Noise Theory for Broadband Detection with Bolometers, Pair-Breaking, and Coherent Detectors

Sunil Golwala

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Contents

Section 1

Review of Fundamentals of Noise Theory

A Few Fourier Transform Relations

We'll use the following as our Fourier transforms. Our "analytic" case will consist of an infinitely long timestream that is sampled at infinite frequency, with Fourier transforms

$$\tilde{g}(f) = \int_{-\infty}^{\infty} dt\, g(t)\, e^{-j\omega t} \tag{1.1}$$

$$g(t) = \int_{-\infty}^{\infty} df\, \tilde{g}(f)\, e^{j\omega t} \tag{1.2}$$

For a timestream from $t = 0$ to $t = T$ with an even number N samples at frequency $2\, f_{nyq} = N/T$ where f_{nyq} is the Nyquist frequency, the transforms are

$$\tilde{g}_n = \frac{1}{T} \sum_{k=0}^{N-1} g_k\, e^{-j\omega_n t_k}\, \Delta t = \frac{1}{N} \sum_{k=0}^{N-1} g_k\, e^{-j\omega_n t_k} \tag{1.3}$$

$$g_k \equiv g(t_k) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \tilde{g}_n e^{j\omega_n t_k} \tag{1.4}$$

$$\text{with } \Delta t = \frac{1}{2\, f_{nyq}} \quad \Delta f = \frac{1}{T} = \frac{2\, f_{nyq}}{N} \quad t_k = k\, \Delta t \quad f_n = n\, \Delta f \quad \omega_n = 2\,\pi\, f_n \tag{1.5}$$

Review of Fundamentals of Noise Theory (cont.)

Note that the units of $\tilde{g}(f)$ and $\tilde{g}_n$ are different, the first has a 1/Hz in it. One can construct something like the former from the latter by defining

$$\tilde{g}(f_n) = \frac{\tilde{g}_n}{\Delta f} \tag{1.6}$$

If one is translating from analytic formulae for Fourier transforms to the discrete coefficients to put into a calculational routine, this is a useful relation.

We note that IDL, in its usual perverse fashion, considers the one Fourier coefficient at the Nyquist frequency to be at positive f_{nyq} rather than $-f_{nyq}$. One can eliminate this ambiguity by using an odd number of samples, but the FFT algorithm is much faster for an even number of samples.

Review of Fundamentals of Noise Theory (cont.)

Random noise in the timestream is characterized by its autocorrelation function. For the analytic case, this is

$$R(\tau) = \langle g(t)\, g(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, g(t)\, g(t+\tau) \tag{1.7}$$

For the discrete case, this is

$$R_m = R(t_m) = \frac{1}{N} \sum_{k=0}^{N-1} g_k g_{k+m} \tag{1.8}$$

where the subscript is assumed to wrap around ($k+m$ is taken modulo N). The argument of R is called the lag. $R(0)$ is the timestream variance. Notice that the noise depends on more than just its variance; the shape of R describes how correlated the noise is with itself in time, hence the name "autocorrelation function." The autocorrelation function has units of (timestream units)^2.

It is conventional to define the (noise) power spectral density by the Fourier transform of the autocorrelation function:

$$J(f) = \int_{-\infty}^{\infty} dt\, R(t)\, e^{-j\omega t} \tag{1.9}$$

$$J_n = J(f_n) = \frac{1}{N} \sum_{k=0}^{N-1} R_k\, e^{-j\omega_n t_k} \tag{1.10}$$

The convolution theorem tells us that the Fourier transform of a correlation function of the type calculated for R is the product of the Fourier transforms of the two functions being correlated (actually, the product of one with the complex conjugate of the other.) Using this, we find

$$J(f) = \lim_{T\to\infty} \frac{1}{T} |\tilde{g}(f)|^2 \tag{1.11}$$

$$J_n = J(f_n) = \frac{1}{T}\, |\tilde{g}(f_n)|^2 = \Delta f\, |\tilde{g}(f_n)|^2 = T\, |\tilde{g}_n|^2 = \frac{1}{\Delta f}\, |\tilde{g}_n|^2 \tag{1.12}$$

Note that J_n and $J(f_n)$ are the same, unlike $\tilde{g}_n$ and $\tilde{g}(f_n)$. The power spectral density has units of (timestream units)^2/Hz. The power spectral density characterizes the shape of the noise in frequency space. This is of course intimately connected to the autocorrelation function. Noise that has a constant power spectral density — equal noise at all frequencies — has a δ-function-like timestream autocorrelation function (no autocorrelation except at zero lag). Noise that decreases at high frequency has an autocorrelation function that is large at small lag and falls off at larger lag.

