Solution: HW 2
AY 123, Spring 2003

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Problem 1: Prove that the gravitational potential energy for a polytrope is $\Omega = -\frac{3}{(5-n)}\frac{GM^2}{R}$

Solution:

[First, we work out the following relation, which we will use later. Begin with hydrostatic equilibrium:

$$\frac{dP}{dr} = \frac{-GM_r(r)\rho_r(r)}{r^2}. \tag{1}$$

Multiply by $4\pi r^3 dr$ and integrate $0 \to R$:

$$\int_0^R 4\pi r^3 dP = -\int_0^r \frac{GM_r}{r}\rho 4\pi r^2 dr = -\int_0^M \frac{GM_r}{r} dM_r, \tag{2}$$

where we have used $dM_r = 4\pi r^2 \rho dr$. This last term is just Ω. We integrate the first term by parts:

$$\int_0^R 4\pi r^3 dP = 4\pi P r^3 \Big|_0^R - 4\pi \int_0^R 3r^2 P dr = -3\int_0^R P 4\pi r^2 dr, \tag{3}$$

where we have used the approximation $P(R) \approx 0$. So, finally,

$$-\int_0^R P 4\pi r^2 dr = \frac{1}{3}\Omega \quad] \tag{4}$$

Moving on, we will start with the definition of Ω:

$$\Omega = -\int_0^M \frac{GM_r}{r} dM_r. \tag{5}$$

We integrate once by parts ($u = 1/r$, $dv = M_r dM_r$) to obtain

$$\Omega = -\frac{GM^2}{2R} + \frac{M_r^2}{2r}\Big|_{M_r=0} - \int_0^R \frac{GM_r^2}{2r^2} dr. \tag{6}$$

You can verify for yourself using L'Hopital's rule that the second term goes to zero, assuming ρ is finite at $M_r = r = 0$. We again make use hydrostatic equilibrium to rewrite

$$-\frac{GM_r}{r^2} = \frac{1}{\rho}\frac{dP}{dr} \;\Rightarrow\; \Omega = -\frac{GM^2}{2R} + \frac{1}{2}\int_0^R \frac{M_r}{\rho}\frac{dP}{dr} dr. \tag{7}$$

Integrate again by parts ($u = M_r/\rho$, $dv = dP/dr$) to obtain

$$\Omega = -\frac{GM^2}{2R} - \frac{1}{2}\int_0^R \frac{P}{\rho}\frac{dM_r}{dr}dr + \frac{1}{2}\int_0^R \frac{PM_r}{\rho^2}\frac{d\rho}{dr}dr, \tag{8}$$

again making use of $P(R) \approx 0$.

Now, $dM_r = 4\pi r^2 \rho dr$ means that the second term on the RHS is

$$-\frac{1}{2}\int_0^R P4\pi r^2 dr = \frac{1}{6}\Omega, \tag{9}$$

using Eq. 4 from before (see, it really was useful). As for the third term, we finally use the fact that we have a polytropic equation of state, which means that

$$P = K\rho^{1+1/n} = K\rho^{(n+1)/n}. \tag{10}$$

So, lets try to get something useful out of this:

$$dP = \frac{(n+1)}{n}K\rho^{1/n}d\rho \;\Rightarrow\; \frac{dP}{P} = \frac{n+1}{n}\frac{d\rho}{\rho} \;\Rightarrow\; \frac{1}{\rho}\frac{d\rho}{dr} = \frac{n}{n+1}\frac{1}{P}\frac{dP}{dr} \tag{11}$$

Putting together Eqs 8, 9 & 11, we have

$$\frac{5}{6}\Omega = -\frac{GM^2}{2R} + \frac{1}{2}\frac{n}{(n+1)}\int_0^R \frac{M_r}{\rho}\frac{dP}{dr}dr. \tag{12}$$

However, we see from Eq. 7 that we can rewrite this last term in terms of Ω, M and R to get

$$\frac{5}{6}\Omega = -\frac{GM^2}{2R} + \frac{n}{(n+1)}(\Omega + \frac{GM^2}{2R}). \tag{13}$$

A little algebra finally yields the answer we are looking for:

$$\Omega = -\frac{3}{(5-n)}\frac{GM^2}{R} \tag{14}$$

Yeah, done!

