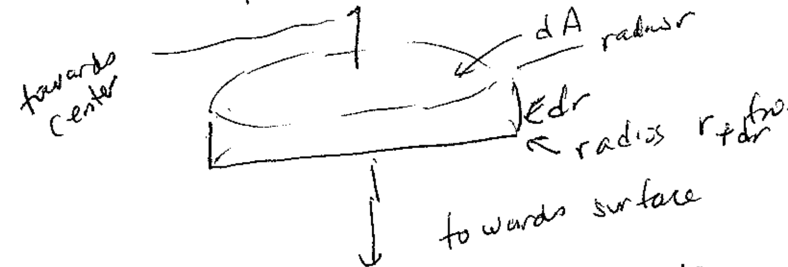
Star Formation (hydrostatic equilibrium, virial theorem, collapse etc)

Hydrostatic eq.

Pressure P = force / unit area = F/A

P = weight of gas above it per unit area

P increases as a function of depth in star



for equilibrium $\uparrow F$ down to center = $\downarrow F$ up to surface

Pressure force from below $P\,dA$

" from above $-(P+dP)\,dA$

gravity (down) $= \rho g\, dA\, dr$ = mass $\cdot\, g$

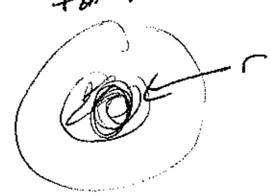
$g = G\,M(r)/r^2$

In ~~ste~~ outer parts of star, $M \approx M_*$, $g \approx G M_* / r^2$

Balance forces

$$P\,dA - (P+dP)\,dA + \rho g\, dr\, dA = 0$$

$M(r)$ = enclosed mass for radius r



$$\frac{dP}{dr} = \rho g = -\rho \frac{G M(r)}{r^2}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho\, dr$$

Column mass $\quad m_c(r) = -\int_{R_*}^{r} \rho(r)\, dr$

$m_c(r) \uparrow$ as $r \downarrow$ so need $-$ sign.

$d\,m_c = -\rho(r)\, dr$. so

hydro eq $\frac{dP}{dr} = -\rho(r)\, g = \cancel{\rho(r)}$ and $dm_c = -\rho(r)\, dr$

So $\frac{dP}{dm_c} = -g = G\frac{m(r)}{r^2}$

at stellar surface; ~~$\rho(r)$~~ g = constant and

$P(m) = g\, m \qquad P(m=0) = 0$

near the surface of the stars radiation + matter decouple
radiation can also produce a pressure outward

So need $\frac{dP}{dr} = \rho(r)\left[g(r) + g_{rad}(r)\right]$

where $g(r)$ is toward center and
$g_{rad}(r)$ is toward surface.

If $g_{rad}(r) \gg g(r)$, matter will be pushed out of the star (and sign of $\frac{dP}{dr}$ will be reversed)
(strong stellar wind)

Virial theorem (~~general treatment~~) (for a star)

Connection between gravitational + internal (kinetic) energy of a system in equilibrium.

statement: $\boxed{E_{kinetic} = -\frac{1}{2} E_{potential}}$ Ω

U internal energy gravitational potential energy

this is derived from basic mechanics, $F = ma$

$\vec{F} = m \cdot \ddot{r}$

The crucial thing is "integrating by parts"

$$\int_{\text{center}}^{\text{surface}} P\,dV = \underbrace{(PV)^{\text{surface}} - PV(\text{center})}_{\downarrow} - \int_{\text{center}}^{\text{surface}} V\,dP$$

Integrate ~~&~~ over a $\underline{\text{closed Volume}}$ $\longrightarrow$ surface term $\mp \int_{\text{volume}} B\,dA$

$$\int_{\text{volume}} A\,dB$$

Start with hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho g = -\rho \frac{G\,M(r)}{r^2} \qquad V = \frac{4}{3}\pi r^3$$

$$dM = 4\pi r^2\,dr$$

So $\frac{4}{3}\pi r^3\,dP = -\frac{4}{3}\pi r^3 \rho \frac{G\,M(r)}{r^2}\,dr$

so $V\,dP = -\frac{1}{3}\frac{G\,M(r)}{r}\,dM$ — (a)