Review of Fundamentals of Noise Theory (cont.)

Parseval's Theorem states that

$$\int_{-\infty}^{\infty} dt\, |g(t)|^2 = \int_{-\infty}^{\infty} df\, |\tilde{g}(f)|^2 \qquad \frac{1}{N}\sum_{k=0}^{N-1} |g_k|^2 = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} |\tilde{g}_n|^2 \tag{1.13}$$

A very important corollary of Parseval's theorem relates the timestream variance to the power spectral density:

$$R(0) = \int_{-\infty}^{\infty} df\, J(f) \qquad R_0 = R(0\,\Delta t) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \Delta f\, J_n = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \Delta f\, J(f_n) \tag{1.14}$$

The utility of defining J to have units of 1/Hz now becomes clear — integrate J over frequency and you get your timestream noise variance! The naming of J is now obvious: it is literally the noise variance per unit frequency, and it can be simply added up over all frequencies to get the total timestream noise variance.

Because the timestream is real in the case of real measurements, $\tilde{g}(-f) = \tilde{g}^*(f)$ and $\tilde{g}_{-n} = \tilde{g}_n^*$, so $J(-f) = J(f)$ and $J_{-n} = J_n$. Therefore, it is standard practice to consider only the positive frequencies and include a 2 in front so that $J(f)$ gives you the sum of the noise power at f and $-f$. If one then integrates or sums J over only positive frequencies, one again recovers the timestream noise variance.

Review of Fundamentals of Noise Theory (cont.)

Aside: one will frequently see $\sqrt{J}$ plotted and called the "power spectral density". This is wrong. J is the power spectral density. $\sqrt{J}$ is a convenience. Note that $\sqrt{J}$ has units of (timestream units)$/\sqrt{\mathrm{Hz}}$, which is infinitely confusing to first-year graduate students because it gives the impression of being something that is directly related to timestream noise, but carries a bizarre $1/\sqrt{\mathrm{Hz}}$.

Review of Fundamentals of Noise Theory (cont.)

What is the use of having J? Consider a linear system, which, by definition, is one whose behavior is characterized by differential equations that are linear in time derivatives. The Fourier transform of the mth-order time derivative $(d/dt)^m g(t)$ is just $(j\,\omega)^n \tilde{g}(f)$. Hence, if you Fourier transform a linear differential equation in time, one gets an algebraic equation in frequency space that may have multiple powers of ω but does not have different ω's.

Review of Fundamentals of Noise Theory (cont.)

A simple example is the response of a thermal system with a thermal heat capacity C and conductance G to a power input $P(t)$. Conservation of energy tells us

$$C \frac{d}{dt} \delta T(t) = \frac{dU}{dt} = P(t) - G\, \delta T(t) \tag{1.15}$$

where G is the thermal conductance across the link from the heat capacity C to a fixed-temperature bath. The left side relates the rate of change in energy content U of the system to the power flowing in, $P(t)$, and the power flowing out via the link, $G\, \delta T(t)$. The above is a linear differential equation. Its Fourier transform is

$$j\,\omega\, C\, \widetilde{\delta T}(f) = \widetilde{P}(f) - G\, \widetilde{\delta T}(f) \tag{1.16}$$

with solution

$$\widetilde{\delta T}(f) = \frac{1}{G} \frac{\widetilde{P}(f)}{1 + j\,\omega\,\tau} \tag{1.17}$$

where $\tau = C/G$. The behavior of the temperature response at frequency f depends only on the power input at frequency f. If one has a random noise power (*e.g.*, photon noise) coming in at frequency f, then we can calculate the resulting temperature noise at the frequency without reference to any other frequency. This is a result of the linearity of the original differential equation.

Review of Fundamentals of Noise Theory (cont.)

Another obvious example of interest to us are the fluctuations in the cosmic microwave background, which are broken down using the Fourier transform on the sphere rather than for a one-dimensional timestream. But the principle is the same.

Simple Derivation of Shot Noise for a Generic Flow

Another basic derivation we will find useful is the noise on a flow that consists of a set of discrete carriers each transporting some quantum of the flowing quantity. For electrical current flowing through a tunnel barrier, for example, the carriers are individual electrons that tunnel at some average rate Γ resulting in an electrical current $I = e\,\Gamma$. For power flowing through a thermal link, you can think of it as a spectrum of phonons with typical energy $k\,T$ flowing through a thermal link at a typical rate Γ, yielding a power flow $P = k\,T\,\Gamma$. What is the noise on this kind of flow due to the discrete nature of the carriers and assuming their flow is not autocorrelated aside from the overall constraint that the average flow be known?

