Non relativistic degeneracy

Heisenberg's uncertainty principle $\Delta V \Delta^3 p \geq h^3$ (location vs momentum)

Bosons can have any # of particles/energy state

$f(p)$ distribution function in 6D space 3D p (p_x, p_y, p_z), 3D (x, y, z) location

$$f(p) = \frac{1}{e^{(\mathcal{E}(p) - \mu)/kT} - 1}$$ (from statistical mechanics)

(lowest energy is $\mathcal{E}(p) = mc^2$)

for $\frac{\mathcal{E}(p) - \mu}{kT} << 1$, $f(p) \propto e^{-(\mathcal{E}_p - \mu)/kT}$

this leads to the Maxwell Boltzman distribution

$$n(v)\, dv\, dV \propto \frac{e^{-mv^2/2kT}}{(2\pi m_e kT)^{3/2}} 4\pi v^2\, dv\, e^{\mu/kT}\, dV$$

What is μ? (chemical potential)

$$dE = T\,dS - p\,dV + \mu\, dN$$

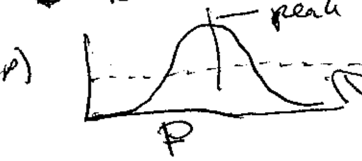
internal energy; $dS = \frac{dQ}{T}$ (entropy)

If gas is not ionizing, no chemical/fast nuclear reactions, μ is fixed. If gas has same chemical composition throughout, μ can be ignored.

This leads to usual ideal gas conditions.

But fermions have only 1 allowed particle/quantum state. Electrons are fermions. (photons are bosons)

If $f(p)$/quantum state is $<<1$ throughout, no difference between Maxwell-Boltzman (Bose-Einstein) vs Fermi-Dirac ~~Bose-Einstein~~ distribution



peak near kT

$\cancel{\tfrac{1}{2}p^2}\ p^2/2m$

max allowed by QE

$$f(\varepsilon_p) = \frac{1}{\exp[(\varepsilon_p - \mu)/kT) + 1]}$$ for fermions

$f(\varepsilon_p) \leq 1$ always

The cell size in momentum space is $\frac{1}{h^3}$, and there are 2 spin states, fill lowest energy states first, so

$$g(p) = \underset{\text{spin}}{2} \frac{(4\pi p^2 dp)}{h^3} V \text{ — volume}$$

Fill up to p_F, 0 / cell above p_F so

$$N = \int_0^{p_F} g(p)\, dp = \frac{8\pi}{3h^3} V p_F^3$$

and

$$\boxed{p_F = \left(\frac{3n}{8\pi}\right)^{1/3} h} \quad \boxed{n = \left(\frac{p_F}{h}\right)^3 \frac{8\pi}{3}}$$

to find the internal energy we integrate

$$E = \int n(p)\, \varepsilon(p)\, 4\pi p^2\, dp$$

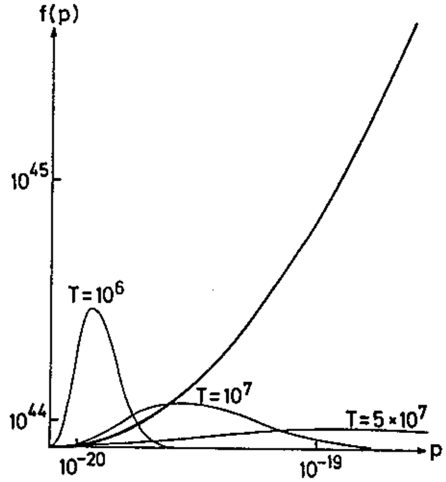
and pressure $P = \frac{1}{3} \int n(p)\, pv\, (4\pi p^2)\, dp$

We find

$$\frac{E}{V} = \int_0^{p_F} \varepsilon_p\, g(p)\, dp = \int_0^{p_F} \left(\frac{p^2}{2m}\right) 2 \frac{(4\pi p^2)\, dp}{h^3}$$

This assumes Energy/electron $= \frac{p^2}{2m}$, valid for a non-relativistic particle

$$\frac{E}{V} = \frac{4\pi}{mh^3} \left(\frac{p_F^5}{5}\right) = n p_F^2 \left(\frac{4\pi}{5m h^3}\right) \left(\frac{3}{8\pi}\right) h^3$$