Problem 2: Derive the mass-radius relation for an $n = 1$ polytrope and apply to Jupiter and Saturn

Solution:

I'll work out the general mass-radius relation for a polytrope, although it's somewhat simpler for the $n = 1$ case. We start with the following equations of stellar structure:

$$\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \tag{15}$$

and

$$\frac{dM_r}{dr} = 4\pi r^2\rho, \tag{16}$$

together with the polytropic EOS:

$$P = K\rho^{(n+1)/n} \tag{17}$$

Now, if we solve Eq. 15 for M_r, take the derivative with respect to r and plug this into Eq. 16, we get Poisson's equation:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{18}$$

To make analysis easier (and because this is the way it seems to always be done) we define the following dimensionless variables ξ and θ:

$$r = \alpha\xi \quad \text{and} \quad \rho = \rho_c\theta^n \quad \Rightarrow \quad P = K\rho_c^{(n+1)/n}\theta^{(n+1)}. \tag{19}$$

Also, ξ_1 is defined as the surface, such that $R = \alpha\xi_1$. If we plug these expressions for r, ρ and P into Eq. 18, with a little algebra, and a convenient definition of α, we will end up with the Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \quad \text{with} \quad \alpha = \left[\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2}. \tag{20}$$

The total radius is easy to write down, it's

$$R = \alpha\xi_1 = \left[\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2}\xi_1. \tag{21}$$

Now we will solve for ρ_c in terms of the total mass in order to get a relationship between radius and mass. The total mass is

$$M = M_\xi(\xi_1) = 4\pi\alpha^3\rho_c\int_0^{\xi_1}\xi^2\theta^n d\xi, \tag{22}$$

which follows directly from Eq. 16 when the expressions for ξ and θ are substituted for r and ρ. We use Eq. 20 to rewrite θ^n:

$$M = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(-\xi^2\frac{d\theta}{d\xi}\right) d\xi = 4\pi\alpha^3\rho_c \left[-\xi^2\theta'\right]_0^{\xi_1}. \tag{23}$$

You can check that $\theta'(0) \propto \frac{dP}{dr}(0)$. Therefore, if we are to have spherical symmetry at the center ($\frac{dP}{dr} = 0$ at $r = 0$), it must be that $\theta'(\xi = 0) = 0$. Which means

$$M = 4\pi\alpha^3\rho_c \left(-\xi^2\theta'\right)_{\xi_1} = \frac{1}{\sqrt{4\pi}}\left(\frac{K(n+1)}{G}\right)^{3/2} \rho_c^{(3-n)/2n}(-\xi^2\theta')_{\xi_1} \tag{24}$$

$$\Rightarrow \quad \rho_c = (4\pi)^{n/(3-n)}\left(\frac{G}{K(n+1)}\right)^{3n/(3-n)}\left(\frac{M}{(-\xi^2\theta')_{\xi_1}}\right)^{2n/(3-n)}. \tag{25}$$

We take this expression for ρ_c and plug it back into the expression for R. After crunching through some annoying algebra we find:

$$R = \left(\frac{1}{4\pi}\right)^{\frac{1}{3-n}}\left(\frac{K(n+1)}{G}\right)^{\frac{n}{3-n}} \xi_1^{\frac{n+1}{3-n}} \; (-\theta'(\xi_1))^{\frac{n-1}{3-n}} \; M^{\frac{1-n}{3-n}} \tag{26}$$

$$\Rightarrow \quad R \propto M^{\frac{1-n}{3-n}} \tag{27}$$

So, for a polytrope with $n = 1$, we have $R \propto M^0$, the radius is independent of mass. Which is approximately true for low mass brown dwarfs and giant gas planets like Jupiter.

Application to Jupiter and Saturn:
The radius of Jupiter is $R_J = 7 \times 10^9$ cm, and the mass is $M_J = 2 \times 10^{30}$ g.
For Saturn, $R_S = 6 \times 10^9$ cm and $M_S = 6 \times 10^{29}$ g.
While it is clear that radius is not exactly constant for these two planets, we can work out the dependence of radius on mass, assuming there are no other dependencies present:

$$\frac{R_S}{R_J} = \left(\frac{M_s}{M_J}\right)^x \quad \Rightarrow \quad x = 0.13 \tag{28}$$

So $R \propto M^{0.13}$, which is a pretty weak dependence, weak enough that we can say Jupiter and Saturn are reasonably well described by an $n = 1$ polytrope.