Let's first consider a situation in which we have carriers of fixed quantum size q flowing at an average rate Γ to obtain a current $\dot{Q}(t)$ with $\langle \dot{Q} \rangle = q\,\Gamma$. Let's write the current as a sum of events occurring at random times t_p:

$$\dot{Q}(t) = \lim_{T \to \infty} \sum_{p \approx -\Gamma\,T/2}^{\Gamma\,T/2} q\,\delta(t - t_p) \tag{1.18}$$

where the limits of the sum indicate that approximately $\Gamma\,T$ flow events happen in the time T, which we will take to infinity in the end. Recall that a δ-function has units of the inverse of its argument, so the right side indeed has units of q/time.

Review of Fundamentals of Noise Theory (cont.)

The analytic Fourier transform of the above is

$$\tilde{\dot{Q}}(f) = \sum_{p \approx -\Gamma\, T/2}^{\Gamma\, T/2} q\, e^{-j\omega_p t} \tag{1.19}$$

The noise power spectral density (PSD) is given by (considering only positive frequencies and therefore including the factor of 2 discussed before)

$$J(f) = \lim_{T\to\infty} \frac{2}{T} |\tilde{\dot{Q}}(f)|^2 = \lim_{T\to\infty} \frac{2}{T} \left| \sum_{p \approx -\Gamma\, T/2}^{\Gamma\, T/2} q\, e^{-j\omega\, t_p} \right|^2 \tag{1.20}$$

Review of Fundamentals of Noise Theory (cont.)

There are three kinds of terms to consider in the squaring. The first term consists of the multiplication of each term with its conjugate. This just yields q^2 for each such term. The second term consists of the sum of all the cross-terms of the form $q^2\, e^{-j\,\omega\,(t_p - t_s)}$ with $p \neq s$. Since the events are uncorrelated, the time differences $t_p - t_s$ will take on all values with equal probability, resulting in phases $\omega\,(t_p - t_s)$ that are uniformly distributed between 0 and $2\,\pi$. The resulting complex numbers $e^{-j\,\omega\,(t_p - t_s)}$ will thus be uniformly distributed on the unit circle in the complex plane. Averaging together an infinite set of such numbers yields zero for $\omega \neq 0$. For $\omega = 0$ the averaging does not happen and one just ends up with an infinity of terms. The result is

$$J(f \neq 0) = \lim_{T \to \infty} \frac{2}{T}\, q^2 \sum_{p \approx -\Gamma\, T/2}^{\Gamma\, T/2} 1 = \lim_{T \to \infty} \frac{2}{T}\, q^2 \Gamma\, T = 2\, q^2\, \Gamma = 2\, q\, \langle \dot{Q} \rangle \tag{1.21}$$

$$\begin{aligned} J(f = 0) &= \lim_{T \to \infty} \frac{1}{T} q^2 \left[\sum_{p \approx -\Gamma\, T/2}^{\Gamma\, T/2} 1 + \sum_{p \approx -\Gamma\, T/2}^{\Gamma\, T/2} \sum_{s \approx -\Gamma\, T/2}^{\Gamma\, T/2} 1 \right] \\ &= q^2\, \Gamma + q^2\, \Gamma^2 \lim_{T \to \infty} T = q\, \langle \dot{Q} \rangle + \langle \dot{Q} \rangle^2 \delta(f) = q\, \langle \dot{Q} \rangle + \langle \dot{Q} \rangle^2 \delta(f) \end{aligned} \tag{1.22}$$

where we have used the fact that the sum over flow events gives $\Gamma\, T$ on average by the way the sum's limits are defined, that this average becomes exact in the limit $T \to \infty$, and that $\delta(f) = \lim_{T \to \infty} T$.

Review of Fundamentals of Noise Theory (cont.)

For later reference, note that $\sum_{p \approx -\Gamma T/2}^{\Gamma T/2} 1$ arises from the sum of the squares of the quantum in each flow event. That is, it is really more like a variance than an average. In this case, that quantum was always q because we assumed a simple Poisson shot noise process, where each event consisted of zero or one uncorrelated quanta flowing. There is thus no difference between the sum of quanta and the sum of the squares of the quanta. We will return to this term later for photon noise, for which that is not true.

The expression for $J(f \neq 0)$ gives the power spectral density of the noise for such flow processes. The expression for $J(f = 0)$ is different because it also include the mean flow as the $\delta(f)$ term. That piece should be ignored for the purposes of calculating noise. Note that $J(f = 0)$ includes no factor of 2 for positive and negative frequencies because there is a single $f = 0$ frequency.

