefore consider the joint eigenvector-eigenvalue problem of L^2 and L_z and determine how it reduces the Schrödinger Equation to a differential equation in the radial coordinate only.

We will refer back frequently to material on continuous symmetry transformations that we covered in Section 12, so please review that material.

Passive Classical Rotation Transformations in Two Dimensions

A *passive* coordinate system rotation in two dimensions by an angle θ counterclockwise yields the following relationship between the components of a vector $\vec{a}$ in the untransformed system (a_x, a_y, a_z) and its components in the transformed system $(a_{x'}, a_{y'}, a_{z'})$:

$$a_{x'} = a_x\, c_\theta + a_y\, s_\theta \qquad a_{y'} = -a_x\, s_\theta + a_y\, c_\theta \qquad a_{z'} = a_z$$

where $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$ as usual. The x' and y' axes are obtained by rotating the x and y axes counterclockwise by the angle θ. The rotation is termed passive because we are not changing the vector $\vec{a}$, we are simply writing its representation in terms of a new set of coordinate axes. The above may be written as a matrix operation:

$$\begin{bmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{bmatrix} = \begin{bmatrix} c_\theta & s_\theta & 0 \\ -s_\theta & c_\theta & 0 \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \equiv \mathbf{R}_{P,\theta\widehat{z}} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

where we use the $_P$ subscript to indicate a passive transformation (as we did in the QM case) and the $_{\theta\widehat{z}}$ subscript to indicate the rotation angle from the untransformed to the transformed system.

Let us emphasize here the concept of **coordinate representations** of classical vectors. The unprimed and primed coordinate systems are just two different ways of labeling space. The vector $\vec{a}$ has not changed by relabeling space. However, the components of $\vec{a}$ in the two coordinate systems are different. We thus call (a_x, a_y, a_z) and $(a_{x'}, a_{y'}, a_{z'})$ two different **coordinate representations** of the same vector $\vec{a}$. This is very much the same idea as our discussion of different position-basis representations of a state $|\psi\rangle$ depending on whether we project it onto the position-basis elements for the original coordinate system $\{|x, y\rangle\}$ or those of the transformed coordinate system $\{x', y'\}$, giving position-basis representations $\langle x, y|\psi\rangle$ and $\langle x', y'|\psi\rangle$, respectively.

Rotations Transformations in Two Dimensions (cont.)

Active Classical Rotation Transformations in Two Dimensions

The classical analogue of an *active* coordinate transformation is to change the vector; that is, to fix the coordinate system and to change the vector by changing its coordinate representation (components) in that coordinate system. If we denote the **new vector** by $\vec{a}\,'$, then the coordinate representation (components) of $\vec{a}\,'$ are related to those of $\vec{a}$ by

$$\begin{bmatrix} a_x' \\ a_y' \\ a_z' \end{bmatrix} = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \equiv \mathbf{R}_{A,\theta\widehat{z}} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

or

$$a_x' = a_x\, c_\theta - a_y\, s_\theta \qquad a_y' = a_x\, s_\theta + a_y\, c_\theta \qquad a_z' = a_z$$

where both are being represented in the untransformed coordinate system. This transformation corresponds to physically rotating $\vec{a}$ by θ CCW about $\widehat{z}$. $\vec{a}\,'$ is a **different** vector than $\vec{a}$ because its coordinate representation in this fixed coordinate system is different from that of $\vec{a}$. Again, this is in direct analogy to our active transformations in QM, where we kept the position basis unchanged but transformed the state, $|\psi\,'\rangle = T|\psi\rangle$, and saw that the states had different position-basis representations in the same basis, $\langle x, y\,|\psi\rangle$ and $\langle x, y\,|\psi\,'\rangle$.

Rotations Transformations in Two Dimensions (cont.)

Passive vs. Active Classical Rotation Transformations

The key difference between active and passive transformations is that the active transformation rotates the vector $\vec{a}$, creating a new vector $\vec{a}\,'$, while the passive transformation rotates the coordinate system so that the representation of the vector $\vec{a}$ changes from (a_x, a_y, a_z) to $(a_{x'}, a_{y'}, a_{z'})$, but the vector $\vec{a}$ is unchanged. This is in exactly analogy to what we considered for QM states: for a passive transformation, we consider the projection of the untransformed state $|\psi\rangle$ onto the transformed position basis $\{|q'\rangle = T|q\rangle\}$ by looking at $\langle q'|\psi\rangle$, while, for an active transformation, we consider the projection of the transformed state $|\psi'\rangle = T|\psi\rangle$ onto the untransformed basis $\{|q\rangle\}$ by looking at $\langle q|\psi'\rangle$.

