Solution: HW 3
AY 123, Spring 2003

by Melissa Enoch (with some help from previous solutions)

Problem 1: Solar Model

Solution:

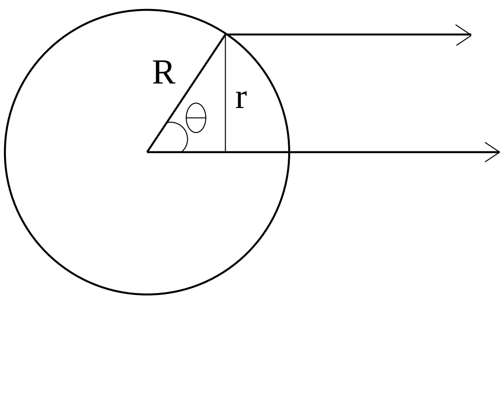
a) The solar model from Bob Kurucz gives us values of $\int \rho dz$, T, P, n_e and κ_R for each of 64 values of optical depth τ. We read in the model and plot T vs τ, comparing it to the grey case solution $T^4 = (3/4)[T_{eff}^4(\tau + 2/3)]$. Plots for both linear and log scales are on the next page (Fig. 1). We can see that the model matches the grey case solution fairly well for moderate values of $10^{-3} \lesssim \tau \lesssim 10$. Following this is the same plot for $0.1 < \tau < 0.2$ (Fig. 2). The largest percent deviation $((T_{Kurucz} - T_{grey})/T_{Kurucz})$ in this range is $\approx 4\%$, and occurs at $\tau \approx 1$.

b) The limb darkening is described by the intensity as a function of angle $I(\mu)$ where $\mu = \cos\theta$. Limb darkening occurs because lines of sight toward the edge of the sun reach an optical depth of $\tau = 2/3$ at a somewhat larger radius from the *center* of the sun (as compared to $\theta = 0$ lines of sight), and thus at a slightly lower temperature. This lower temperature means that we see a slightly smaller emergent intensity, an effect which grows stronger as we move toward the edge of the sun. For a nice simple picture see **http://sim.jpl.nasa.gov/michelson/iss2000/limbdark.pdf**, page 7, and for a real image of the sun see **http://solar-center.stanford.edu/images/wl.gif**.

To calculate the limb darkening we use HK Eq. 4.12

$$I(\mu) = I(\tau_0, \mu \geq 0) = \int_{\tau_0}^{\infty} e^{-(\tau-\tau_0)/\mu} \frac{S(\tau)}{\mu} d\tau, \tag{1}$$

where we take $\tau_0 = 0$, so we are integrating $0 \to \infty$. $S(\tau)$ is the source function, which we assume is a blackbody $B(T)$. I am going to assume that we are looking at bolometric