One might be worried by the fact that the noise PSD is constant and thus, integrated over infinite frequency, it would given an infinite timestream noise variance. This makes sense. Because we have modeled the current as being an infinite number of δ-function events, the variance of the timestream, which would end up as a sum over the products of δ functions, would be infinite. In practice, the way this is circumvented is that the flow events are not truly δ-function-like — they take some finite time — and they cannot be spaced an infinitely small time apart. This results in a falloff of the noise PSD at some high frequency, making the timestream noise variance finite.

Examples of Application of Shot Noise Power Spectral Density

Shot noise of a tunnel barrier: Given a tunnel barrier with a current I consisting of charges flowing in one direction, the noise power spectral density is

$$J(f \neq 0) = 2\,e\,I \tag{1.23}$$

If the current I is the difference of two currents going in opposite directions, $I = I_{\rightarrow} - I_{\leftarrow}$, then the noises on those two currents should be considered independently and added in quadrature. This gives

$$J(f \neq 0) = 2\,e\,(I_{\rightarrow} + I_{\leftarrow}) \tag{1.24}$$

Phonon noise in an isothermal weak thermal link: As before, let's suppose that we have heat capacity C linked to a thermal bath by a conductance G, with C having the same temperature as the bath. There is no net power flow in this case. However, to maintain the absorber in thermal equilibrium with the bath, there are two power flows $P = G\,T$ in opposite directions maintaining this. The form is obtained by simply noting that the average energy of a phonon emitted by the absorber or the bath will be $k\,T$ because they sit at temperature T (assuming $T \ll$ Debye temperature) and that G characterizes the rate at which energy flows through the link. The above expression is the only combination of G and T with the right units. The average carrier energy is $k\,T$, so the noise PSD is

$$J(f \neq 0) = 2\,k\,T\,(G\,T + G\,T) = 4\,k\,T^2 G \qquad (1.25)$$

This was a bit of a handwavy calculation, but one can show that it is correct by integrating over the emission rate as a function of energy, using the usual shot noise expression at each energy and adding the contributions at different energies and in the two directions in quadrature.

When C and the bath are not at the same temperature, there is a correction factor in the front of order unity. Mather calculates this factor for a continuous weak link G. It can be calculated for discrete weak links (*e.g.*, electron-phonon decoupling), too. On dimensional grounds, one always ends up with a formula of the above kind.

Review of Fundamentals of Noise Theory (cont.)

Photon shot noise: Consider an incoming optical power Q in a narrow frequency band centered around *nu*. The shot noise formula gives

$$J(f \neq 0) = 2\,(h\,\nu)\,Q \tag{1.26}$$

because $h\,\nu$ is the energy of the photons and $Q/h\,\nu$ is their arrival rate. For a broadband optical power, we add the contributions at different frequencies in quadrature:

$$J(f \neq 0) = 2\int_{\nu_1}^{\nu_2} d\nu\,(h\,\nu)\,\frac{dQ}{d\nu} = 2\,h\,\langle\nu\rangle\,Q \tag{1.27}$$

where $dQ/d\nu$ is the power per unit spectral bandwidth, $Q = \int_{\nu_1}^{\nu_2} d\nu\,(dQ/d\nu)$, and $\langle\nu\rangle$ is the spectrum-weighted mean spectral frequency. This is the standard result, neglecting the "Bose term", which we will come to shortly.

Section 2

Photon Noise

Derivation of Noise-Equivalent Power

Let's use the above kind of formalism to derive the full result for photon noise. We neglect optical efficiency to begin with.

Because photons are bosons, quantum mechanics tells us that the variance on the number of photons occupying a given spatial and spectral mode is

$$\langle(\delta N)^2\rangle = N(N+1) = N + N^2 \tag{2.1}$$

Photon Noise (cont.)

One can show that N gives the arrival rate of photons per unit spectral bandwidth. For a quick derivation, consider the Planck blackbody law:

$$B(\nu, T) = \left(\frac{2\,h\,\nu^3}{c^2}\right)\frac{1}{e^{h\nu/kT} - 1} = \frac{2}{\lambda^2}\,\frac{h\,\nu}{e^{h\nu/kT} - 1} \tag{2.2}$$

which gives the brightness in W/m^2/ster/Hz. For a single spectral mode, the throughput (area $\times$ solid angle) is λ^2 and the energy is $h\,\nu$. Also, the above includes both polarizations. So the photon arrival rate per unit spectral bandwidth and per polarization is

$$\Gamma(\nu, T) = B(\nu, T)\,\frac{\lambda^2}{2\,h\,\nu} = \frac{1}{e^{h\nu/kT} - 1} \tag{2.3}$$

This is just the occupancy function at frequency ν for photons, which we denoted above as N. The units work out because we are left with (W/J)/Hz, which is unitless.

Photon Noise (cont.)