[SIDENOTE: What if we want to know what the actual predicted radius for both Jupiter and Saturn is? For $n = 1$ Eq. 26 becomes

$$R = \left(\frac{K}{2\pi G}\right)^{1/2} \xi_1. \tag{29}$$

In the next problem we will integrate the equations of stellar structure to find ξ_1, which is easy enough to repeat for $n = 1$ (or, you can actually solve the $n = 1$ case analytically). However, if we really wanted to know what this "constant" radius should be, we would have to find out what K is, and thus would need to know more about the equation of state.]

Problem 3: Write a computer program to solve the equations of stellar structure for an $n = 3$ polytrope and plot the ρ/ρ_c vs r/R for a solar model. Compare with a real solar model. You need only hand in your results but do bear in mind that you will eventually be computing a solar model using all of the equations of stellar structure.

Solution:

Luckily, we have already worked out some of the preliminaries in Problem 2. It was shown above that solving the equations of stellar structure (Eqs. 15 and 16) together with the EOS (Eq. 17) is equivalent to solving the second order Lane-Emden equation (Eq. 20). We solve this subject to the following boundary conditions:

$$\theta = 1 \text{ and } \theta' = \frac{d\theta}{d\xi} = 0 \text{ at } \xi = 0, \qquad \theta = 0 \text{ at } \xi = \xi_1 \tag{30}$$

What we will want in the end is $\rho/\rho_c = \theta^n$. I wrote a fourth order Runge-Kutta (see e.g., Numerical Recipes) procedure in IDL to solve this problem. There are, of course, other ways to do it. [NOTE: there might be an easier way to do this considering we just want θ^n and ξ_1, but, as the problem points out, we will want a more general solution later, so I'll just do it out now.]

In order to use the Runge-Kutta technique we define a new variable $\eta(\xi) = -\xi^2\theta'(\xi)$, and use this to make two first order differential equations out of our single second order equation:

$$\frac{d\eta}{d\xi} = -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \xi^2\theta^n \quad \Rightarrow \quad \frac{d\eta}{d\xi} = \xi^2\theta^n \tag{31}$$

and

$$\frac{d\theta}{d\xi} = -\frac{1}{\xi^2}\eta \tag{32}$$

Now we would be ready to integrate, stepping through ξ to get η and θ at each ξ, if it weren't for the annoying fact that we seem to have a divergent point in our equations at $\xi = 0$. To deal with this, we start instead at some small $\xi_0 \ll 1$, and Taylor expand to get

$\theta(\xi_0)$ and $\theta'(\xi_0)$. Write θ as a power series:

$$\theta = \sum_{i=0}^{\infty}[a_i(\xi)]^i = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + ... \tag{33}$$

Take the first two derivatives of θ (θ' and θ'') and apply the boundary conditions $\theta(0) = 1 \Rightarrow a_0 = 1$ and $\theta'(0) = 0 \Rightarrow a_1 = 0$. Now write out θ^n and expand, using the fact that $\xi_0 \ll 1$:

$$\theta^n \approx 1 + na_2\xi^2 + \mathrm{O}(\xi^4). \tag{34}$$

Write the Lane-Emden equation as

$$\frac{1}{\xi^2}\left[2\xi\frac{d\theta}{d\xi} + \xi^2\frac{d^2\theta}{d\xi^2}\right] = -\theta^n. \tag{35}$$

Plug everything in and gather terms of the same order in ξ. Thus we get the starting conditions

