Homework 8, due Monday Nov 30 by noon

1. (4 points) **Mass-Radius Relationship**

Assuming Kramer's opacity $\kappa \sim \rho^n \, T^{-\alpha}$ and a nuclear energy generation rate per mass
$\epsilon \sim \rho^{u-1} T^s$, and $T \sim \mu G M R^{-1}$, derive

$$\mu^{\alpha+4} R^{3n-\alpha} M^{\alpha+3-n} \sim M \, (\frac{M^{u-1}}{R^3}) \, (\frac{\mu G M^s}{R}),$$

and a mass-radius relationship $R \sim M^{???}$ for $n = 1$, $\alpha = 3.5$ (i.e. Kramer's opacity) and
$s = 4$, $u = 2$ (appropriate values for the PP-chain). What is ??? ? Such power law relations
among the key stellar parameters are called homology relations; they provide a useful way of
characterizing stars in similar stages of evolution with slightly different masses. Compare the
relation you derive with the slope of that shown in Fig. 1.13 of LeBlanc near $M = 1 \, M_\odot$.

2. (10 pts) **Polytropic Approximation of the Sun**

The structure of the sun can be approximated by a polytrope of index 3, i.e. $P = K\rho^{4/3}$,
also known as the Eddington model.

    a) Show that the gravitational potential $\phi(r)$ for this model can be written as

$$\phi(r) = -4\frac{P(r)}{\rho(r)} - C,$$

where C is a positive constant independent of r. Also determine the value of C.

    b) The virial theorem states that

$$\int_0^M \frac{P}{\rho} dm = -\frac{1}{3}\Omega(M).$$

Using this with the above result, determine the total gravitational potential energy for such
a star.

c) Under the assumption of an ideal gas equation of state, show that the mass-averaged temperature $\overline{T} \equiv \frac{1}{M} \int_0^M T \, dm$ is given by

$$\overline{T} = \frac{1}{2} \frac{\mu m_H}{k_B} \frac{GM}{R}.$$

[Hint: find $\overline{T}$ as a function of Ω, then use your result from (**b**)]

d) Estimate $\overline{T}$ for the sun, in K.

3. (10 pts) **Polytropes and the Lane Emden Equation**

(a) Solve the Lane Emden equation for a few values of n not listed in the tables in the notes on polytropes, use $n = 0.7, 2.7, 3.4$. For those solutions calculate the first three entries in the table on page 6 of the polytrope note (i.e. ξ_{max}, $-\xi^2 d\Theta/d\xi|_{\xi=\xi_{max}}$ and $\rho_c/\bar{\rho}$). Prove that $\bar{\rho}/\rho_c = -3\xi^{-1} d\Theta/d\xi|_{\xi=\xi_{max}}$.

(b) If the Sun was radiative throughout with Kramer's law opacity, what would have been its polytropic index n? Given the mass of the Sun and its radius, what would have been it central density? Compare this to the correct value.

(c) Derive the relation between mass, the constant K of the polytropic relation and the radius. Show that for the special case of $n = 3$ the radius cancels out, and there is a relation between M and K. What is K for non relativistic degenerate matter with mass per electron μ_e ? Find the Chandrasekhar mass in terms of fundamental constants and μ_e.

(d) Evaluate the limiting mass for a neutron star, the Oppenheimer-Volkoff limit. Indicate the values of μ_e and of the particle mass that you use.

4. (5 points) **Brown Dwarfs**

Investigate, then summarize in your own words and equations (roughly 2 paragraphs, perhaps 0.5 - 0.75 pages), the evolution of brown dwarfs (go into detail on their thermonuclear reactions). Derive an approximate quantitative explanation for why $0.08 M_\odot$ is the upper mass limit of brown dwarfs (and thus the lower mass limit for normal stars). You may find the discussion in LeBlanc of brown dwarfs useful.