Let's consider photons arriving over a time T. This defines the spectral bandwidth of a "mode" as $\delta\nu = 1/T$: in a measurement that takes a time T, we cannot distinguish photons with spectral frequencies separated by less than $1/T$. The power in one polarization and one spectral mode is

$$Q_\nu = h\,\nu\,\Gamma(\nu, T)\,\delta\nu = N\,h\,\nu\,\delta\nu \qquad (2.4)$$

(Recall, $\Gamma(\nu, T)$ is an arrival rate per unit spectral bandwidth.) Now, let's calculate the variance on Q_ν over this time T:

$$\langle(\delta Q_\nu)^2\rangle = (h\,\nu\,\delta\nu)^2\,\langle(\delta N)^2\rangle = (h\,\nu\,\delta\nu)^2\left(N + N^2\right) = h\,\nu\,\delta\nu\,Q_\nu + Q_\nu^2 \qquad (2.5)$$

To get the total variance over a spectral band in one polarization, we sum:

$$\langle(\delta Q)^2\rangle = \sum_\nu \langle(\delta Q_\nu)^2\rangle = \sum_\nu (h\,\nu\,Q_\nu\delta\nu) + \sum_\nu Q_\nu^2 \qquad (2.6)$$

Photon Noise (cont.)

Let's define $dQ/d\nu = Q_\nu/\delta\nu$ to be the power per unit spectral bandwidth. Then we have

$$\langle(\delta Q)^2\rangle = \delta\nu \left[\sum_\nu h\,\nu\,\delta\nu\,\frac{dQ}{d\nu} + \sum_\nu \delta\nu \left(\frac{dQ}{d\nu}\right)^2\right] \tag{2.7}$$

$$\frac{\langle(\delta Q)^2\rangle}{\delta\nu} = \int_{\nu_1}^{\nu_2} d\nu\,h\,\nu\,\frac{dQ}{d\nu} + \int_{\nu_1}^{\nu_2} d\nu \left(\frac{dQ}{d\nu}\right)^2 \tag{2.8}$$

$$\frac{\langle(\delta Q)^2\rangle}{\delta\nu} \approx h\,\langle\nu\rangle\,Q + \frac{Q^2}{\Delta\nu} \tag{2.9}$$

where we converted the sums to integrals in the second line and where $\Delta\nu = \nu_2 - \nu_1$ is the spectral bandwidth that is accepted by the system. In the last step, we have made the approximation that $dQ/d\nu$ is constant and takes on value $Q/\Delta\nu$; obviously, if it is not, the appropriate integral needs to be done. Clearly, though, the above result carries the main features of the noise.

Photon Noise (cont.)

Let's rewrite the last equation using the fact that $\delta\nu = 1/T$ is not just the spectral bin width, it is also the frequency bin width for the Fourier transform of a measurement taken over a time T:

$$J(f=0) = \frac{\langle(\delta Q)^2\rangle}{\Delta f} \approx h\,\langle\nu\rangle\,Q + \frac{Q^2}{\Delta\nu} \qquad (2.10)$$

That is, we recover the noise spectral density at $f = 0$. Now, we recall from our generic shot noise derivation that the DC bin gets half the noise power that the other bins do because there is only one DC bin but there are bins at f and $-f$ for other non-DC frequencies. So we may infer

$$\mathrm{NEP}_\gamma^2 = J(f \neq 0) = 2\,J(f=0) \approx 2\,h\,\langle\nu\rangle\,Q + 2\,\frac{Q^2}{\Delta\nu} \qquad (2.11)$$

where in the last line we have defined the noise-equivalent power, NEP, to be the square root of the noise spectral density derived from the variance on the incoming power.

Photon Noise (cont.)

The above result is modified for a system that is insensitive to polarization: the same optical power is split into twice as many modes. One can carry that factor of 2 through the whole derivation to find

$$\text{NEP}_\gamma^2 \approx 2\,h\,\langle \nu \rangle\,Q + \frac{Q^2}{2\,\Delta\nu} \tag{2.12}$$

where the critical differences are that now $\langle (\delta Q)^2 \rangle = 2 \sum_\nu \langle (\delta Q_\nu)^2 \rangle$ and $Q = 2 \int_{\nu_1}^{\nu_2} d\nu \frac{dQ_\nu}{d\nu}$ where we maintain (δQ_ν) and $dQ_\nu / d\nu$ as single-polarization powers.

Photon Noise (cont.)

