Ay 101 - The Physics of Stars – fall 2015 - J. Cohen

Homework 5, due Friday Nov 6 by 5 pm

1. (3 points) **Equivalent Width** (Le Blanc 4.9) Calculate the equivalent width of an absorption atomic line of triangular shape, where the flux in the center is a quarter of the corresponding value in the continuum, and the width of the base of the triangle is $6\mathring{A}$.

2. (12 points) **Maxwell Distribution** The Maxwell distribution gives the distribution of speeds of particles with mass m and statistical weight $g = 1$ for an ideal classical gas. Getting this right is very important for computing line shapes in stellar atmospheres The probability $f(v)dv$ of having a speed in $[v, v + dv]$ is given by

$$f(v)dv = (\frac{m}{2\pi kT})^{3/2}\ 4\pi v^2\ exp(-\frac{1}{2}\frac{mv^2}{kT})\ dv.$$

a) (3 points) Start with Boltzmann's law, which tells you that the probability for a particle to have energy ϵ_i (note that the ϵ_i may be continuous or discrete) is given by

$$P(\epsilon_i) = \frac{n_i}{n} = \frac{exp(-\epsilon_i/kT)}{\sum_j exp(-\epsilon_j/kT)}.$$

You should treat the denominator as a normalization constant Z. Replace ϵ_i with the continuous velocity-dependent kinetic energy of the particles. Show that you arrive at $f(v)$ given in Eq. 1 by integrating this probability over phase space and ensuring that $\int_0^\infty f(v)\ dv = 1$. Assume that the distribution in velocity space is isotropic.

b) (2 points) Compute the expression for the mean speed $\bar{v}$ (the first moment of the distribution $f(v)\ dv$.

c) (2 points) Compute the expression for the most probable speed $\tilde{v}$ defined as the speed at which $f(v)\ dv$ has its maximum.

d) (2 points) Compute the expression for the rms speed $v_{\rm rms}$ of $f(v)\ dv$.

e) (1 points) Using parameters that are meaningful in the context of stellar atmospheres, make a plot of $f(v)$ and mark $\bar{v}$, $\tilde{v}$, and $v_{\rm rms}$.

f) (2 points) Only the velocity in line of sight of the observer is relevant for Doppler broadening, so the velocity distribution for a single velocity component rather than the speed is what is important. Find the correctly normalized expression $f(v_i)\, dv_i$ by repeating part (a) for only one component of the velocity

3. (20 points) **Monte Carlo Spectral Lines**

In the homework for last week, we used "Monte Carlo transport" to demonstrate that photons with a constant mean free path diffuse according to a random walk. This week, we will use a more sophisticated version to demonstrate the formation of stellar absorption lines.

Since the photosphere is much smaller than the stellar radius, we assume a planar geometry. The bottom boundary plane represents the location in the photosphere at an optical depth of $\tau \approx 2/3$, where about half of the photons have already decoupled and will escape without further interaction. (The bottom of the photosphere is much deeper, but we place the boundary here for computational efficiency). It emits a blackbody spectrum of photons at $T = 6600$ K and absorbs any photons that pass back through it. The upper boundary plane represents the edge of the photosphere, where $\tau = 0$. Photons emitted from the bottom boundary propagate through a layer of gas and escape through the upper boundary. Typical values for the temperature and density of the photospheric gas are $T = 5800$ K and $\rho = 2 \times 10^{-7}$ g/cm^3, and we will assume the temperature, density, and opacity do not depend on position within the layer.

There are a few major differences with our upgraded transport. This time, what happens to any given photon depends on the frequency, so we have to keep track of the frequency of each photon. By default, we will use 50 spectral bins (because we do not have infinite computing time and memory, we must "bin" photons into finite-size frequency ranges). Additionally, we will account for the fact that the distance a real photon travels before scattering is random.

We will use a frequency grid with 30 bins defined as follows:

set up frequency grid:

frequency = np.linspace(0.01,1,num=n$_{frequencies}$) * 1.5e15 (Hz)

You will find a template Python code created by Christian Ott, who wrote this problem for Ay 101, which he taught last year, in my anonymous ftp directory:

gringo://home/jlc/public_ftp/ay101/lines.py.

If you do not have an account on the astronomy cluster, you can reach this directory via anonymous ftp.

ftp ftp.astro.caltech.edu

... log in as "anonymous" ... use your email as password

cd users/jlc/

(a) (2 points) **How thick is the photosphere?**

In the photospheres of moderately hot stars like our Sun, a large abundance of atomic hydrogen is present and the dominant source of opacity in the visible spectrum is H^- ion bound-free absorption. Though in reality the resulting opacity does depend somewhat on frequency, to first order we can model it as a constant at $k_{H^-} = 0.07$ cm^2 g^{-1}. Using the definition of optical depth, determine the thickness (vertical height) of the photospheric gas layer we should use in our simulations (i.e. the thickness such that $\tau = 2/3$).