Integrate by parts

$$\int_{P_0}^{0} V\,dP = VP\Big]_{\text{center}}^{\text{surface}} - \int_0^{V_*} P\,dV$$

but $VP\Big]_{\text{center}}^{\text{surface}} = 0$ as $V=0$ at center + $P=0$ at surface

$$E_{grav} = -\int_0^M \frac{GM}{r}\,dm = 3\int V\,dP \quad \text{from (a)}$$

$$-\frac{1}{3}\Omega = \int_0^{V_*} P\,dV = \int nkT\,dV \quad \text{but } E_{thermal} = \frac{3}{2}nkT\,(\text{Vol})$$

$$= \frac{2}{3}\,U \qquad U = \text{total thermal energy}$$

$$\boxed{U = -\frac{\Omega}{2}} \qquad \boxed{2U + \Omega = 0} \qquad \boxed{\text{Virial theorem}}$$

Applications – Kelvin-Helmholtz Time Scale

W = total energy = $\int_M \epsilon \, dM_r + \Omega = U + \Omega$

Ω = grav potential energy

$$d\Omega = -\int_r^\infty \frac{G M_r}{(r')^2} \, dM_r \, dr' = -\frac{G M_r}{r} \, dM_r$$

$$\Omega = -\int_0^M \frac{G M_r}{r} \, dM_r = -q \frac{G M^2}{R}$$

q = constant, depends on $\rho(r)$. If ρ = constant, $M_r = \frac{4}{3}\pi r^3 \rho$

(uniform density)

$$\Omega = -\int_0^R G \frac{\left(\frac{4}{3}\pi r^3 \rho\right)}{r} 4\pi r^2 \rho \, dr$$

$$= -G \, \frac{16}{3}\pi^2 \rho^2 \int_0^R r^4 \, dr \rightarrow \frac{R^5}{5}$$

$$= -G \frac{M^2}{R} \frac{3}{5} \qquad q = 3/5$$

$$L \xrightarrow{\frac{d\Omega}{dt}} -\frac{d}{dt}\left(\frac{3}{5} \frac{G M^2}{R}\right) \quad \text{from grav contraction only}$$

$$= \left(\frac{3}{5}\right) G M^2 \frac{(dR/dt)}{R^2}$$

$$= -\frac{3}{5} \frac{G M^2}{R} \frac{d \ln R}{dt}$$

R decreases exponentially with time

$$\frac{d \ln R}{dt} = -\frac{5 R L}{3 G M^2}$$

$R = R_0 e^{-t/t_{KH}}$

$\ln R = -t/t_{KH}$

$\frac{d \ln R}{dt} = \frac{1}{t_{KH}}$

$$t_{KH} = \frac{3}{5} \frac{G M^2}{R L}$$

$$\approx 2 \times 10^7 \text{ years} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{L}{L_\odot}\right)^{-1} \left(\frac{R}{R_\odot}\right)^{-1}$$

But life on earth/fossils no drastic change for hundreds of millions of years, could not have a large change in $L_\odot$, which would change T(Earth).

Conclude Sun cannot be fueled by gravitational contraction

Dynamical time Scale

By magic take away ~~gravitational~~ pressure support, star will collapse

$$\frac{dP}{dr} - \rho g = \rho \ddot{r} \approx 0 \qquad \text{but now} \quad \rho g = \rho \ddot{r}$$

$$\ddot{r} = g = GM/R^2 \qquad \text{time to collapse } t_c$$

$$R = \frac{1}{2} a t_c^2 = \frac{1}{2}\frac{GM}{R^2} t_c^2 \Rightarrow t_c \approx \sqrt{\frac{R^2}{GM}}$$

$$\text{but } \rho = \frac{M}{R^3}, \quad R^2 = \frac{M}{\rho R} \quad \text{so} \quad \boxed{t_c \approx \sqrt{\frac{1}{G\langle\rho\rangle}}}$$