Kippenhahn + Weigert

Fig. 15.1. For an electron gas with $n_e = 10^{28}$ cm^{-3} (corresponding to a density of $\varrho = 1.66 \times 10^4$ g cm^{-3} for $\mu_e = 1$), the Boltzmann distribution function $f(p)$ is shown by thin lines over the absolute value of the momentum p (both in cgs units) for 3 different temperatures (in K). The heavy line shows the parabola that gives an upper bound to the distribution function owing to the Pauli principle. (Note that the coordinates are not logarithmic, but linear as in Figs. 15.2 and 15.5)

$p \rightarrow$ (momentum)

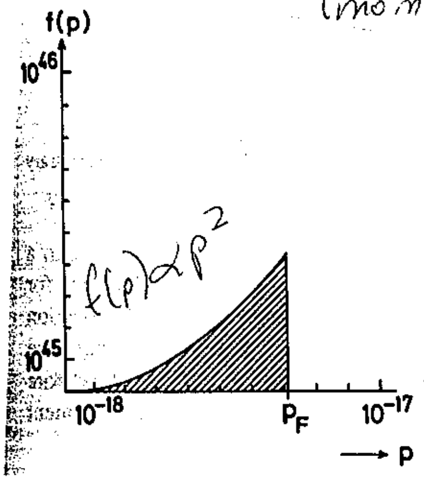


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Fig. 15.2. The distribution function $f(p)$ against the momentum p (both in cgs units) in the case of a completely degenerate electron gas with $T = 0$ K and $n_e = 10^{28}$ cm^{-3} (cf. Fig. 15.1)

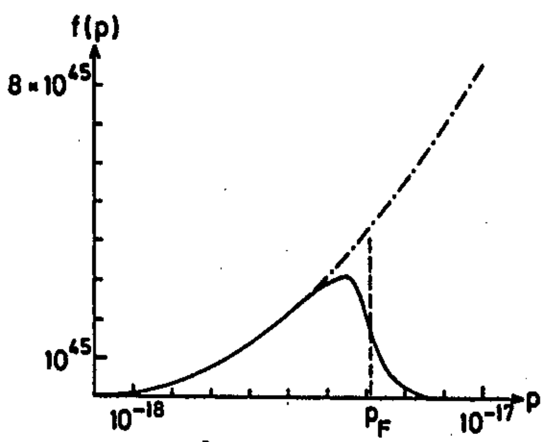

Fig. 15.5. The solid line gives the distribution function ($f(p)$ and p in cgs) for a partially degenerate electron gas with $n_e = 10^{28}$ cm^{-3} and $T = 1.9 \times 10^7$ K, which corresponds to a degeneracy parameter $\psi = 10$ (cf. the case of complete degeneracy of Fig. 15.2). The dot-dashed line shows the further increase of the parabola that defines an upper bound for the distribution function

Kippenhahn + Weigert

$$\boxed{\frac{E}{V} = \frac{3n\,p_F^2}{10m_e} = \frac{4\pi}{mh^3}\frac{p_F^5}{5}}\quad (\text{non-rel})$$

$\boxed{\text{(instead of } \tfrac{3}{2}nkT \text{ classical gas}}$

Pressure is found via a similar integral

$$\boxed{P = \frac{n}{5m}p_F^2 = \frac{2}{3}\left(\frac{E}{V}\right)}\quad \boxed{P \propto n^{5/3}}\quad \boxed{P \propto p_F^5}$$

When is the electron gas degenerate? (Non rel)

$$kT << \frac{h^2 n^{2/3}}{2\pi m_e} \approx \frac{p_F^2}{2m_e}$$

If raise T for a fixed n, high enough, degeneracy can be lifted.

What happens when the electrons near p_F change ~~for~~ from non-relativistic to relativistic?

when $n >> \left(\frac{mc}{h}\right)^3$

Then the particle energy is pc instead of $\frac{p^2}{2m}$

$$E = \int_0^{p_F} (pc)\,\frac{2V}{h^3}\,4\pi p^2\,dp = \frac{2\pi V c}{h^3}\left(p_F^4\right)$$

and $n = \left(\frac{p_F}{h}\right)^3 \frac{8\pi}{3}$ so $\frac{E}{V} = \frac{3n}{4}\,p_F c$

So the relativistic electron degenerate gas has

$$\boxed{\frac{E}{V} = \frac{2\pi c}{h^3} p_F^4 \text{ — not } p_F^5 \text{ as for the NR case}}$$