(b) (3 points) **Implement the H_α line opacity.**

We will deal only with the H_α spectral line, which comes from the excitation and relaxation of electrons between the $n = 2$ and $n = 3$ levels of the neutral hydrogen atom. The center of this line is at a frequency of $\nu_{23} = 4.57 \times 10^{14}$ Hz (red, 6562 Å).

LeBlanc equation 4.86 describes how we can get the line opacity as a function of frequency

$$\kappa_\nu = \frac{\pi e^2}{m_e m_H c} f_{23} \, [1 - exp(\frac{-h\nu_{23}}{kT}] \, \phi_\nu,$$

where ϕ_ν describes the line profile and $f_{23} = 0.6407$ is the oscillator strength given in LeBlanc table 4.1. The line profile is determined in reality by both natural and doppler broadening, but since doppler broadening dominates everywhere except at the tails of the line, we will only use doppler broadening for simplicity. The corresponding line profile is a simple gaussian

$$\phi_\nu = \frac{1}{\sigma\sqrt{2\pi}} \, exp[-\frac{1}{2}\frac{(\nu - \nu_{23})}{\sigma}]^2,$$

where the width is $\sigma = \nu_{23}\sqrt{\frac{kT}{mc^2}}$. The width of this line is much smaller than our spectrometer in the code can resolve, so we are going to artificially amplify the width by a factor of 10^3 for visibility.

In your code, implement the function line_opac(nu) to return the value of the H_α line opacity as a function of frequency, assuming only doppler broadening, all in cgs units. The necessary constants (oscillator strength: f_{23}, line center ν_{23}, and line width (width_{23}) are given below.

c. (3 points) **Implement random free paths.**

The random free paths should follow an exponential distribution so photons are "attenuated" in the same way as the intensity. That is, for the random path length d and some distance x,

$$P(d < x) = 1 - e^{-\rho k x}.$$

Replace the left side of the above cumulative distribution function (CDF) with a uniform random variable and analytically invert to get a random path length with the appropriate distribution (known as inverse transform sampling). In your code, implement the function $random\_free\_path(opac)$ to return a random free path given the opacity in $cm^2\ g^{-1}$.

d) (3 pts) **Randomly choose a frequency.**

Photons are emitted with random frequencies, but the probability a photon is emitted from any frequency bin is related to the emissivity in that frequency bin. The function emissivity_CDF(T, opacity, frequency) (a Python version of this function is given below) returns the emissivity CDF, where each point represents

$$\epsilon(\nu_i) \;=\; \frac{\int_0^{\nu_i} B_\nu(T)\kappa_\nu d\nu}{\int_0^{\nu_{max}} B_\nu(T)\kappa_\nu d\nu},$$

where ν_i is the frequency at the top of the frequency bin, so the last value is always 1. This is useful for generating an appropriately distributed random frequency. Choose a uniform random number between 0 and 1. Loop forward through the emissivity array as long as the array value is smaller than the uniform random number. The highest bin where the random number is larger than the emissivity CDF is the frequency bin you will emit from.

Implement the function random_frequency_bin(emis) that returns the randomly-selected bin index from which the photon will be emitted.

e. (4 pts) **What do the emissivity CDF and opacities look like?** (4 points) Set up your program to print various data. Run the script and make a plot of the opacity and emissivity CDF. Submit the plots with descriptions. Why does the "emissivity" only increase with frequency? What feature in the emissivity reflects the spectral line? Why is it that the opacity outside the spectral line is constant in frequency, but the emissivity is not?

f. (5 points) **Absorption Lines.** Set your program to print the spectrum collected from escaping photons. Run with a small number of particles ($\sim 10^3$) to make sure everything works, and then increase the number of particles to 10^5, or until you get a decently smooth spectrum. Plot the normalized spectrum as points on top of the two normalized blackbody curves from the boundary and photosphere, respectively. Outside of the spectral line, does the spectrum match one of the blackbodies or is it a combination of the two?

Heavy metals have very complicated atomic structures, and hence lots of spectral lines. If star A has a relatively high concentration of metals in its photosphere and star B has a relatively low concentration, for which would the Eddington luminosity be larger? Assuming the two stars are the same size, mass, temperature, and luminosity, which will drive a stronger stellar wind?