$\approx$ 1 hour for Sun

~~Dynamical~~ Estimate of ~~Stellar~~ ~~Temperature~~ ~~Interior~~ Average Temp Inside Star

$$E = \frac{3}{2} n kT = \frac{3}{2}\rho \frac{N_A kT}{\mu} \qquad N_A = \text{Avogadro's number} = 1/m_H$$

$$U = EV = E\left(\frac{4}{3}\pi R^3\right) \qquad \text{and } \rho\left(\frac{4}{3}\pi R^3\right) = M_*$$

$$U = \frac{3}{2}\frac{M N_A kT}{\mu} = -q\frac{GM^2}{R} \simeq \Omega$$

$$\rho \propto M/R^3$$

$$\rho^{1/3} \propto M^{1/3}/R$$

$$\frac{M N_A kT}{\mu} = GM^2/R$$

$$kT \propto GM/R = G\rho^{1/3} M^{2/3}$$

$$\boxed{\langle T\rangle = 4.1\times10^6 \mu \left(\frac{M}{M_\odot}\right)^{2/3}\left(\frac{\rho}{\rho_\odot}\right)^{1/3} K}$$

Time Scale for Small (Adiabatic) Sound Waves to travel from center of star to surface & back

Related to periods of pulsating variable stars

V_s = stellar speed of sound

Period $\Pi = 2R/V_s$

$$V_s^2 = \frac{dP}{d\rho} = \Gamma \frac{P}{\rho} \qquad \Gamma = \text{constant}$$

$$-\Omega = \frac{GM^2}{R} = 3\int \frac{P}{\rho}\, dM_r = 3\int \frac{V_s^2}{\Gamma}\, dM_r \approx \frac{3V_s^2}{\Gamma} M$$

$$V_s^2 = \frac{\Gamma GM}{3R} \qquad \Pi = 2R\sqrt{\frac{3R}{\Gamma GM}} \propto \sqrt{\frac{R^3}{GM}} = \sqrt{\frac{1}{\rho G}}$$

$$\Pi \approx \frac{1}{\sqrt{G\rho}} \approx \frac{0.04 \text{ days}}{[\langle\rho\rangle / \langle\rho_{sun}\rangle]^{1/2}}$$

average ρ over star

Star Formation - the Jeans Criterion (2-7)

Basic * formation picture:

a) cloud of gas in ISM becomes gravitationally bound

b) cools + contracts, eventually contraction accelerates into collapse

c) half of ΔE_{grav} is radiated, other half heats up *, eventually T_c high enough for H burning

reality is messier: stars form in clusters within 1 large gas cloud (Orion nebula) will fragment into smaller clouds during contraction/collapse → distribution of stellar masses → IMF (initial mass function).

angular momentum issues: centrifugal force must be (L) overcome → excess L drives fragmentation.

P_{rad} can regulate star formation

magnetic field a role (P_{mag} + torques)

Timescale for * formation = free fall time (dynamical time scale)

(Calculated this earlier) $\tau_{ff} \approx \sqrt{\frac{R^3}{GM}} \simeq \frac{0.05}{\sqrt{G<\rho>}}$

P drops to 0 → collapse

$\tau_{ff} \sim 1$ hour for Sun, very fast!

for a ISM cloud, $n \sim 10^6/cm^3$ (gas cloud), assume neutral H $\Rightarrow \bar{\rho} \sim 1.68 \times 10^{-18}\, gm/cm^3 \Rightarrow \tau_{ff} \sim 50{,}000\, yr$

Very fast from astronomy point of view

Jeans Instability

Virial theorem $2E_{kin} = -E_{grav}$ (equilibrium)

$>$ (expansion)

$<$ (contraction)

$$2E_{kin} = 3NkT = \frac{3MkT}{\mu m_H}$$ μ mean atomic weight

$$E_{grav} = -\int_0^M \frac{Gm}{r}\,dm$$

assume constant ρ ($\rho = \bar{\rho}$)

$M = \frac{4}{3}\pi\bar{\rho}r^3$

$dm = 4\pi\bar{\rho}r^2 dr$

$$= -\int_0^M G\bar{\rho}\frac{4\pi}{3}r^3 \underbrace{4\pi\bar{\rho}r^2 dr}_{dm}$$

$$= -\frac{16}{3}\pi^2\bar{\rho}^2 G\int r^4\,dr = -\frac{3}{5}\frac{GM^2}{R}$$

Need $2E_{kin} < -E_{grav}$ for contraction

$$\frac{3}{5}\frac{GM^2}{R} > \frac{3MkT}{\mu m_H}$$

Substitute $R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$

$$M > \left(\frac{5kT}{G\mu m_H}\right)^{3/2}\left(\frac{3}{4\pi\bar{\rho}}\right)^{1/2} = M_J \text{ (Jeans mass)}$$

