1. Polytropes – Derivation and Solutions of the Lane-Emden Equation

Polytropes are useful as they provide simple solutions (albeit in some cases via numerical integration) for the internal structure of a star that can be tabulated and used for estimates of various quantities. They are much simpler to manipulate than the full rigorous solutions of **all** the equations of stellar structure. But the price of this simplicity is assuming a power law relationship between pressure and density which must hold (including a fixed constant) throughout the star.

We begin with the equations of mass continuity and of hydrostatic equilibrium. Since there are three unknowns (pressure, density, and mass as a function of radius) and only two equations, in order to get a solution we must either add more equations (i.e. energy generation and transfer) or introduce an additional assumption.

For a polytrope, one assumes that gas pressure $P = K\rho^\gamma = K\rho^{(n+1)/n}$, where γ is the adiabatic index (a parameter characterizing the behvaior of the specific heat of a gas) and n is called the polytropic index. K is a constant. $\gamma = (n+1)/n$. This adds a third equation, and the set of three equations can then be reduced to a single differential equation whose terms depend on n and on K, and solved.

The resulting equation is the called the Lane-Emden equation after the first people who worked this out. A derivation is given below. It basically requires changing of variables and manipulation of the three equations, but its derivation is otherwise straightforward. The radius variable r is multipled by a constant which depends on n, K and other constants to be rescaled into the variable ξ.

The constants for a specific model are P_c and ρ_c (the central density and pressure, related to the constant K, as well as to the polytropic index n. From the solution we hope to derive the total mass and radius, and the density, pressure, and temperature as a

function of radius for the star.

We begin with the equation of continuity,

$$\frac{dM}{dr} = 4\pi r^2 \rho(r),$$

and the equation of hydrostatic equilibrium,

$$\frac{dP}{dr} = -GM(r)\rho(r)/r^2.$$

Eliminating $M(r)$ between these two equations, and replacing P by $K\rho^\gamma$, we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 K}{\rho}\gamma\rho^{\gamma-1}\frac{d\rho}{dr}\right) = -4\pi G\rho.$$

First we rescale the radial variable, by the constant α, so $r = \alpha\xi$, where

$$\alpha = [\frac{n+1}{4\pi G}K\rho_c^{1/(n-1)}]^{1/2}.$$

Then we replace $\rho(r)$ by $\theta(\xi)$, where $\rho(r) = \rho_c\theta^n(\xi)$.

This requires that at the center of the star where $\xi = 0$, $\theta(0)$ must be one. Furthermore, since dP/dr approaches 0 as $r \to 0$, we need $d\theta/dr = 0$ at $\xi = 0$. These are the boundary conditions for the solution. The outer boundary (the surface) is the first location where $\rho = 0$, or equivalently $\theta(\xi) = 0$. That location is called ξ_1. The formal solution may have additional zeros at larger values of ξ, but $\xi > \xi_1$ is not relevant for stellar models.

We then get the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n.$$

The solutions of the Lane-Emden equation, which are known as polytropes, are functions of density versus r expressed as $\theta(\xi)$. The index n determines the order of that solution. In particular, the solution only depends on n, and can be scaled by varying P_c and ρ_c to give solutions for stars over a range of total mass and radius.

For $n = 0$, the density of the solution as a function of radius is constant, $\rho(r) = \rho_c$. This is the solution for a constant density incompressible sphere.

$n = 1$ to 1.5 approximates a fully convective star, i.e. a very cool late-type star such as a M, L, or T dwarf.

$n = 3$ is the Eddington Approximation discussed below. There is no analytical solution for this value of n, but it is useful as it corresponds to a fully radiative star, which is, as we will see below, a useful approximation for the Sun.

1.1. Solutions of the Lane-Emden Equation

The Lane-Emden equation has analytical solutions for $n = 0, 1$, and 5 which are given in Fig. 2. For $n = 5$, the first zero of $\theta(\xi)$, which is proportional to the radius of the polytrope, occurs at infinity. For $n > 5$, the binding energy is positive, and hence such a polytrope cannot represent a real star.

For all other polytrope indices n, a numerical solution to the Lane-Emden equation must be calculated. A display of solutions for several values of n between 0 to 6 is given in Fig. 2. Note that the radius of the star is defined by the first zero in the solution, and the solution at larger values of ξ is not relevant for computing stellar models.