Pressure:

$$\boxed{P = \frac{1}{4} n p_F c = \frac{1}{4} n \left(\frac{3n}{8\pi}\right)^{1/3} h \propto n^{4/3}}$$

$$\boxed{P \propto p_F^{4} \propto n^{4/3}}$$

Demarcation NR vs R

$$E(p_F)_{NR} \approx m_e c$$

or $\frac{\rho}{\mu_e} \approx 10^6 \text{ gm/cm}^3$

Demarcation ~~degenerate NR~~ → classical — degenerate NR

$$\frac{3}{2} kT \approx E_F \sim p_F^2/(2m_e)$$

$$\rho/\mu_e = 6 \times 10^{-9} T^{3/2} \text{ gm/cm}^3$$

Why electrons and not atoms?

Because $\frac{E}{\text{particle}} = \frac{1}{2} m v^2$

$m_e << m_{atom}$
so $v_e >> v_{atom}$
$p_e >> p_{atom}$

Much easier when momentum is large to have degeneracy.

Classical → rel degenerate $kT \sim p_F c$
$\rho/\mu_e \sim 4.6 \times 10^{-24} T^3 \text{ gm/cm}^3$

"Perfect gas" ← no interaction between particles

When does this begin to fail as $n\uparrow$?

Coulomb energy (e^- + atom interaction)
(positive charge

$\frac{Ze^2}{a}$ a = typical separation of the ions

$\frac{4\pi}{3} a^3 = 1/n_{ion}$

If $\Gamma = \frac{(Ze^2/a)}{kT} \gg 1$ then Coulomb

effects are important.

This happens for fully ionized pure H gas

at $\rho \approx 8.5 \times 10^{-17}\ T^3$ gm/cm^3

Above this ρ, the gas is not "perfect"

Degeneracy / Crystalization

At some large Γ, gas crystalizes

$(\Gamma \sim 170 \equiv \rho \sim 4.2 \times 10^{-10} T^3 \text{ gm/cm}^3)$

$$\frac{\rho(\text{crys})}{\rho(\Gamma=1)} \propto \left(\frac{a_{\Gamma\sim 1}}{a_{\Gamma\sim 170}}\right)^3 \propto 1.7\times 10^6 \text{ for pure H gas}$$

For a WD, need to use pure He or C

Ionization effects at high pressure

normal ionization: bound $e^- + \gamma \to$ free e^-

In 1st level of H, $n=1$, a = Bohr radius = ~~0.5~~ 0.5×10^{-8} cm

$$\frac{4\pi a^3}{3} = \frac{1}{n_1} \qquad n_1 = \frac{3}{4\pi}\,\frac{8}{10^{-24}}\ /\text{cm}^3 = 1.9\times 10^{24}/\text{cm}^3$$

When $n \gtrsim n_1$, free electron is within 1 Bohr radius of an atom, electrons cannot tell to which nucleus they belong, and the gas is effectively completely ~~ionized~~ ionized.

$$\boxed{\rho_1 = \mu m_H \simeq 1 \text{ gm/cm}^3}$$

This is common in stellar interiors. The light elements are fully ionized irrespective of the radiation field for $\rho > 1$ gm/cm^3.