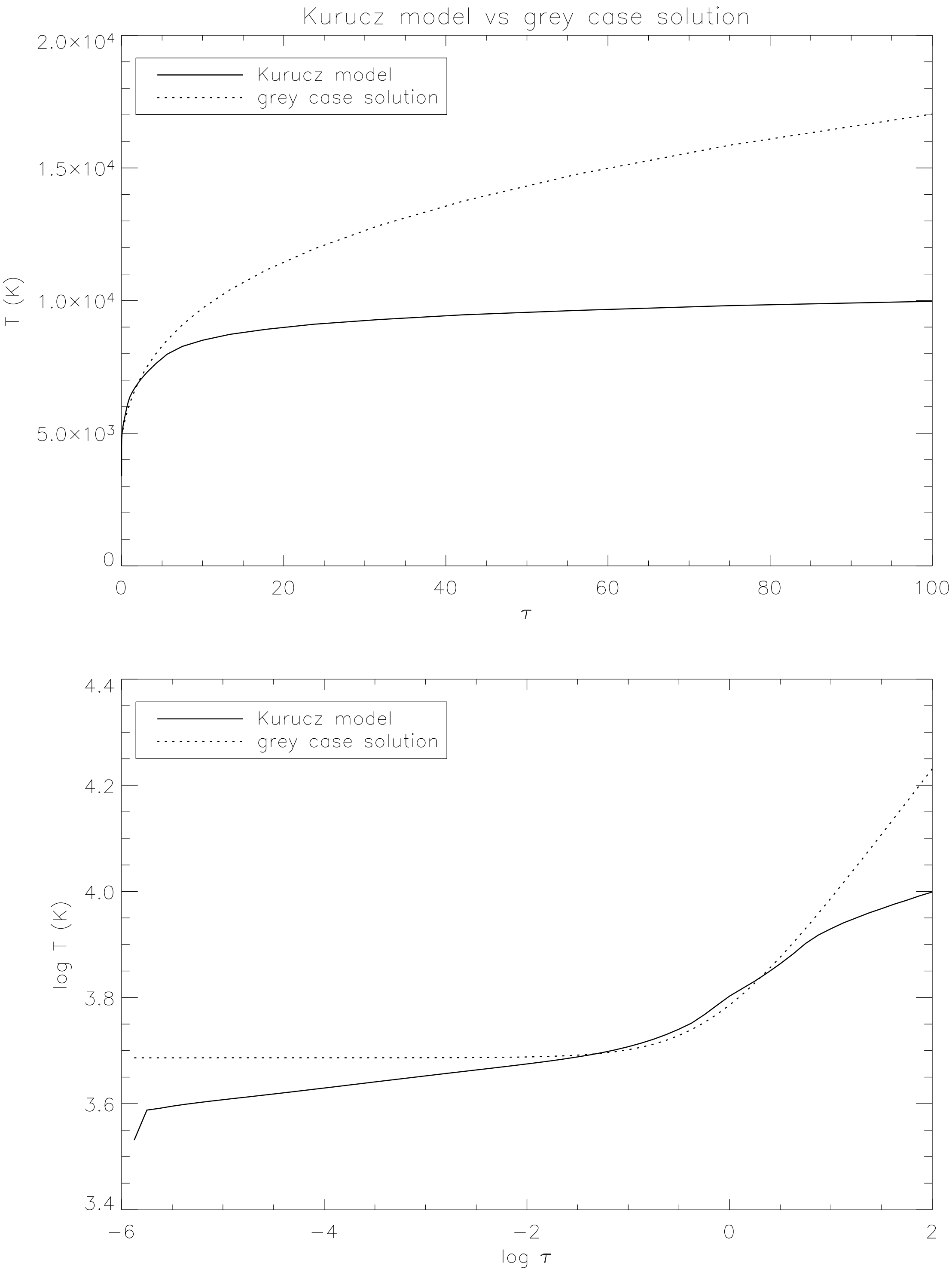


Fig. 1.— The Kurucz solar model compared to the analytic grey case solution, for all tabulated τ.