An alternative derivation takes a short cut by using our generic shot noise derivation along with the photon "flow rate". We need to recognize that, where previously we had $\sum_{p \approx -\Gamma\,T/2}^{\Gamma\,T/2} 1 = \Gamma\,T$, the sum of the squares of the quantum in each flow event, now we should have $\Gamma T + (\Gamma T)^2$ because photons have Bose statistics, not Poisson statistics. That is, a flow event need not have 0 or 1 quantum, but can have 2 or more because photons like to clump into the same mode. This is expressed by the N^2 term in the occupation number variance. Making that ansatz, we have

$$J(f \neq 0) = \lim_{T \to \infty} \frac{2}{T}\,(h\,\nu)^2(\Gamma\,T + \Gamma^2\,T^2) \tag{2.13}$$

$$= 2\,(h\,\nu)^2\Gamma + 2\,T\,(h\,\nu\,\Gamma)^2 \tag{2.14}$$

$$= 2\,h\,\nu\,Q_\nu + 2\,\frac{Q_\nu^2}{1/T} \tag{2.15}$$

$$= 2\,h\,\nu\,Q_\nu + 2\,\frac{Q_\nu^2}{\delta\nu} \tag{2.16}$$

where, as above, $1/T = \delta\nu$. Clearly, this is of the same form as our result derived in the other way, and one can check that it is identically the same when the same integral over spectral band is done.

Photon Noise (cont.)

When optical efficiency — the fact that the system only lets a fraction of photons through and that they have less than unity probability of being absorbed — is taken into account, the relation between the occupancy number N and the power absorbed in a given spectral bin Q_ν changes to

$$Q_\nu = \eta\, N\, h\, \nu\, \delta\nu \tag{2.17}$$

Carrying this factor through the calculation yields (for a single polarization)

$$\mathrm{NEP}_\gamma^2 \approx 2\, \eta\, h\, \langle \nu \rangle\, Q + 2\, \frac{Q^2}{\Delta\nu} \tag{2.18}$$

Now, recognize that this NEP is the noise on the *absorbed* power — after the factor η — not on the *incident power*. One would need to divide both sides by η^2 to get the latter.

It may seem counterintuitive that the first term decreases — the noise gets smaller — as the optical efficiency decreases. This is because the above equation is for the NEP^2 on the absorbed power. If one divides by η^2 to get the NEP^2 on the incident power, one will see that both terms degrade as η decreases.

Photon Noise (cont.)

Noise Temperature and Coherent vs. Incoherent Detector Photon Noise Limits

It is interesting to convert the above formula into a noise equivalent temperature for the purpose of comparing with coherent detectors. Let's use Rayleigh-Jeans temperature, as the conversion from RJ to thermodynamic temperature fluctuations is just a function of frequency and does not care about the type of detector. Recall that the Rayleigh-Jeans brightness function for a single polarization is

$$B(\nu, T) = \frac{k\,T}{\lambda^2} \tag{2.19}$$

The power in a single mode in a spectral bandwidth $\Delta\nu$ is

$$Q(\nu, T) = B(\nu, T)\,\lambda^2 \Delta\nu = k\,T\,\Delta\nu \tag{2.20}$$

We assume the detector is subject to optical loading Q that corresponds to a Rayleigh-Jeans temperature T_{load} by the above equation.

Photon Noise (cont.)

We can relate NEP to noise-equivalent temperature (NET) via $dQ/dT = k\,\Delta\nu$ (we again neglect optical efficiency to start with):

$$\mathrm{NET}^2 = \frac{\mathrm{NEP}^2}{(dQ/dT)^2} = \frac{\mathrm{NEP}^2}{(k\,\Delta\nu)^2} = \frac{2\,h\,\nu\,Q}{(k\,\Delta\nu)^2} + \frac{2}{(k\,\Delta\nu)^2}\,\frac{Q^2}{\Delta\nu} = 2\,\frac{T_{load}(T_Q + T_{load})}{k\,\Delta\nu} \tag{2.21}$$

where $T_Q = h\,nu/k$ and $Q = k\,T_{load}\,\Delta\nu$, or

$$\mathrm{NET}_\gamma = \sqrt{2}\,\frac{\sqrt{T_{load}\,(T_Q + T_{load})}}{\sqrt{\Delta\nu}} \tag{2.22}$$

(units of $\mathrm{K}/\sqrt{\mathrm{Hz}}$).

Photon Noise (cont.)

The Dicke radiometer equation for coherent detectors is

$$\text{NET}_{Dicke} = \sqrt{2}\,\frac{T_{sys}}{\sqrt{\Delta\nu}} = \sqrt{2}\,\frac{\xi T_Q + T_{load}}{\sqrt{\Delta\nu}} \tag{2.23}$$

where ξT_Q is the coherent detector's noise temperature, with ξ indicating the degradation relative to the quantum limit T_Q. Note that NET is frequently quoted in K $\sqrt{s}$, which requires dividing the above formulae by $\sqrt{2}$ (for reasons we won't go into here...).