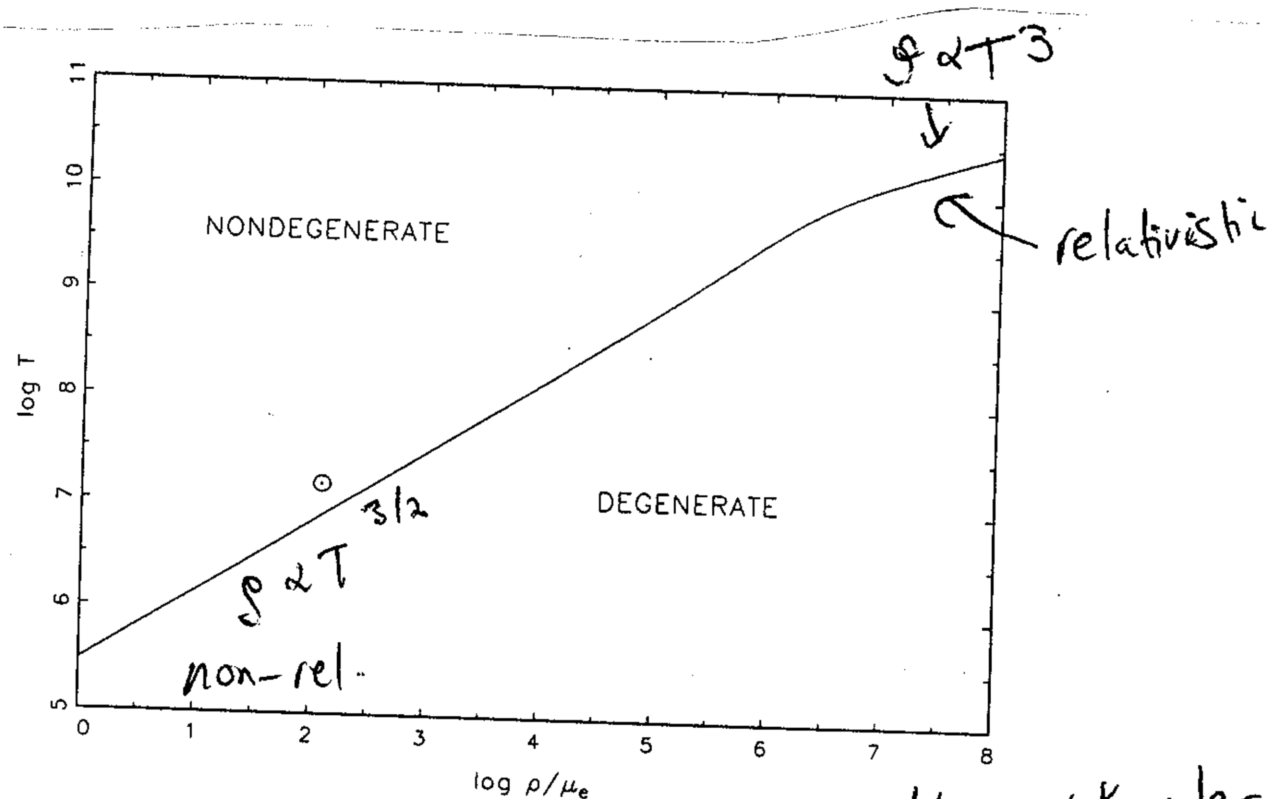

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FIGURE 3.4. The domains of nondegenerate and degenerate electrons in the T-ρ/μ_e plane. The location of the center of the present-day sun in these coordinates is indicated by the $\odot$ sign.

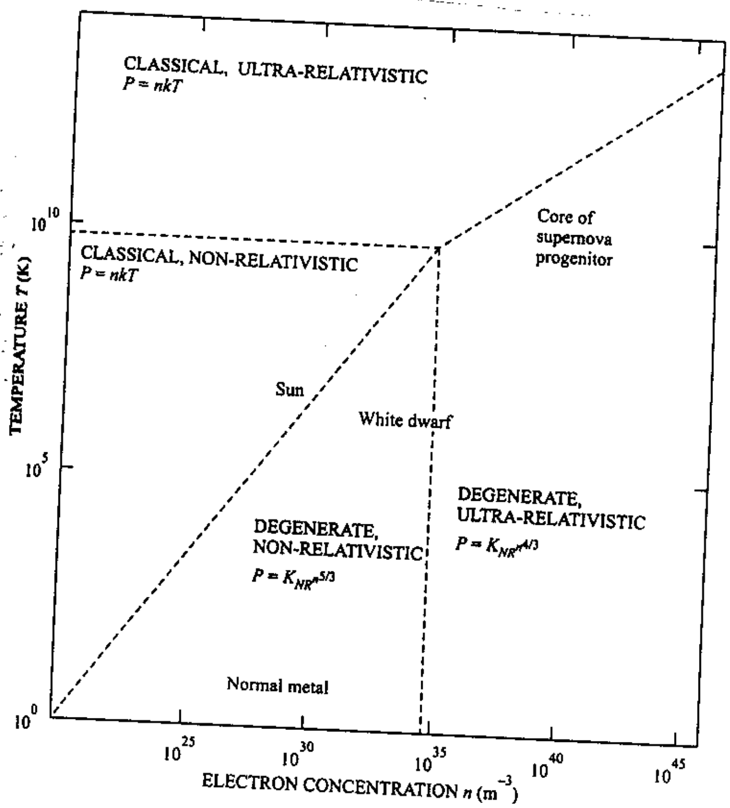

Fig. 2.2 Equation of state regimes for an ideal electron gas at a temperature T and at a density of n electrons per cubic metre. Typical values are shown for the temperature and density for electrons in a normal metal, in the sun, in a white dwarf and in the iron core of an evolved star just prior to a supernova