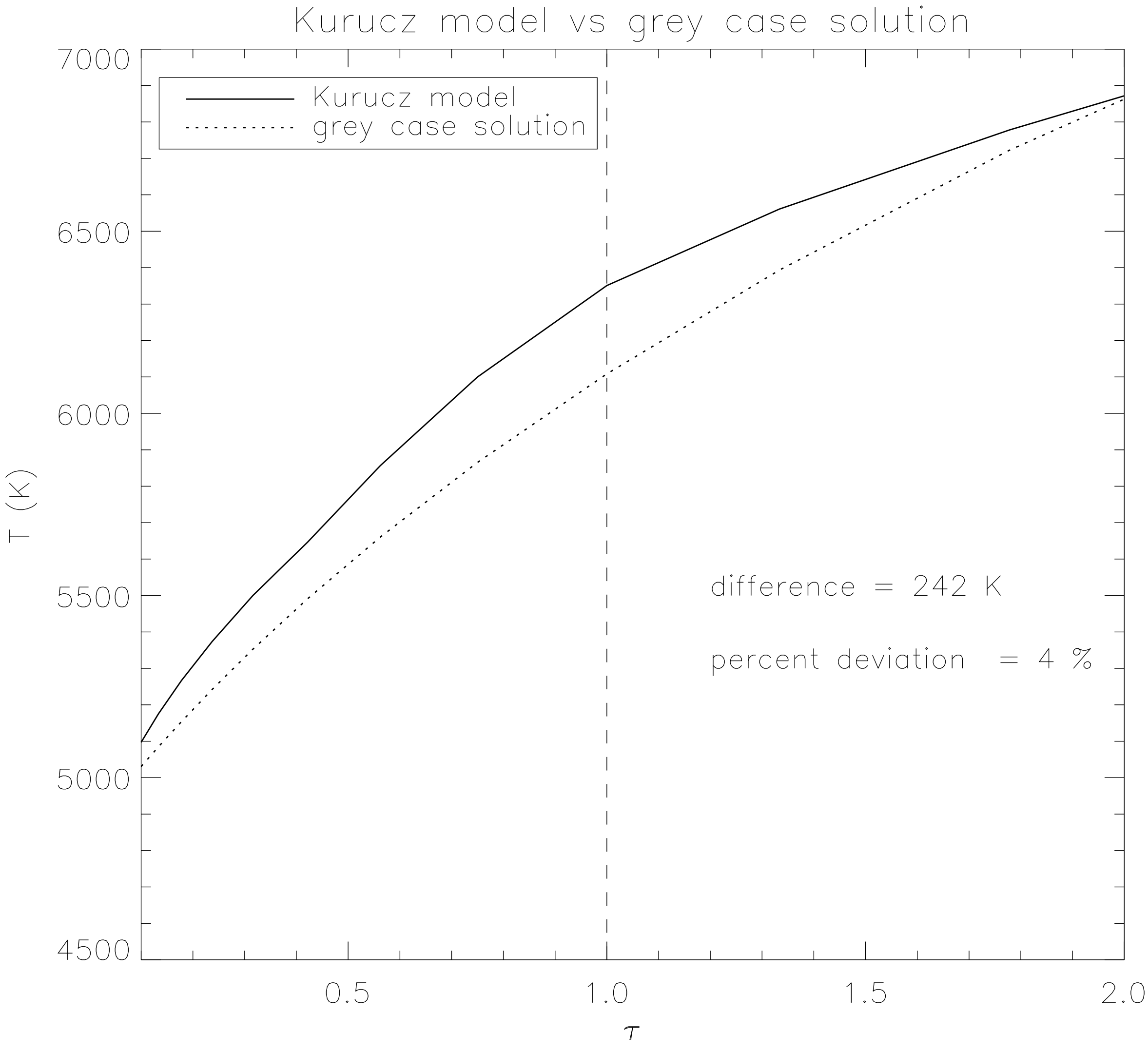


Fig. 2.— Same as Fig. 1 but for $0.1 < \tau < 0.2$ only

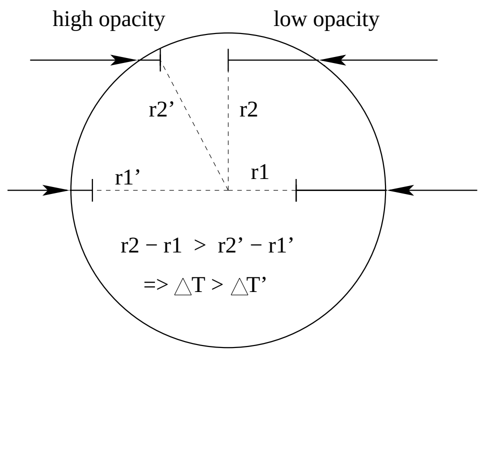
intensity (integrated over all wavelengths) so that $S(\tau) = B(T, \tau) = (\sigma/\pi)T^4(\tau)$. We need to numerically integrate this over all the tabulated τ, and then plot $I(\mu)/I(1)$ (normalizing by the intensity at the center) vs μ. I use IDL to integrate (int_tabulated), taking the last τ at 100 to be $\sim \infty$. This seems to work OK for $\tau \gtrsim 0.4$, but funny things happen below that.