If $M_{cloud} > M_J$, cloud can collapse

Jean's length: radius of cloud that has $M = M_J$

$$R_J = \lambda_J =$$ Jean's length

$\cancel{M_J = \frac{4\pi}{3}\bar{\rho}\,l^3}$

$M_J = \frac{4}{3}\pi\lambda_J^3\bar{\rho}$

Solve for λ_J

$$\lambda_J = \left[\frac{15kT}{4\pi\bar{\rho}\mu m_H G}\right]^{1/2}$$

Timescale for collapse $\approx \tau_{ff}$ ff = free fall

Can also calculate $t_J \approx \frac{R_J}{c_s}$ ← speed of sound $= \frac{P}{\rho}$

yields $\approx 2 \times \tau_{ff}$

Example ISM cloud
For neutral gas, H + He $\mu = 2.3$ $\rho = n \mu m_H$
n $10^5/cm^3$ $T \sim 50\,K$

$$M_J = 18\, M_\odot \left(\frac{\mu}{2.3}\right)^{-3/2} \left(\frac{T}{50K}\right)^{3/2} \left(\frac{n}{10^5/cm^3}\right)^{-1/2}$$

18 $M_\odot$ is large, probably will end up fragmenting & forming many stars

$M_J \uparrow$ if $T \uparrow$
~~M_J~~ if $n\uparrow$, $M_J \downarrow$

Problem: ISM cloud often exceed 1000 $M_\odot$, so reach criticality at low n, high M_J, how can $1 M_\odot$ stars form?

$M_J \propto T^{3/2} \mu^{-3/2} \rho^{-1/2}$

Consider 2 cases:
a) T constant, all compression heat is radiated away, as star collapses, $M_J \downarrow$, so smaller regions inside unstable large cloud become unstable → fragmentation
b) adiabatic collapse, half of heat is radiated away
$T \propto \rho^{\gamma - 1} \propto \rho^{2/3}$ (perfect monatomic gas)
$M_J \propto (\rho^{2/3})^{3/2} \rho^{-1/2} \propto \rho^{1/2}$ so $M_J \uparrow$ as compresses & collapse can be shut off.

Conclusions

Initial collapse must be cooled so that
fragmentation is possible
eventually contraction timescale becomes
slower than cooling timescale
contraction becomes adiabatic,
star becomes stable.

Details of star formation ~~is~~ are still very uncertain.

Star Formation Phases

Cloud in ISM collapses (fragments) → protostar

Protostar collapses via free fall very quickly.

H + He become ionized inside star →

protostar enters hydrostatic eq. →

pre-MS star:
- grav. collapse with HE (Hayashi track) →
- formation of a radiative core →
- further collapse along Henyey track →

ignition of H in core (star on main sequence)

or

→ brown dwarf (degenerate, T_c too low for H burning

We ignore accretion after initial formation of protostar

A collapsing cloud has

$\Omega > U$ (thermal energy)

as it collapses, $\frac{1}{2}$ of $\Delta\Omega$ goes to increase U, and $1/2$ is radiated (virial theorem).

ΔU goes into ionization of atoms (+ before that, dissociation of molecules) until these processes are complete

$$E_i = N_0 X (IP_H) + \frac{1}{4} N_0 Y (IP_{He}) + \frac{1}{2} N_0 X (IP)_D$$

when ionization is complete

$$\frac{1}{2}\frac{M_s^2 G}{R_s} \sim M_s E_{ionization}$$

ergs / gm of material

$$R_s = M_s \frac{G}{2 E_{ion}}$$

$$\left(\frac{R_s}{R_\odot}\right) \approx \frac{43}{(1-0.2X)}\left(\frac{M}{M_\odot}\right)$$

This is the maximum radius for a stable star as it finishes ionizing + dissociating.

For $R > R_s$ the star is not completely ionized + dissociated + cannot be in equilibrium.

For $R < R_S$, ~~q~~ quasi equilibrium while star contracts + heats up.