from Phillips – the Physics of Stars

the boundary lines between the different regimes in this n–T diagram are set by these four equations:

$$
\begin{aligned}
n &= n_{QNR} \approx 2 \times 10^{21}\ T^{3/2}\ \mathrm{m}^{-3}, \\
n &= n_{QUR} \approx 8 \times 10^{6}\ T^{3}\ \mathrm{m}^{-3}, \\
n &= (mc/h)^3 \approx 7 \times 10^{34}\ \mathrm{m}^{-3}, \\
T &= mc^2/k \approx 6 \times 10^{9}\ \mathrm{K}.
\end{aligned}
$$

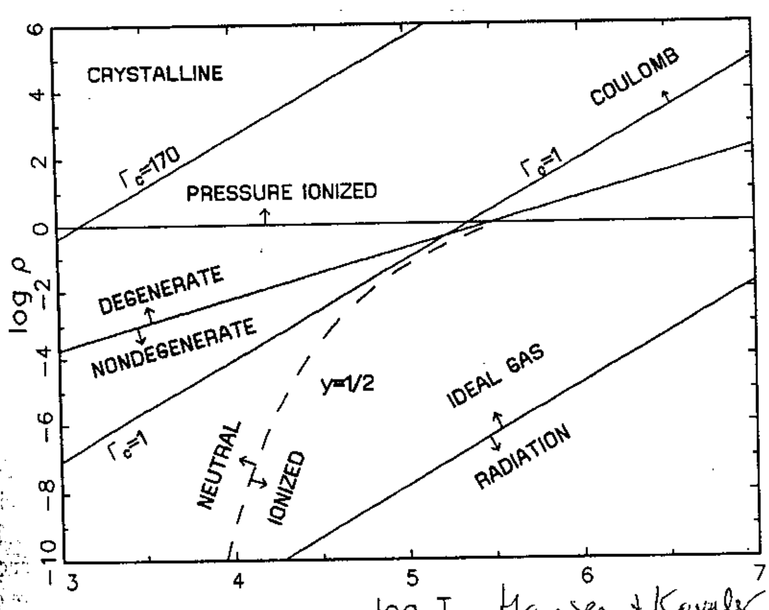

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FIGURE 3.6. A composite showing how the ρ–T plane is broken up into regions dominated by pressure ionization, degeneracy, radiation, ideal gas, crystallization, and ionization-recombination. The gas is assumed to be pure hydrogen.

Degeneracy & white dwarfs

WD – use the virial theorem

$$3(\gamma-1)\,U = -\Omega$$

Assume ρ = constant, then $\Omega = -\frac{3}{5}\frac{GM^2}{R}$

$$U = V\bar{E} = \frac{4}{3}\pi R^3\, \bar{E}_e$$

in the non-rel. case, $E_e = \frac{3}{10} n\, p_F^2 = \frac{4\pi}{m h^3}\frac{p_F^5}{5}$

So $U = \frac{4\pi R^3}{3} n\, p_F^5 \left(\frac{4\pi}{m h^3 5}\right) \propto \rho^{5/3}$

$$3(\gamma-1)\frac{4\pi R^3}{3} A \rho^{5/3} = \frac{3}{5}\frac{GM^2}{R}$$

$$R^3 \frac{M}{R^5}^{5/3} \propto M^2/R$$

$$\boxed{M^{1/3} \propto 1/R}$$

$$\frac{M}{M_\odot} = 10^{-6}\left(\frac{R}{R_\odot}\right)^{-3}\left(\frac{2}{\mu_e}\right)^5$$

For $M \sim 0.6\, M_\odot$ (WD), μ_e: He, CNO fully ionized, but no H, get $R \sim 10^{-2} R_\odot$ — close to observed R

For neutron star, with $\mu = 1$, get $\frac{M}{M_\odot} \sim 5\times10^{-15}\left(\frac{R}{R_\odot}\right)^{-3}$

$M = 1 M_\odot \rightarrow R = 11$ km.