In addition to our comparison of NET's above, we can present T_{sys} for incoherent and coherent detectors:

$$T_{sys}^{incoh} = \sqrt{T_{load}\left(T_Q + T_{load}\right)} \qquad T_{sys}^{coh} = \xi T_Q + T_{load} \tag{2.24}$$

Photon Noise (cont.)

Let's recalculate taking into account optical efficiency, η. The optical efficiency modifies the relation between power and load temperature to be

$$Q = \eta\, k\, T_{load}\, \Delta\nu \tag{2.25}$$

A consequence of the above is that, when η is included, NEP is calculated at the detector (noise on the *absorbed* power) while NET is calculated at the input to the instrument (noise on the *incident* power). No additional correction is needed, we simply need to remember that η is now included in dQ/dT. Thus, we have

$$\mathrm{NET}_\gamma = \sqrt{2}\,\frac{\sqrt{T_{load}\,(T_Q + \eta\, T_{load})}}{\sqrt{\eta\,\Delta\nu}} \tag{2.26}$$

which shows that the NET scales as $1/\sqrt{\eta}$ — larger η is better — unless one is in the high-loading regime where the second term dominates.

Photon Noise (cont.)

For a coherent system, the efficiency-corrected version is

$$\mathrm{NET}_{Dicke} = \sqrt{2}\,\frac{\frac{\xi}{\eta}\,T_Q + T_{load}}{\sqrt{\Delta\nu}} \tag{2.27}$$

under the assumption that the amplifier noise has been specified at its input, not at the input to the instrument. In practice, $\eta \approx 1$ for coherent systems because essentially no optical filtering is needed; such filtering is what degrades η for bolometric systems.

The corresponding system temperatures are

$$T_{sys}^{incoh} = \sqrt{\frac{T_{load}}{\eta_{incoh}}\left(T_Q + \eta_{incoh}\,T_{load}\right)} \qquad T_{sys}^{coh} = \frac{\xi}{\eta_{coh}}\,T_Q + T_{load} \tag{2.28}$$

A clear advantage of incoherent systems is that the minimum possible system temperature is $\sqrt{T_{load}\,T_Q/\eta_{incoh}}$ whereas for coherent systems it is $(\xi/\eta_{incoh})\,T_Q + T_{load}$. The former is in general lower as one goes into space where T_{load} can be reduced to < 3 K and η can be substantially increased relative to a ground-based instrument that must operate in a 300 K thermal environment. On the other hand, $\xi\,T_Q$ is a fundamental limit for coherent systems.

Photon Noise (cont.)

All of our calculations including optical efficiency assume that the atmosphere itself is highly transparent so that, while it may be the dominant load, it does not reduce η appreciably from unity. One can trivially include a separate atmospheric transtmission η_{atm} and divide the right sides by it to project the NET to above the atmosphere.

Section 3
“G”-Noise in Various Systems

Introduction

"G"-noise, or the noise due to the thermal conductance from a detector to the bath, is well known for bolometers. It is fundamental in that there must always be a connection between the absorber and thermal bath to remove the optical power being absorbed. It can be usually reduced to the point where it is lower than the photon noise. It is interesting to show how there is an equivalent noise in pair-breaking detectors and to present the two in a unified form.

Rewriting G-Noise for Bolometers

The standard form for G-noise in bolometers was calculated earlier and is

$$\mathrm{NEP}_G^2 = 4\,\gamma\,k\,T^2\,G \tag{3.1}$$

where G is the thermal conductance to the bath and γ is a factor of order unity to absorb corrections mentioned earlier. Now, it holds that $Q = G\,\Delta T$ where $\Delta T = T_{abs} - T_{bath}$ is the temperature difference between the absorbing material and the bath. Thus, we can write the above as

$$\mathrm{NEP}_G^2 = 4\,\gamma\,k\,T^2\,\frac{Q}{\Delta T} = 4\,\gamma\,Q\,k\,T\,\frac{T}{\Delta T} \tag{3.2}$$

That is, the G-noise is simply related to the optical load, the operating temperature, and the ratio of the temperature to the temperature difference.

“G”-Noise in Various Systems (cont.)