T_c at $R_S \approx 3\times10^5 \mu(1-0.2X)$ Kelvin

This is too low for nuclear reactions, so star contracts

$$E_{tot} = \Omega + U$$

$$L = \frac{dE_T}{dt} = \frac{1}{2} q \frac{d}{dt}(GM^2/R)$$

$$= -\frac{q}{2} GM^2 \frac{dR}{dt}/R$$

$L > 0$ so $\frac{dR}{dt}$ must be < 0, ie star contracting

timescale for contraction set by luminosity

$$\Delta t \sim \frac{\Delta t}{L} \approx -\frac{1}{2}\frac{\Delta \Omega}{L} \approx q\frac{M^2 G}{2RL}$$

$$\Delta t \approx 1.6\times10^7 \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right)\left(\frac{L_\odot}{L}\right) \text{ years}$$

Since R and L can be big, Δt can be short

Aside for later once we discuss convection:

The stars in the contracting stage are fully convective ~~with~~ except for a thin outer envelope

For fully convective stars, it is not possible to have a hydrostatic model if T falls below a certain value. There is thus a "forbidden region" in the M-R diagram which boundary is almost vertical. Stars contract staying on the blue edge of this Hayashi region - called a Hayashi track. Eventually nuclear reactions turn on - beginning with $D \to He^4$ - at $\sim 2 \times 10^6$ K. This slows down the contraction briefly & then the main sequence is reached.