$n =$	0	1	5
$\theta =$	$1 - \dfrac{\xi^2}{6}$	$\dfrac{\sin \xi}{\xi}$	$\left(1 + \dfrac{\xi^2}{3}\right)^{-\frac{1}{2}}$
$\xi =$	$\sqrt{6}$	π	∞

Fig. 1.— Analytical solutions for the three cases of n for which such exist. The second line is ξ_1, the value of ξ at which the first zero of $\theta(\xi)$ happens, which defines the radius of the polytrope.
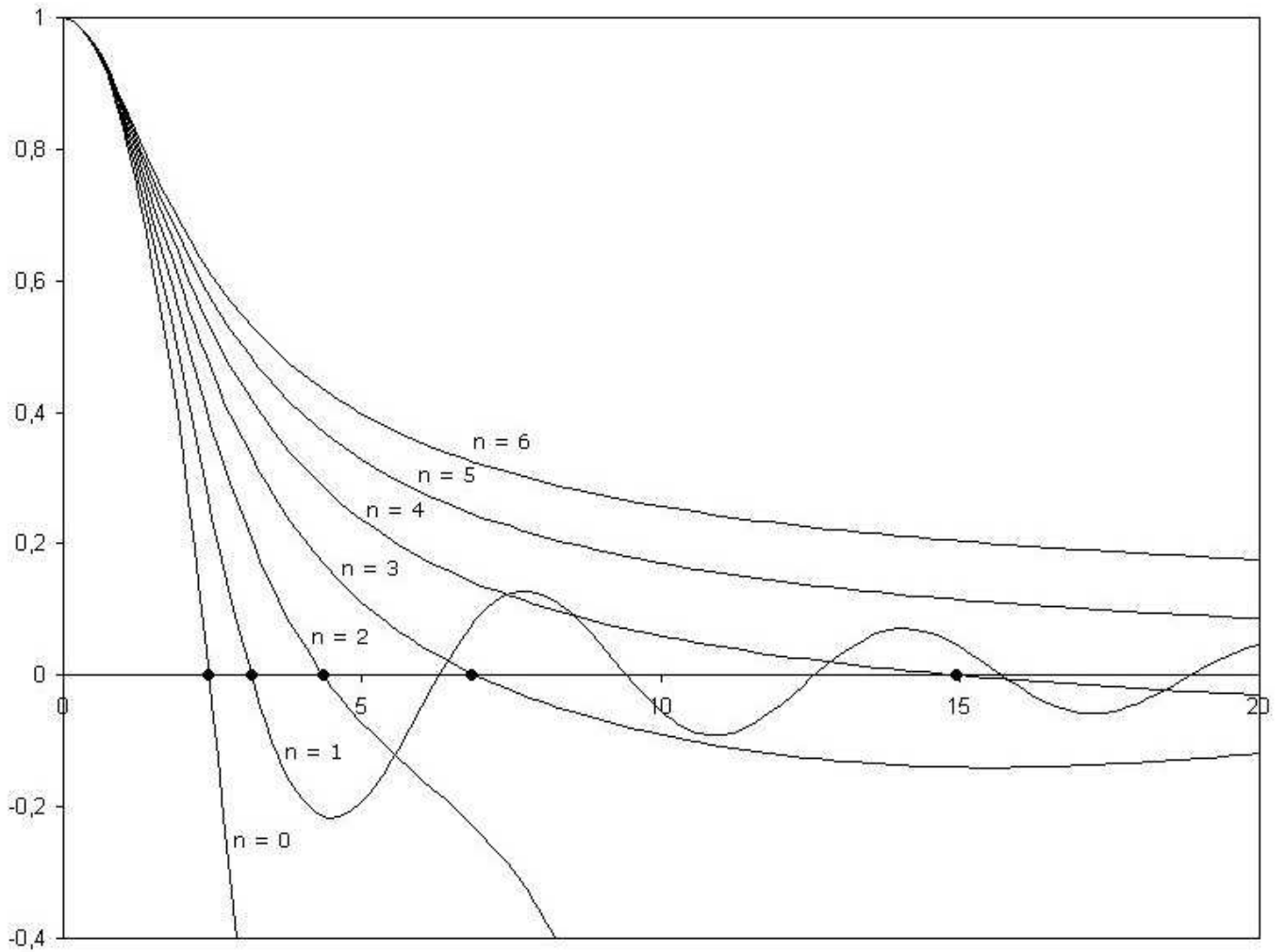
Fig. 2.— Solutions $\theta(\xi)$ to the Lane-Emden equation for various values of n. The black dot along the horizontal line (denoting $\theta = 0$) marks the value of ξ_1 for each solution displayed.