For the Eddington approximation $S(\tau) = S_0 + S_1\tau$, we can do the integral Eq. 1 analytically to find $I(\mu) = S_0 + S_1\mu$, or the intensity is linear in μ. (NOTE – I numerically integrated this also to check that the answers are the same. They were close for $\tau > 0.4$, but the numerically integrated curve is very wrong for $\tau < 0.4$. Moral of the story is, I'm not at all sure about the integrated curve for the solar model.)

The limb darkening curves for the solar model and the Eddington approximation are compared in Fig. 3. Note that this figure also contains the curves for part (c). It is obvious that the Kurucz solar model predicts a much steeper dropoff in intensity toward the edge of the sun than does the Eddington approximation.

c) See Fig. 3. It seems that the observed curve from Allen is matched much more closely by the Eddington approximation than by the Kurucz model. But again, there could be a problem with my integration of the Kurucz model.

d) For opacities larger than the Rosseland mean, similar optical depths will be reached at smaller actual depths z from the surface. This means that there will be less of a difference between the distance from the center of the sun to the $\tau = 2/3$ point for $\theta = 0^o$ and $\theta \sim 90^o$. Thus the temperatures seen at these two points will be more similar than in the Rosseland mean opacity case, and the limb darkening effect will be dampened. So, *for wavelengths were the opacity is larger than the Rosseland mean, the limb darkening curve flattens.* Conversely, if the opacity is smaller than the Rosseland mean, $\tau = 2/3$ is reached at larger depths, and the limb darkening effect is enhanced. *For wavelengths were the opacity is smaller, the limb darkening curve steepens.* In other words, there is a greater contrast between $I(\mu \sim 0)$ and $I(1)$ at these wavelengths. See below (this is an exaggeration, obviously).