White dwarfs pg 5

Case of extreme relativistic degeneracy

$$U = \frac{4}{3}\pi R^3 n \bar{E} \qquad \bar{E} = p_F c \;\alpha\; n^{1/3}$$

$$\Omega \approx -q G M^2/R \qquad (q \approx 1)$$

So $$3(\gamma-1)\,\frac{4}{3}\pi R^3 A n^{4/3} = q G M^2/R$$

$n \propto M/R^3$

$$R^3\left(\frac{M}{R^3}\right)^{4/3} \propto M^2/R$$

R drops out

$$M^{4/3} \doteq \text{constant, ind of } R$$

$$\frac{M}{M_0} = 1.46\left(\frac{2}{\mu_e}\right)^2$$ Chandrasekhar limiting mass

fixed mass for relativistic degenerate gas

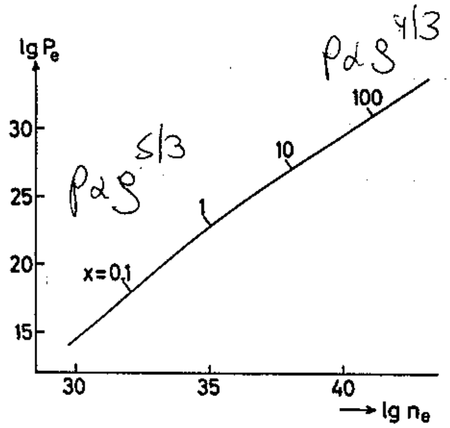

Fig. 15.4. The equation of state for the fully degenerate electron gas. On logarithmic scales the pressure P_e (in dyn cm^{-2}) is plotted against the number density n_e (in cm^{-3}). The relativity parameter $x = p_F/m_e c$ increases along the curve from the lower left to the upper right; values of x are indicated above the curve

Kippenhahn + Weigert

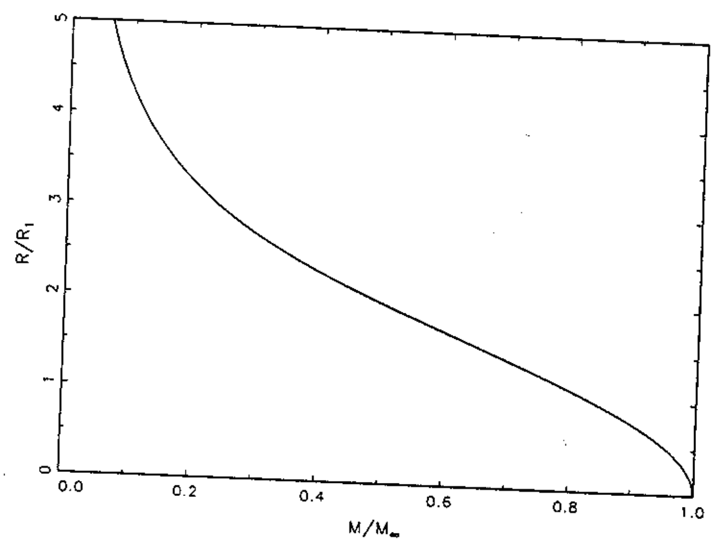

FIGURE 3.3. The mass-radius relation for zero temperature white dwarfs with constant μ_e. (See eqs. 3.64–3.65.)

Hansen + Kawaler

Examples of Degeneracy for Stars

1) White dwarfs + neutron stars are so compact + high density that they are degenerate

2) He flash in red giants

If a star has depleted its H in its core, the next thing to burn will be He ($3\,{}^4He \rightarrow {}^{12}C$) but this requires a much higher T.

To reach this T, a ~~red g~~ star will contract the core, but if T does not get high enough before the core becomes degenerate, ignition will occur in a degenerate gas.

Then suddenly there is a big input of energy + T of the gas rises, but since the gas is degenerate, P is independent of T, so P does not change, T rises still more, the nuclear energy rate $\propto T^n$ where n is a large number, and there is a runaway until T gets high enough that the gas is no longer degenerate, the P rises, the core expands out, + stable He burning can begin.

This is called the He flash – see Le Blanc pg 242. ~~Nothing much happens if the star is~~ massive

Nothing much happens if the Star is massive enagh so theat T(core) can reach the value required for He ignition <u>without</u> the gas in the core becomins degenerate.