Ay 101 - The Physics of Stars – fall 2015 - J. Cohen

Homework 4, due Friday Oct 30 by 5 pm

1. (20 pts) **The Opacity of H$^-$**

In this problem you will consider the opacity due to the negative hydrogen ion H$^-$ in stellar atmospheres. In cool stars, bound-free (photoionization) of this ion provides most of the opacity at wavelengths above the Balmer limit ($\lambda = 3647\,\AA$, below which opacity is due to photoionization of hydrogen atoms in the $n = 2$ level). The ionization energy of H$^-$ is $\chi(H^-) = 0.754$ eV, meaning that this ion can absorb photons with wavelengths $\lambda < 16,444\,\AA$. H$^-$ has only one bound state.

a. First, use the Saha equation to write an expression for $n(H^-)/n(H)$, the ratio of the hydrogen-ion to neutral-hydrogen abundances, in terms of $\chi(H^-)$, the temperature T, and the electron pressure, $P_e = n_e kT$ (where n_e is the free-electron number density; these free electrons are provided in cool stellar atmospheres presumably by metals, which have lower ionization thresholds than hydrogen or helium).

b. Next, calculate the ratio $n(H; n = 2)/n(H; n = 1)$ of the abundance of hydrogen atoms in the first excited state ($n = 2$) to the abundance in the ground state ($n = 1$); then do the same for $n(H; n = 3)/n(H; n = 1)$.

c. Assuming an electron pressure, $P_e = 10^{1.5}$ dyne/cm^2, at what temperature is $n(H; n = 3) \simeq n(H^-)$? Assuming the bound-free absorption coefficients for H^- and the $n = 3$ level of hydrogen are comparable, this gives the temperature below which H^- absorption dominates the opacity above the Balmer break.

d. Write an expression for $\kappa(3647^+)/\kappa(3647^-)$, the ratio of the opacities just above and below the Balmer break for (i) low-temperature atmospheres where the opacity above the Balmer break is due to the H^- ion, and (ii) high-temperature atmospheres where the opacity above the Balmer break is due to the $n = 3$ level of neutral hydrogen. Your results should show that at low temperatures, the Balmer discontinuity depends on both temperature and electron pressure, while at high temperature, it depends only on temperature. (You do not need to find numerical values for this ratio, just the dependence on T and P_e.)

2. (15 pts) **Limb Darkening in the Sun**

Normally stars are so distant that their surface cannot be resolved, and one can only measure their flux (emission integrated over the surface). However, because the Sun is so close, we can spatially resolve its disk, and measure I from the center to the edge (limb) of the Sun. Let θ be the angle between the Sun's radius vector and the line of sight. Then at the center of the disk, $\theta = 0$ and μ, which is $\cos(\theta)$, is 1, while going towards the edge of the Solar disk (the limb), θ approaches 90 deg while μ approaches 0.

The limb darkening curve is $I_\lambda(\theta)/I_\lambda(\theta = 0)$.

We are going to use the Solar limb darkening to study the properties of the atmosphere of the Sun, including the variation of opacity with frequency for the Solar atmosphere. Recall that for the Sun we can resolve the surface. Thus we can measure $I(\mu, \tau = 0)$, not just the flux (which is the integral of $I(\mu, \tau = 0)$, over the surface). Recall that we derived an approximate solution for $T(\tau)$ for a plane parallel atmosphere, which should be used for this problem.

$$T^4 = \frac{3}{4} T_{eff}^4 (\tau + 2/3).$$

We also derived in class an appropriate mean optical depth (the Rosseland mean) to evaluate $I(\mu, \tau = 0)$.

a) Predict the form of the limb darkening curve for the Sun assuming this T, τ relation.

b) What happens to the limb darkening curve at wavelengths where the opacity is larger overall than the Rosseland mean ? smaller ?

c) Assume LTE. The emergent specific intensity at the center of the Sun's disk in the continuum is (units are ergs/cm^2/sec/steradian/micron) is 0.014×10^{10} at 0.2 μm, 3.63×10^{10} at 0.5 μm, 1.21×10^{10} at 1.0 μm, 0.18×10^{10} at 2.0 μm, and 0.0057×10^{10} at 5.0 μm. Use the emergent intensity as a function of wavelength at the center of the solar disc to find temperatures at $\tau_\nu = 1$ for each of the five wavelengths listed above.

3. **Random Walk (derivation)** (10 points)

Photon transport through a pure scattering atmosphere can be interpreted as a random walk of particles through a medium.

Consider the 1D case and constant mean free path λ. A photon starting at the origin $x_0 = 0$ has equal probability to go left or right (towards positive or negative x). Each step size is $\delta x = \lambda$. After how many steps $N_{\rm out}$ will the photon have reached the outer boundary located at distance d from the origin?

Hint: Realize that the expectation value $\overline{x}_{n+1} = 0 = \overline{x}_N = ... = \overline{x}_0$ and work with the expectation value of x^2. Show via induction that $\overline{x_N^2} = N\lambda^2$ and compute $N_{\rm out}$ given d.

5. **Numerical Implementation of a Random Walk in 3D** (10 points)

(Note this part can be done without having solved the previous problem)

Take a sphere of radius $d = 1$ and a mean free path $\lambda = 10^{-3}$ and let particles (photons) starting from the origin move in a random walk until they reach the sphere. Record the number $N_{\rm out}$ of scatterings (the number of times they must go a length λ) needed to reach the surface. Do this with a (strongly) varying number of photons and note that the error decreases with $\sqrt{N_{\rm photons}}$. Check the dependence of $N_{\rm out}$ on d and λ. You will find that the 3D and 1D cases give the same dependence. Discuss why this is the case.

Implementation help : seed the random number generator, which subsequently will output a random number between 0.0 and 1.0 each time it is called. Let it run until the photon reaches the surface of the sphere. Note that you should use a different seed for each run, as if you use the same seed you will get the same sequence of random numbers.

At each scattering step, the photon is scattered isotropically. This means the probability is the same for being scattered into any direction (x, y, z). In spherical coordinates, this transforms to equal probability of scattering into directions given by $\sin\theta$ and ϕ. So randomly draw values of $sin(\theta)$ and ϕ at each step:

$sin(\theta) \;=\; 2.0 * \;$ random $\; - \; 1.0$

$\phi \;=\;$ random * 2.0 * π

Then propagate the photon a length λ into direction (θ, ϕ) from its current location (x, y, z). Recall

$$dx = \lambda \sin\theta \cos\varphi$$

$$dy = \lambda \sin\theta \sin\varphi$$

$$dz = \lambda \cos\theta.$$

Update to new position $(x + dx, y + dy, z + dy)$ and compute the new radial location r. Iterate until $r \geq d$ and record how many steps it takes. Finally compute the average number of steps $\overline{N}_{\text{out}}$ based on all photons.