2-15

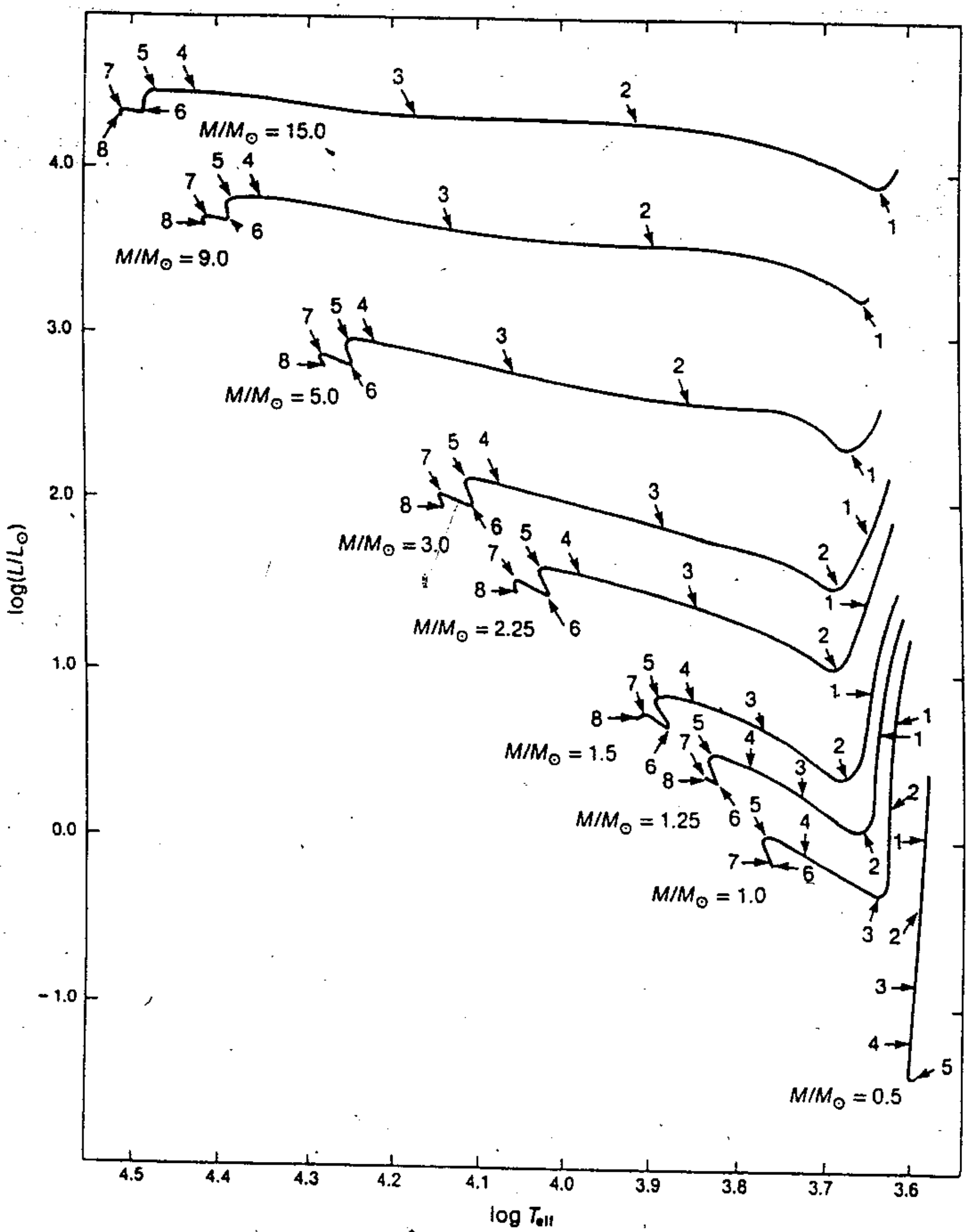


Figure 8.2. Evolutionary tracks for stars before the main sequence. The time intervals required for stars to reach the numbered points are given in Table 8.2. The locus of the terminal points of the paths defines the zero age main sequence.

Iben (1965)
ApJ, 141

Table 8.2
Elapsed time in years of selected points along evolutionary tracks leading to the main sequence, as shown in Figure 8.2.

	$M/M_\odot$								
Point	15.0	9.0	5.0	3.0	2.25	1.5	1.25	1.0	0.5
1	6.740×10^2	1.443×10^3	2.936×10^4	3.420×10^4	7.862×10^4	2.347×10^5	4.508×10^5	1.189×10^5	3.195×10^5
2	3.766×10^3	1.473×10^4	1.069×10^5	2.078×10^5	5.940×10^5	2.363×10^6	3.957×10^6	1.058×10^4	1.786×10^4
3	9.350×10^3	3.645×10^4	2.001×10^5	7.633×10^5	1.883×10^6	5.801×10^6	8.800×10^6	8.910×10^4	8.711×10^6
4	2.203×10^4	6.987×10^4	2.860×10^5	1.135×10^6	2.505×10^6	7.584×10^6	1.155×10^7	1.821×10^7	3.092×10^7
5	2.657×10^4	7.922×10^4	3.137×10^5	1.250×10^6	2.818×10^6	8.620×10^6	1.404×10^7	2.529×10^7	1.550×10^8
6	3.984×10^4	1.019×10^5	3.880×10^5	1.465×10^6	3.319×10^6	1.043×10^7	1.755×10^7	3.418×10^7	
7	4.585×10^4	1.195×10^5	4.559×10^5	1.741×10^6	3.993×10^6	1.339×10^7	2.796×10^7	5.016×10^7	
8	6.170×10^4	1.505×10^5	5.759×10^5	2.514×10^6	5.855×10^6	1.821×10^7	2.954×10^7		

SOURCE: I. Iben, Jr., *Astrophys. J.*, 141:993 (1965). Reprinted by permission of The University of Chicago Press. Copyright 1965 by The University of Chicago.