G-Noise for Pair-Breaking Detectors

Pair-breaking detectors such as MKIDs or superconducting tunnel junctions suffer from a similar effect due to quasiparticle generation-recombination noise. The incoming optical power breaks Cooper pairs, creating quasiparticles. The Cooper pairs have binding energy $2\,\Delta$ where Δ is the superconducting gap parameters. Those quasiparticles must decay, however, and emit the absorbed energy as phonons that escape into the bath in order for the absorber to sit at some quiescent temperature. Therefore, there is a quasiparticle balance equation

$$\frac{Q}{\Delta} = \frac{N_{qp}}{\tau_{qp}} = \frac{R}{V}\,N_{qp}^2 \tag{3.3}$$

where the rate at which quasiparticles are created by incoming photons on the left side is balanced by the rate at which quasiparticles decay on the right side. In the second step, we have assumed that this decay time is set by pair-recombination, not an intrinsic lifetime τ_0, though we will carry the calculation through for both cases. The quasiparticle recombination rate, $1/\tau_{qp}$, is given by $R\,N_{qp}/V$ where R is the quasiparticle recombination constant (a materials parameter, like Δ) and V is the superconductor's volume. Note that, while it takes energy 2Δ to break a Cooper pair, each pair-breaking produces two quasiparticles, so the factors of 2 in the numerator and denominator of the left-hand side cancel. For the moment, we assume none of the incoming energy is lost in breaking Cooper pairs; such a loss could be included as an efficiency factor η in front of Q.

"G"-Noise in Various Systems (cont.)

Noise arises because there is shot noise on the quasiparticle population: the above equation gives the mean quasiparticle density, but the number fluctuates randomly due to generation and recombination. These fluctuations directly translate into fluctuations on the inferred incoming power via the above equation. Thus, we can calculate the power variance if we know the quasiparticle variance, which we assume is simple Poissonian, $\langle(\delta N_{qp})^2\rangle = N_{qp}$:

$$\langle(\delta Q)^2\rangle = \left|\frac{dQ}{dN_{qp}}\right|^2 \langle(\delta N_{qp})^2\rangle = \begin{cases} \left(2\,\frac{R\,\Delta}{V}\,N_{qp}\right)^2 N_{qp} & \text{pair-recombination limited} \\ \left(\frac{\Delta}{\tau_{qp}}\right)^2 N_{qp} & \text{intrinsic lifetime limited} \end{cases} \tag{3.4}$$

$$= \begin{cases} 4\,Q\,\frac{R\,\Delta}{V}\,N_{qp} & \text{pair-recombination limited} \\ Q\,\frac{\Delta}{\tau_{qp}} & \text{intrinsic lifetime limited} \end{cases} \tag{3.5}$$

$$= \begin{cases} 4\,Q\,\frac{\Delta}{\tau_{qp}} & \text{pair-recombination limited} \\ Q\,\frac{\Delta}{\tau_{qp}} & \text{intrinsic lifetime limited} \end{cases} \tag{3.6}$$

where we have used $1/\tau_{qp} = R\,N_{qp}/V$ in the pair-recombination limited case.

"G"-Noise in Various Systems (cont.)

To convert this to a NEP, we need to divide the left side by a bandwidth. In contrast to our photon noise derivation, there is no measurement time involved. Quasiparticles appear and disappear on the timescale τ_{qp} — for times short compared to τ_{qp}, the quasiparticle number is constant. The noise thus is spread across a bandwidth $1/\tau_{qp}$. We may thus divide both sides by $1/\tau_{qp}$ to obtain a NEP. We do not need to throw in a factor of 2 because we have already accounted for it by saying the bandwidth that the noise is distributed over is $1/\tau_{qp}$ rather than $2/\tau_{qp}$. Also, there is a factor of order unity that has been ignored to go from this approximate bandwidth to the precise noise bandwidth. Thus, we have

$$NEP_G^2 \approx \frac{\langle(\delta Q)^2\rangle}{1/\tau_{qp}} = \begin{cases} 4\,Q\,\Delta & \text{pair-recombination limited} \\ Q\,\Delta & \text{intrinsic lifetime limited} \end{cases} \tag{3.7}$$

$$= \begin{cases} 4\,Q\,k\,T\,\frac{\Delta/k}{T} & \text{pair-recombination limited} \\ Q\,k\,T\,\frac{\Delta/k}{T} & \text{intrinsic lifetime limited} \end{cases} \tag{3.8}$$

where, in the last line, we have rewritten in a form similar to the G-noise for bolometers up to factors of order unity and the replacement of $T/\Delta T$ for bolometers by $(\Delta/k)/T$ for pair-breaking detectors. (Note that the Δ in ΔT has nothing to do with the superconducting gap parameter Δ.) Note also the extremely simple intermediate form $\mathrm{NEP}_G^2 = \alpha\,Q\,\Delta$ with $\alpha = 1$ or 4.

A more careful calculation would take into account the distribution function of the quasiparticles, their energy-dependent decay rate, and the blocking of final states, but to first order the above result will be correct.