TABLE 4
The Constants of the Lane-Emden Functions*

n	ξ_1	$-\xi_1^2\left(\dfrac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$	$\omega_n = -\xi_1^{\frac{n+1}{n-1}}\left(\dfrac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	N_n	W_n	$-\dfrac{1}{(n+1)\xi_1\left(\dfrac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}}$
0............	2.4494	4.8988	1.0000	0.33333	....	0.119366	0.5
0.5..........	2.7528	3.7871	1.8361	0.02156		0.26227	0.53847
1.0..........	3.14159	3.14159	3.28987	....	2.270	0.392699	0.5
1.5..........	3.65375	2.71406	5.99071	132.3843	0.63662	0.770140	0.53849
2.0..........	4.35287	2.41105	11.40254	10.4950	0.42422	1.63818	0.60180
2.5..........	5.35528	2.18720	23.40646	3.82662	0.36475	3.90906	0.69956
3.0..........	6.89685	2.01824	54.1825	2.01824	0.35150	11.05066	0.85432
3.25.........	8.01894	1.94980	88.153	1.54716	0.36394	20.365	0.96769
3.5..........	9.53581	1.89056	152.884	1.20426	0.37898	40.9098	1.12087
4.0..........	14.97155	1.79723	622.408	0.729202	0.40104	247.558	1.66606
4.5..........	31.83646	1.73780	6189.47	0.394356	0.47720	4922.125	3.33100
4.9..........	169.47	1.7355	934800	0.14239	0.65798	3.693×10⁶	16.550
5.0..........	∞	1.73205	∞	0	1.340	∞	∞

* The values for $n = 0.5$ and 4.9 are computed from Emden's integrations of θ_n; for $n = 3.25$ an unpublished integration by Chandrasekhar has been used. $n = 5$ corresponds to the Schuster-Emden integral. For the other values of n the *British Association Tables*, Vol. II, has been used.

From Chandrasekhar's book, An Introduction to the Study of Stellar Structure (1939)

Fig. 3.— Constants of the polytrope solutions for several values of the polytropic index n. From Chandrasekhar, "An Introduction to the Study of Stellar Structure", 1939.

2. The Eddington Solution

Let us assume a star where energy transfer is via radiation throughout. Let us also assume that the ratio between the gas pressure and the total pressure is β, and that β is constant throughout the star. Then $P_{gas} = (\rho k T)/(\mu m_H) = \beta P$. The radiation pressure is $(1 - \beta)P = (1/3)aT^4$.

Eliminating T between these two relations gives:

$$\beta^4 P^4 (\frac{\mu m_H}{\rho k})^4 = \frac{3(1 - \beta)}{a} P.$$

Simplifying this for P gives:

$$P = (\frac{k}{\mu m_H})^{4/3} (\frac{3(1 - \beta)}{a\beta^4})^{1/3} \rho^{4/3}.$$

Thus this is a polytropic equation of state with $\gamma = 4/3$ and hence $n = 3$. This case was first worked out by Arthur Eddington, and hence is called the Eddington Solution. It is appropriate for the fully radiative case, which is a reasonable approximation for the Sun. The constants and solution are given in Fig. 4.

Note from the two appended figures how well the $n = 3$ polytrope fits the Solar relation from the best available solar interior models even though nuclear processes and details of energy transfer are not included.

Table 4.3

Lane-Emden function θ_3 and its derivative for values of ξ between 0 and ξ_1 6.9011.

ξ	θ_3	θ_3^4	θ_3^5	$-d\theta_3/d\xi$	$-\left(\frac{\xi}{3}\right)\left(\frac{d\xi}{d\theta_3}\right)$	$-\xi^2\left(\frac{d\theta_3}{d\xi}\right)$
0.00	1.00000	1.00000	1.00000	.00000	1.00000	.0000
0.25	.98975	.96960	.95966	.08204	1.0158	.0051
0.50	.95987	.88436	.84886	.15495	1.0756	.0387
0.75	.91355	.76242	.69650	.21270	1.1754	.1196
1.00	.85505	.62513	.53451	.25219	1.3218	.2522
1.25	.78897	.49111	.38747	.27370	1.5224	.4276
1.50	.71948	.37244	.26797	.27993	1.7862	.6298
1.75	.64996	.27458	.17847	.27460	2.1243	.8410
2.00	.58282	.19796	.11538	.26149	2.5495	1.0450
2.50	.46109	.09803	.04520	.22396	3.7210	1.3994
3.00	.35921	.04635	.01665	.18393	5.4370	1.6553
3.50	.27629	.02109	.005828	.14859	7.8697	1.8203
4.00	.20942	.009185	.001923	.11998	11.113	1.9197
4.50	.15529	.003746	.000582	.09748	15.387	1.9740
5.00	.11110	.001371	.000152	.08003	20.826	2.0007
6.00	.04411	.000086	.000004	.05599	35.720	2.0156
6.80	.00471	.000001	.000000	.04360	51.987	2.0161
6.9011	.00000	.000000	.000000	.04231	54.360	2.0150