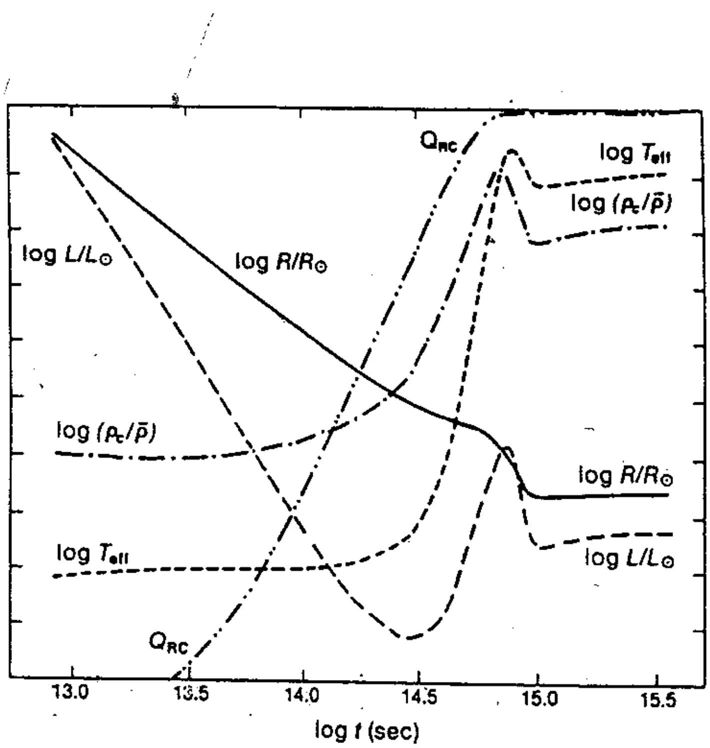
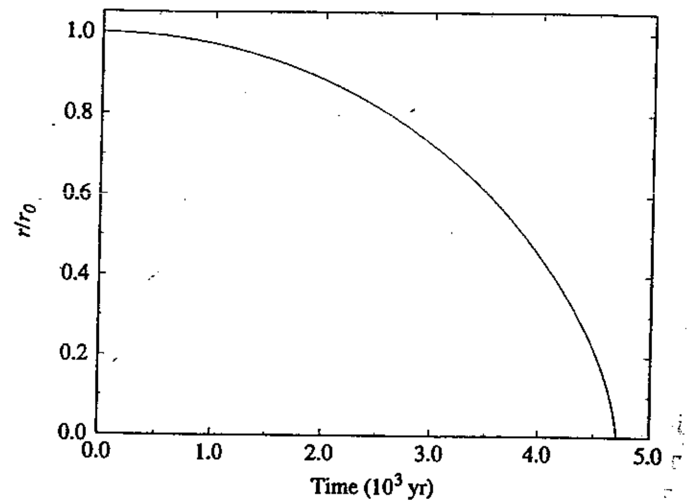


Figure 8.3. Time-variation of stellar parameters for a 1 $M_\odot$ star as it evolves down the Hayashi track; the time is in seconds. The curve Q_{RC} gives the mass fraction in the radiative core, with the ordinate scaled such that $0 \le Q_{RC} \le 1.0$. For other curves, $3.58 \le \log T_{eff} \le 3.78$; $0 \le \log(\rho_c/\bar{\rho}) \le 2.0$; $-0.4 \le \log(L/L_\odot) \le +0.6$; and $-0.4 \le \log(R/R_\odot) \le +0.6$.



ıre 12.5 The ratio of the radius relative to its initial value as a function ne for the homologous collapse of a molecular cloud. The collapse is as- :d to be isothermal, beginning with a density of $\rho_0 = 2\times10^{-16}$ g cm^{-3}.

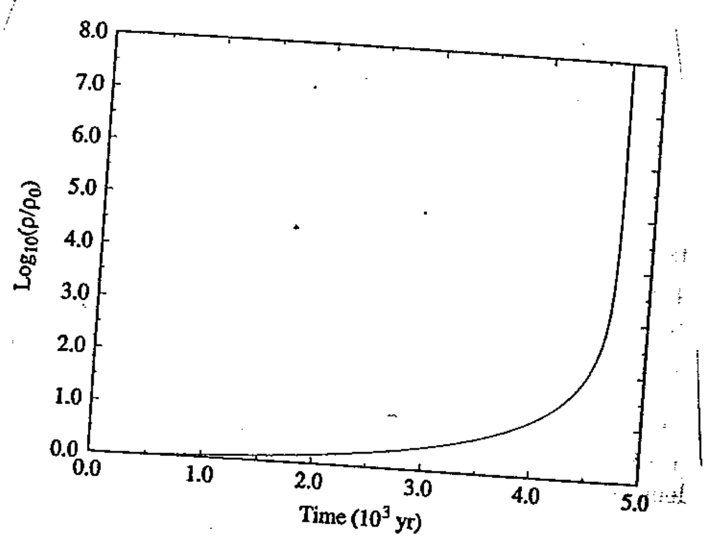
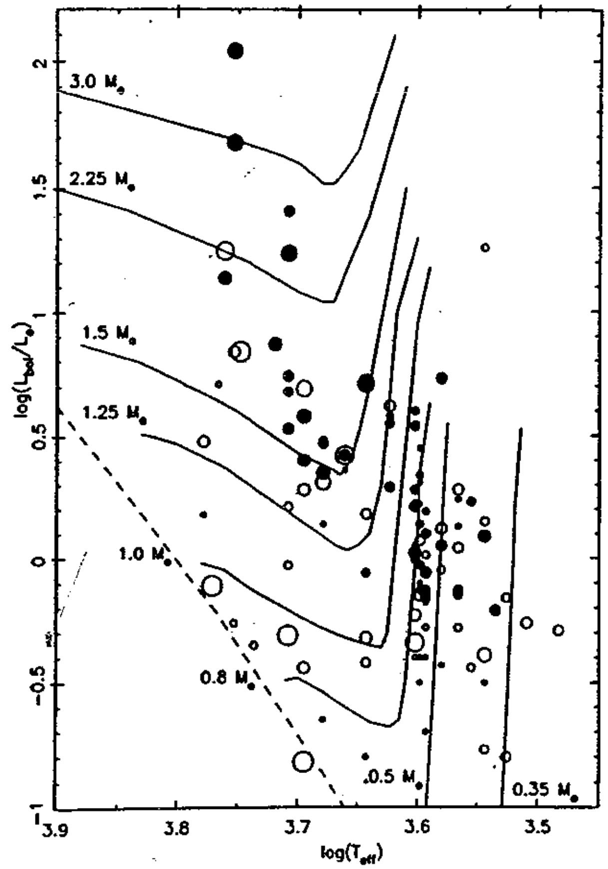