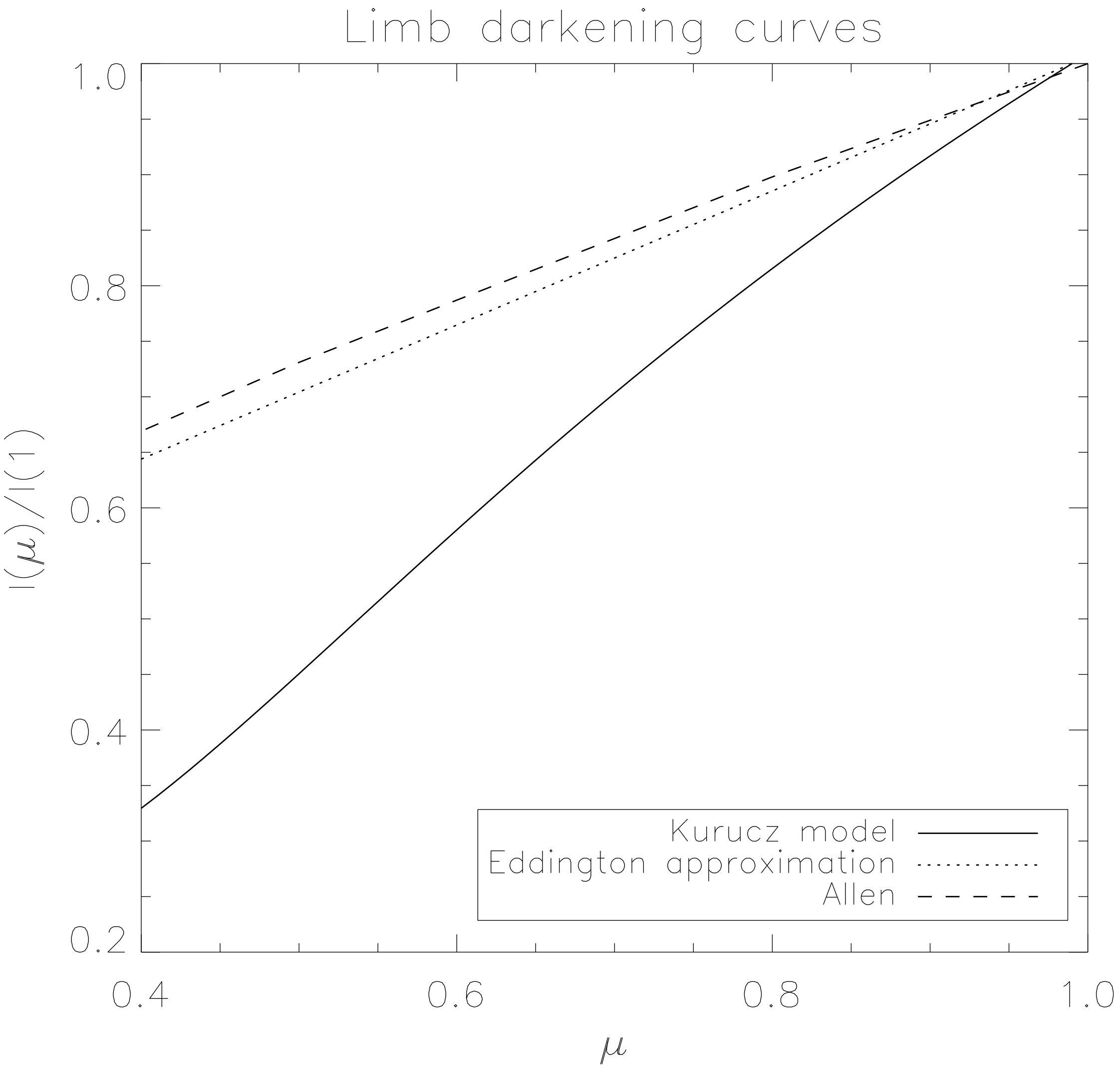


Fig. 3.— Limb darkening curves for the Kurucz model, Eddington approximation, and an observed curve from Allen, Table 14.16

Problem 2: Pressure-Temperature relation

Solution:

a) To solve this problem we will need the following three standard equations. The "energy equation" (see e.g. HK 1.55) which describes the rate of energy generation within the star:

$$\frac{dL_r}{dM_r} = \epsilon. \tag{2}$$

The equation describing how energy is transfered through the star (by diffusion) to the surface:

$$L_r = -\frac{4\pi r^2 c}{3\kappa\rho}\,\frac{d(aT^4)}{dr}. \tag{3}$$

And hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}. \tag{4}$$

We also know that the opacity κ is given by Kramers law:

$$\kappa = \kappa_0 \rho T^{-s},\;\; s = 3.5. \tag{5}$$

If we substitute the opacity law into Eq. 3 and do out the d/dr differentiation, we find

$$\frac{L_r}{4\pi r^2} = -\frac{4ac}{3\kappa_0\rho^2}T^{3+s}\frac{dT}{dr}. \tag{6}$$

Now, as we want a relation between T and P, we can invert Eq. 4 and multiply by the above equation to get

$$-\frac{r^2}{GM_r\rho}\frac{L_r}{4\pi r^2} = -\frac{4ac}{3\kappa_0\rho^2}T^{3+s}\frac{dT}{dr}\frac{dr}{dP} = -\frac{4ac}{3\kappa_0\rho^2}T^{3+s}\frac{dT}{dP} \tag{7}$$

or

$$\frac{dT}{dP} = \frac{L_r}{M_r}\frac{3\kappa_0\rho}{16\pi acG}T^{-(s+3)}. \tag{8}$$