Fig. 4.— The solution for a $n = 3$ polytrope (the Eddington Standard Model). This figure is from Astrophysics I: Stars by Bowers and Deeming, but the original source is the book by A. Eddington, "The Internal Constitution of the Stars", 1959.
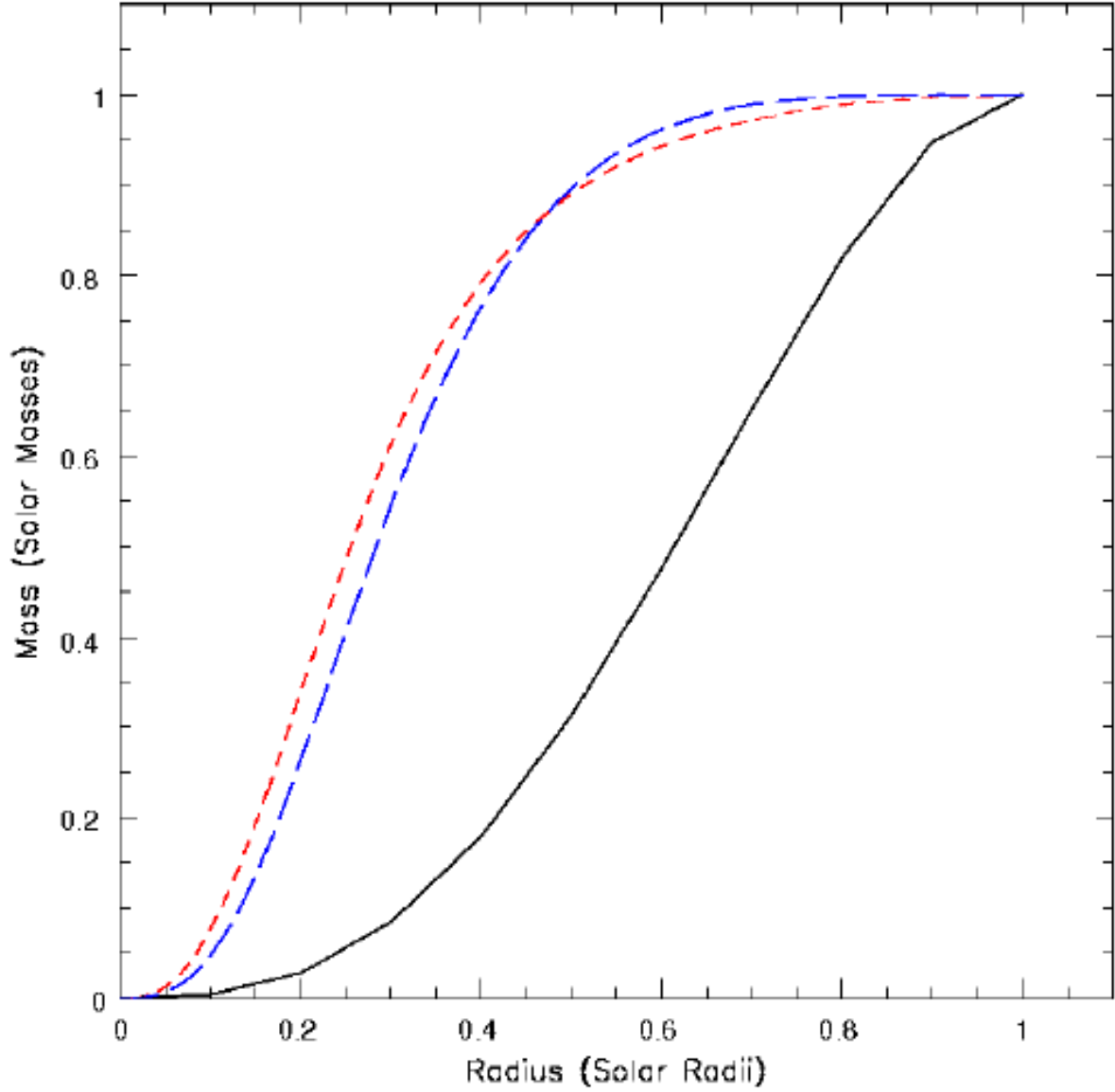
Fig. 5.— A comparison of the behavior of mass as a function of radius for the Sun: red line – the standard Solar model, blue line – $n = 3$ polytrope, black line – linear density law. Note how well the $n = 3$ polytrope fits the Solar relation even though nuclear processes and details of energy transfer are not included.
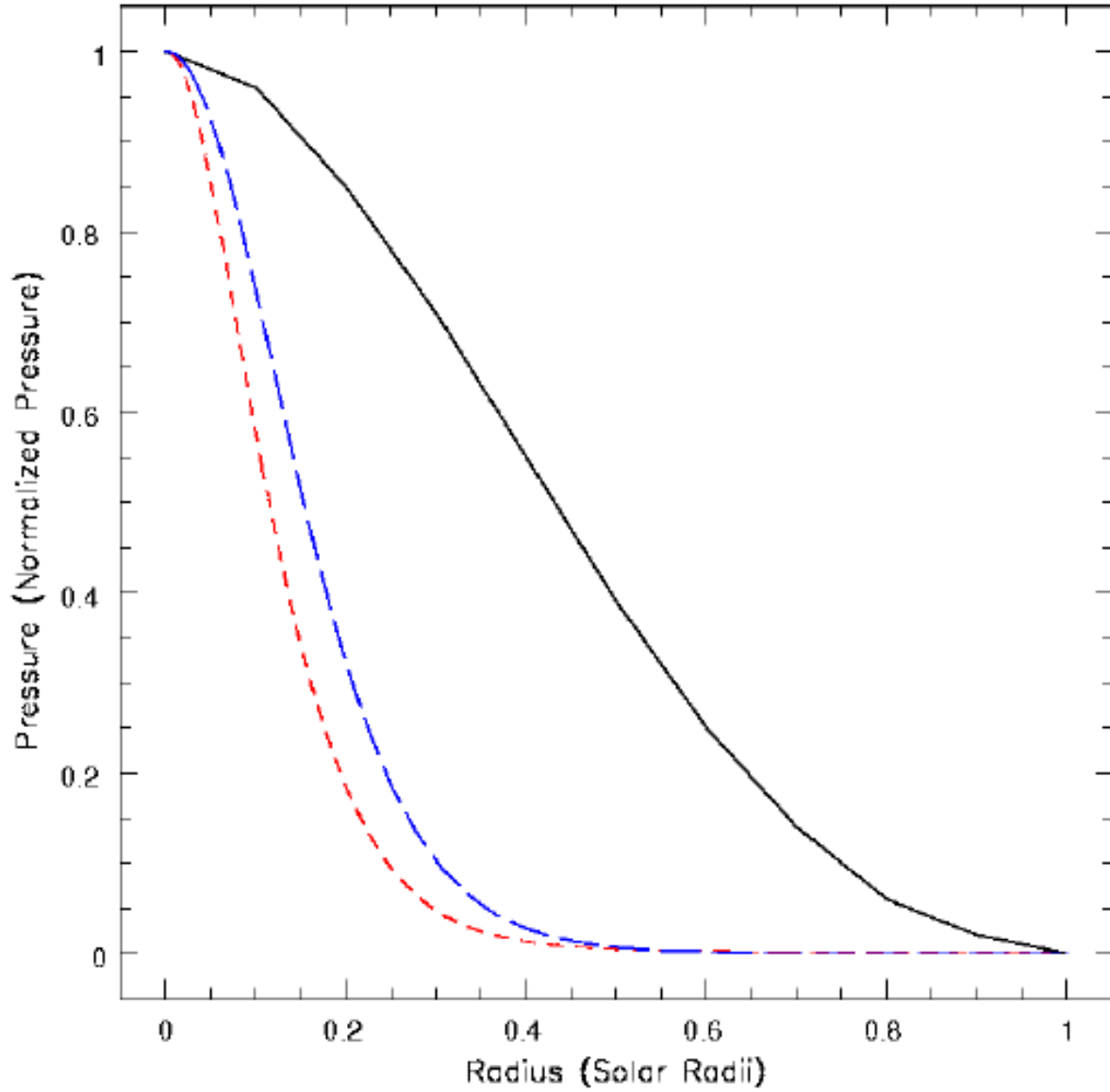
Fig. 6.— A comparison of the behavior of pressure as a function of radius for the Sun: red line – the standard Solar model, blue line – $n = 3$ polytrope, black line – linear density law. Note how well the $n = 3$ polytrope fits the Solar relation.