Figure 12.6 The ratio of the cloud's density relative to its initial value as a function of time for the isothermal, homologous collapse of a molecular cloud with an initial density of $\rho_0 = 2 \times 10^{-16}$ g cm^{-3}.

2-18



igure 12.11 The positions of T Tauri stars on the H–R diagram. The of the circles indicate the rate of rotation. Stars with strong emission are indicated by filled circles and weak emission line stars are represented by open circles. Theoretical pre-main-sequence evolutionary tracks also included. (Figure from Bertout, *Annu. Rev. Astron. Astrophys.*, 351, 1989. Reproduced with permission from the *Annual Review of* *onomy and Astrophysics*, Volume 27, ©1989 by Annual Reviews Inc.)

Initial Mass Function

Collapse + fragmentation → initial mass function

Distribution of stellar masses, (empirical, not from theory),
origin not fully understood

We understand why many more low mass stars than
high mass ones, but beyond that, ...

Find IMF by counting stars in young clusters,
bin by mass, fit a function (usually a
power law)

problems: most massive stars die very quickly
low mass * faint + hard to count
measure colors, luminosity, not mass, must
infer mass

detailed studies limited to nearby & clusters

IMF appears to be independent of metallicity &
environment

Is the IMF universal?

$$dN = n M^{\alpha} dM$$

power law exponent

(normalization

Salpeter (1955) α -2.35

Kroupa (2001) different αs for different mass ranges

α - 0.3 $\quad m < 0.08\, M_\odot$ (substellar)

-1.3 $\quad 0.08 < m < 0.5\, M_\odot$

-2.3 $\quad m > 0.5\, M_\odot$

Use IMF to find fraction of stars in various mass ranges

$$f = \frac{N(1 \text{ to } 2\, M_\odot)}{N_{total}} = \frac{\int_1^2 n\, m^{-2.35}\, dm}{\int_{0.08}^{120} n\, m^{-2.35}\, dm} = 0.02$$

$$\bar{M} = \text{mean mass} = \frac{M_{tot}}{N_*} = \frac{\int_{0.05}^{120} n\, m\, m^{-2.35}\, dm}{\int_{0.05}^{120} n\, m^{-2.35}\, dm} = \cancel{2}\ 0.28\, M_\odot$$

$$\bar{L} \text{ (assume } L \propto m^3 \text{ on main sequence)} = \frac{L_{total}}{M_{total}} = \frac{L_\odot \int n\, m^{-2.35}\, m^3\, dm}{\int n\, m^{-2.35}\, dm}$$

$\bar{L}$ (on MS) $\approx 73\, L_\odot$ ($\equiv 4\, M_\odot$ star)

even though $4\, M_\odot$ star is rare, it is so bright it contributes significantly to the total L