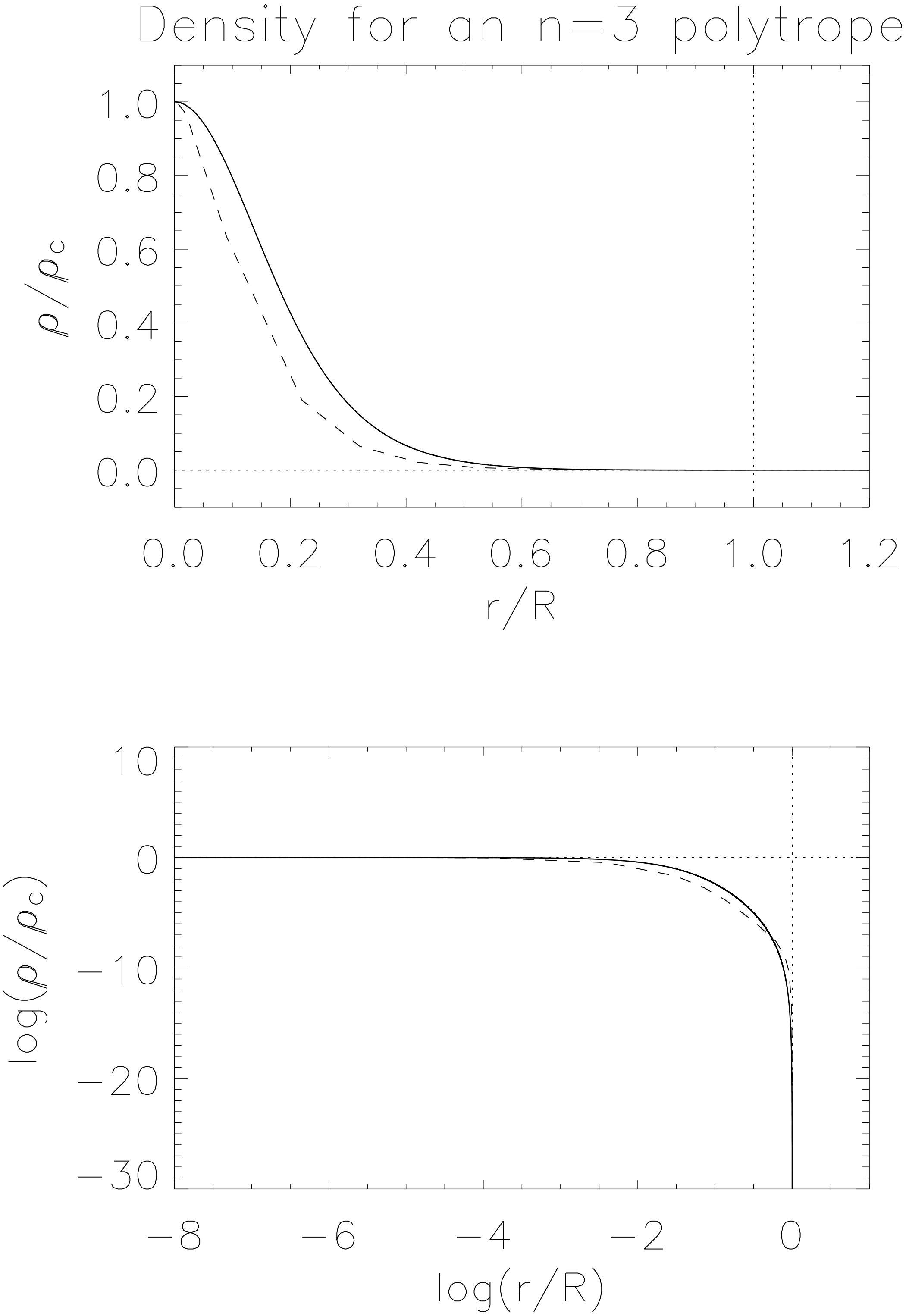
$$\theta(\xi_0) \approx 1 - \frac{1}{6}\xi_0^2 + \frac{n}{120}\xi_0^4 \tag{36}$$

and

$$\eta(\xi_0) \approx \frac{1}{3}\xi_0^3 - \frac{n}{30}\xi_0^5 \tag{37}$$

OK, so we throw this all into the numerical integration program, and out comes θ and η as a function of ξ. The value of ξ at which we reach $\theta = 0$ is ξ_1, the surface. I have done the integration with $\xi_0 = 0.001$ and $\Delta\xi = 0.001$ as the stepsize. For $n = 1$ I find $\xi_1 = 6.9$.

Both liner and log plots of $\rho/\rho_c = \theta^n$ vs $r/R = \xi/\xi_1$ are on the following page (solid lines). Also plotted as a dashed line is a solar model (Allen's Astrophysical Quantities) with a central density $\rho_c = 151\mathrm{g\ cm}^{-3}$. The polytrope model matches the standard solar model fairly well for very small and for large radii, but overestimates the density for $r \sim 0.1 - 0.5R$.



Density for an n=3 polytrope
ρ/ρc
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.2
r/R
log(ρ/ρc)
10
0
−10
−20
−30
−8
−6
−4
−2
0
log(r/R)