To get rid of L_r/M_r we simply integrate Eq. 2 assuming ϵ is approximately constant, which gives $L_r/M_r = \epsilon$. Finally, we need eliminate ρ. We are told that radiation pressure is negligible, so we will take the pressure to be that of an ideal gas:

$$P = \frac{\rho kT}{\mu m_u} \Rightarrow \rho = \frac{\mu m_u P}{kT}. \tag{9}$$

Substituting these in we find

$$\frac{dT}{dP}T^{(s+4)} = \alpha P, \quad \alpha = \frac{3\mu m_u \kappa_0 \epsilon}{16\pi kacG}, \tag{10}$$

which can be integrated to yield $P(r) = f(T(r))$. Multiply by dP and integrate to get

$$P(r) = \left[\frac{2}{(s+5)\alpha}T(r)^{(s+5)} + C_1\right]^{1/2}, \tag{11}$$

where C_1 is determined by $P(R_*)$ and $T(R_*) \sim T_{eff}$.

b) Let's write out that constant of integration C_1:

$$C_1 \approx P(R_*)^2 - \frac{2}{(s+5)\alpha}T_{eff}^{(s+5)}. \tag{12}$$

We can generally assume that $P(R_*) \approx 0$, the pressure goes to zero at the surface. Now we compare T_{eff} to a temperature $T(r_1)$ deep within the star. In general $T(r_1) \gtrsim 5 \times 10^6$ and $T_{eff} \sim 6 \times 10^3$, so we have

$$\left(\frac{T(r_1)}{T_{eff}}\right)^{s+5} > (10^3)^5 = 10^{15} \tag{13}$$

for any $s \geq 0$. Therefore the surface boundary conditions are indeed unimportant relative to the relation in the deep interior, and we can just write

$$P(r) = \left[\frac{2}{(s+5)\alpha}T(r)^{(s+5)}\right]^{1/2}. \tag{14}$$

c) Suppose $s = 3$, in which case we have $P \propto T^4$. How does this compare to a polytrope of index γ, where $P \propto \rho^\gamma$? From our ideal gas equation of state, $T \propto P/\rho$, which means $P \propto (P/\rho)^4$ or $P \propto \rho^{4/3}$, which is exactly the relation for a $\gamma = 4/3$ polytrope. As desired.

Let's repeat for $s = 3.5$ now,

$$P \propto T^{4.25} \propto (P/\rho)^{4.25} \Rightarrow P \propto \rho^{1.31}. \tag{15}$$

As 1.31 is not far from 4/3, the relation for $s = 3.5$ is *nearly* the same as a $\gamma = 4/3$ polytrope.

Problem 3: More solar models

Solution:
a) The radiation energy density is given by

$$\epsilon_{rad} = aT^4, \tag{16}$$

while the gas energy density is that of an ideal gas with 3 degrees of freedom:

$$\epsilon_{gas} = \frac{3}{2}nkT = \frac{3\rho kT}{2\mu m_u}. \tag{17}$$

The ratio is

$$f \equiv \frac{\epsilon_{rad}}{\epsilon_{gas}} = \frac{2am_u}{3k}\left(\frac{T^3\mu}{\rho}\right) = 6.1\times10^{-23}\left(\frac{T^3\mu}{\rho}\right). \tag{18}$$

At the center of the sun ($r = 0$) the needed values are $\rho = 1.53\times10^2\frac{g}{cm^3}$, $T = 1.54\times10^7 K$ and $\mu = 0.829$. Plugging these in we find

$$f(r=0) = 1.2\times10^{-3}. \tag{19}$$

At $r = 0.5R_\odot$, we have $\rho = 1.26\frac{g}{cm^3}$, $T = 3.80\times10^6 K$ and $\mu = 0.605$. If necessary one can interpolate to find these. Thus,

$$f(r=0.5R_\odot) = 1.6\times10^{-3}. \tag{20}$$

So, in both regions the sun is predominantly supported by gas pressure.