3.   The Isothermal Sphere

An isothermal object has $T =$ constant, so $P \propto \rho$. This is a polytrope with $\gamma = 1$ corresponding to $n = \infty$. The proceedure used above to derive the Lane-Emden equation is not valid for $n = \infty$. So we go back to the equation of hydrostatic equilibrium combined with that for mass continuity, which becomes:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{kT}{\mu m_H}\frac{d\rho}{dr}\right) = -4\pi G\rho.$$

We need a different change of variable than that used earlier; it is: $\rho = \rho_c e^{-\phi}$ and

$$r = \left[\frac{kT}{4\pi G\mu m_H\rho_c}\right]^{1/2}\xi = \alpha\xi.$$

Here α is a constant as T is constant. The boundary conditions are set as: $\phi = 0$, $\xi = 0$, and $d\phi/d\xi = 0$.

There is no analytical solution to the problem of the isothermal sphere. Numerical integration is required. The density does not reach 0 at a finite value of r, so the solution extends to infinity. The total mass is also infinite. So when an isothermal sphere is used in astrophysics (for example for stars in clusters) the solution must be truncated at some finite radius.

4. White Dwarfs and Neutron Stars

The equation of state for a non-relativistic dengenerate gas is $P \propto \rho^{5/3}$, and for a highly relativistic degenerate gas is $P \propto \rho^{4/3}$. (These will be derived in a separate note dealing with the equation of state of a gas.)

White dwarfs and neutron stars can be approximated as fully degenerate stars, with the lower mass white dwarfs being approximated as a case of non-relativistic degeneracy, and the higher mass white dwarfs and all neutron stars are cases of fully relativistic degeneracy.

We thus can represent their internal structure by polytropes with $n = 1.5$ for the NR case, and $n = 3$ for the relativistic case.

Polytropes have a definite relationship for their total mass and their total radii, in terms of the constants P_c and ρ_c and the polytropic index n.

$$R = a\xi_1 = [\frac{(n+1)K}{4\pi G}]^{1/2} \rho_c^{(1-n)/2n} \xi_1.$$

Recall that ξ_1 is the first zero of the solution to the Lane-Emden equation for $\theta(\xi)$.

For the total mass, we integrate, so that

$$M = \int_0^R 4\pi r^2 \rho \, dr = 4\pi [\frac{(n+1)K}{4\pi G}]^{3/2} \rho_c^{(3-n)/2n} \xi_1^2 \mid \frac{d\theta}{d\xi}_{\xi_1} \mid .$$

Eliminating ρ_c between the solution for the mass and for the radius gives the mass-radius relation for polytropes,

$$M \propto R^{(3-n)/(1-n)}.$$

We thus find that for non-relativistic degenerate stars with $\gamma = 5/3 \equiv n = 1.5$,

$M \propto R^{-3}$. The radius of a more massive white dwarf is smaller ! The final result for the total mass and radius in this case, where μ_e is a constant from the equation of state related to the chemical composition of the material, is

$$M = 0.70 \, (\frac{R}{10^4 \ km})^{-3} \, (\frac{\mu_e}{2})^{-5} \, M_\odot.$$

$$R = 1.12 \times 10^4 \, (\frac{\rho_c}{10^6 \ \mathrm{gm \ cm}^{-3}})^{-1/6} \, (\frac{\mu_e}{2})^{-5/6} \, km.$$

For the highly relativistic (i.e. very high density) case, $\gamma = 4/3 \equiv n = 3$, M is independent of ρ_c and hence of the stellar radius. The mass is FIXED and for white dwarfs is called the Chandrasekhar limit. Thus as the central density increases, the electrons become more relativistic, and the solution for the structure of such a highly degenerate star has a total mass asymptotically approaches this value.

$$M_{Chandra} = 1.457 \, (\frac{2}{\mu_e})^2 \, M_\odot.$$

For neutron stars, the value of μ is different, but the general concept is the sam.e

In order to understand why this is happening, we need to understand degeneracy. Please see the discussion on the equation of state for a qualitative simple arguement of why this happens.