It may be helpful to realize that the unit vectors of the transformed system, $\widehat{x}\,'$, $\widehat{y}\,'$, and $\widehat{z}\,'$, are obtained by performing an active transformation on the unit vectors of the untransformed system, $\widehat{x}$, $\widehat{y}$, and $\widehat{z}$.

The mathematical difference between the passive and active transformations is just the change of sign of the s_θ terms; that is $R_{P,-\theta\widehat{z}} = R_{A,\theta\widehat{z}}$. This sign flip tells us that the coordinate representation of $\vec{a}$ in a transformed coordinate system is literally equal to the coordinate representation in the untransformed coordinate system of the vector $\vec{a}\,'$ that has been obtained from $\vec{a}$ by active rotation by $-\theta\widehat{z}$. Of course, in spite of this equality, we know $\vec{a}$ and $\vec{a}\,'$ are different vectors because the coordinate representations that are equal are coordinate representations in different coordinate systems (the transformed and untransformed systems). This is analogous to the situation in quantum mechanics of a passively transformed state having the same position-basis representation in the transformed basis as an actively transformed state has in the untransformed basis when the actively transformed state has been transformed using the inverse transformation as was used for the passive transformation (see Section 12.3).

It is convention to use $\mathbf{R}_{\theta\widehat{z}}$ for $\mathbf{R}_{A,\theta\widehat{z}}$ and to never use $\mathbf{R}_{P,\theta\widehat{z}}$. We will follow this convention.

Generators for Classical Rotation Transformations in Two Dimensions

Since we are going to be considering generators in the quantum case and for the three-dimensional classical case, it is worth showing how the above transformation can be written as an operator exponential of a generator. As we did in connection with identifying the generator of a continuous coordinate transformation of quantum mechanical states, we will begin by considering an infinitesimal version of the above coordinate transformation:

$$\mathbf{R}_{\delta\theta\hat{z}} = \begin{bmatrix} \cos\delta\theta & -\sin\delta\theta & 0 \\ \sin\delta\theta & \cos\delta\theta & 0 \\ 0 & & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -\delta\theta & 0 \\ \delta\theta & 1 & 0 \\ 0 & & 1 \end{bmatrix}$$

The generic relationship between a classical coordinate transformation and its generator is

$$\mathbf{T}_\epsilon = \mathbf{I} + \epsilon\,\mathbf{G}$$

Instead of relating Hermitian generators to unitary coordinate transformation operators, we must relate antisymmetric generators to orthogonal coordinate transformation operators. (The generator must be antisymmetric, not symmetric, because we have no i in the argument of the exponential as we do for the QM version). Thus, it makes sense to rewrite our infinitesimal rotation operators as

$$\mathbf{R}_{\delta\theta\widehat{z}} = \mathbf{I} + \delta\theta\,\mathbf{M}_z \qquad \mathbf{M}_z \equiv \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\mathbf{M}_z$ is the classical generator of rotations about $\widehat{z}$. The use of the $_z$ subscript of course foreshadows similar operators for rotations about $\widehat{x}$ and $\widehat{y}$.

Rotations Transformations in Two Dimensions (cont.)

We of course recover the finite classical rotation transformation by the appropriate infinite product, yielding an exponential:

$$\mathbf{R}_{\theta\widehat{z}} = \lim_{N\to\infty}\left(\mathbf{I} + \frac{\theta}{N}\,\mathbf{M}_z\right)^N = \exp\left(\theta\,\mathbf{M}_z\right)$$

We may evaluate the above using the fact

$$\mathbf{M}_z^2 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rotations Transformations in Two Dimensions (cont.)

This yields

$$\mathbf{R}_{\theta\hat{z}} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \mathbf{M}_z^n = \mathbf{I} + \theta\,\mathbf{M}_z + \sum_{n=1}^{\infty} \left(\frac{\theta^{2n}}{(2n)!} \mathbf{M}_z^{2n} + \frac{\theta^{2n+1}}{(2n+1)!} \mathbf{M}_z^{2n} \mathbf{M}_z \right)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(\frac{\theta^{2n}(-1)^n}{(2n)!} + \frac{\theta^{2n+1}(-1)^n}{(2n+1)!} \mathbf{M}_z \right)$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (c_\theta + s_\theta\,\mathbf{M}_z)$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as expected.

Rotations Transformations in Two Dimensions (cont.)

What is the significance of the $\mathbf{M}_z$ matrix? See:

$$-\vec{r}^{\,T}\mathbf{M}_z\vec{p} = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$= x\,p_y - y\,p_x = l_z$$

That is, $\mathbf{M}_z$ can be used to compute the z component of the angular momentum when combined with the $\vec{r}$ and $\vec{p}$ vectors. $\mathbf{M}_z$ is in some nontrivial way connected to the z component of angular momentum.