b) I use Fig.1 of Rogers & Iglesias 1992, ApJ, 401, 361. Note that $R = (\frac{\rho}{\mathrm{gcm^{-3}}})(\frac{T}{10^6\mathrm{K}})^{-3}$. At the center of the sun $\rho = 153$ g cm^{-3} and $T = 1.54\times10^7$ K so $R = 0.042 \Rightarrow logR = -1.38$. Thus the $logR = -1$ is the closest curve. Looking at this curve and using $log(T_6) = 1.2$, eyeball extrapolation gives

$$log\kappa_R \approx 0.1 \;\Rightarrow\; \kappa_R \approx 1.3\ \mathrm{cm^2/g}\ \text{at}\ r=0 \tag{21}$$

The electron scattering opacity is $\kappa_{es} = 0.2(1+X)$ cm^2/g ≈ 0.34 cm^2/g. So, at the center of the sun, the Rosseland mean opacity is larger than the electron scattering opacity by about a factor of four.

At $r = 0.5R_\odot$, we have $logR = -1.64$, so using the $logR = -2$ curve and $log(T_6) = 0.58$ we find

$$log\kappa_R \approx 1 \;\Rightarrow\; \kappa_R \approx 10\ \mathrm{cm^2/g},\ \text{at}\ r=0.5R_\odot \tag{22}$$

Thus, at half the solar radius the Rosseland mean opacity is 30 times larger than the electron scattering opacity.

c) The mean free path is given by

$$l = \frac{1}{\kappa\rho} \tag{23}$$

Using the Rosseland mean $\kappa = \kappa_R$, the desired numbers are:

$$l \approx 5.0 \times 10^{-3}\ \mathrm{cm}, \quad r = 0 \tag{24}$$

$$l \approx 8.0 \times 10^{-2}\ \mathrm{cm}, \quad r = 0.5R_\odot \tag{25}$$

These numbers will, of course, vary depending on the opacities you estimated from the graph in part (b)

d) For a n step random walk, with a photon mean free path λ, the rms distance covered is $D = \langle r^2 \rangle^{1/2} = \lambda\, n^{1/2}$. So, if we want to diffuse (or random walk) a distance D, the number of steps it will take is $n \simeq D^2/\lambda^2$, which corresponds to a time of

$$t = \frac{n\lambda}{c} = \frac{D^2}{\lambda c}. \tag{26}$$

Now, there are two ways to go about this problem:

Way 1. CHEAP & EASY

In this method we simply take λ to be an average mean free path, and take it to be the mfp at half the solar radius, $\lambda \sim l(r = 0.5R_\odot) \sim 8.0 \times 10^{-2}$ cm. The distance is just $D = R_\odot$, and the diffusion time is easily calculated.

$$t_{diff} \sim \frac{R_\odot^2}{c\ l(r = 0.5R_\odot)} \sim 6.4 \times 10^4\ \mathrm{yr} \tag{27}$$

I think this is a perfectly fine way of estimating this. BUT, for those of you who love punishment.....

Way 2. NOT SO CHEAP, NOT SO EASY

We know that λ changes rapidly with radius, so the above method is not terribly valid. A better estimate of the diffusion times at each radius can be found by tabulating λ at each of $r = 0, 0.1, 0.2, 0.3, 0.4R_\odot$: $\lambda = 5, 8, 14, 30, 50 \times 10^{-3}$ cm, and then estimating the time to diffuse to the next shell out:

$$t_n = \frac{(r_{n+1}^2 - r_n^2)}{\lambda_{n+1} c}. \tag{28}$$

This yields $t_n = 2, 3, 3, 2, 2 \times 10^{11}$ s. Thus, we estimate the diffusion time from $r = 0$ to $r = 0.5R_\odot$ by summing these up: $t(0 \to 0.5R_\odot) \simeq 10^{12}$ s. Finally, we can estimate the diffusion time from $0.5R_\odot$ to $R_\odot$ using the cheap & easy method from above, which give a time ($0.5R_\odot \rightarrow R_\odot$) of the same order of magnitude. Adding the two together gives

$$t_{diff} \simeq 1 - 2 \times 10^{12} \text{ s}, \quad \text{or} \quad 3 - 6 \times 10^4 \text{ yr}. \tag{29}$$

So, not much different from above.