Solution: HW 5
AY 123, Spring 2003

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Problem 1:

Solution:

We are looking at the reaction

$$^{7}Be + p \ \rightarrow \ ^{8}B + \gamma, \tag{1}$$

which begins the PP-III chain. The reaction rate for this reaction is

$$r_{^7Be\gamma} = \frac{\rho^2}{m_u^2}\frac{X_{^7Be}X_p}{A_{^7Be}A_p} < \sigma v >_{^7Be\gamma} \mathrm{cm}^{-3}\ \mathrm{s}^{-1} \tag{2}$$

The temperature dependence comes in from the $< \sigma v >$:

$$r \ \propto \ < \sigma v >_{^7Be\gamma} \ \propto \ \frac{e^{aT_6^{-1/3}}}{T_6^{2/3}}, \tag{3}$$

with

$$a = 42.49(Z_{^7Be}^2 Z_p^2 \mu)^{1/3} \ \text{ and } \ \mu = \frac{A_{^7Be}A_p}{A_{^7Be} + A_p}. \tag{4}$$

We write the rate as

$$r \propto T^{\nu} \tag{5}$$

and thus we can solve for ν:

$$\nu = \left(\frac{\partial \ln r}{\partial \ln T}\right) = \frac{a}{3T_6^{1/3}} - \frac{2}{3} \tag{6}$$

We have the following values: $A_{^7Be} = 7$, $A_p = 1$, $Z_{^7Be} = 4$ and $Z_p = 1$. We estimate μ as $\mu = 7/8 \approx 1$. Thus at a temperature of $T = 1.5 \times 10^7 = 15 \times 10^6$K, we have:

$$\nu \approx \frac{42.49(16)^{1/3}}{3(15)^{1/3}} - \frac{2}{3} = 13.8 \approx 14 \tag{7}$$

So we indeed find that the reaction rate is approximately proportional to T^{14}.

How much would the centeral temperature of the sun need to change to solve the solar nuetrino problem? The PP-III chain continues as follows:

$$^{8}B \ \rightarrow \ ^{8}Be + e^{+} + \nu_e \tag{8}$$

creating an electron nuetrino. Assuming the rate of the chain is dominated by the first reaction (^8B has a half life of 0.8 s), the rate of production of nuetrinos from this chain is $\propto T^{14}$. Now, Davis' experiment can only detect nuetrinos from the PP-II and PP-III chains, and the majority are expected to come from this B inverse beta decay in PP-III. So let's assume that all the neutrinos detected are from the above reaction. The observed nuetrino flux (from HK Section 8.3) is 2.07 ± 0.3 SNU, while the predicted flux is 7.9 ± 2.4 SNU. So to explain the discrepency we will need to lower the predicted number of nuetrinos produced by approximately 4:

$$r_2 \approx \frac{1}{4} r_1 \quad \Rightarrow \quad T_2^{14} \approx \frac{1}{4} T_1^{14} \tag{9}$$

$$\Rightarrow T_2 \approx 0.9 T_1 \tag{10}$$

So we would have to lower the central temperature of the sun by $\sim 10\%$, or $\sim 1.5 \times 10^6$K.

Problem 4:

Solution:

a) We will first expand $y(x_n + h)$ in a Taylor series, then compare this to the value y_{n+1} predicted by the integration method. The term at which the two disagree will give us the order of the error.

$$y(x_n + h) = y(x_n) + h \cdot y'(x_n) + \frac{h^2}{2} \cdot y''(x_n) + \frac{h^3}{3!} \cdot y'''(x_n) + \cdots \tag{11}$$

Now, the Euler scheme estimates y_{n+1} as

$$y_{n+1} = y_n + h \cdot f(x_n) = y_n + h \cdot y'(x_n) = y(x_n) + h \cdot y'(x_n) \tag{12}$$

The difference between these is

$$y(x_n + h) - y_{n+1} = \frac{h^2}{2} \cdot y''(x_n) + \cdots = O(h^2) \tag{13}$$

so the error in the method is clearly of order h^2.

b) Second order Runge-Kutta works as follows:

$$k_1 = h \cdot f(x_n, y_n) \tag{14}$$

$$k_2 = h \cdot f(x_n + \frac{1}{2} h, y_n + \frac{1}{2} k_1) \tag{15}$$

$$y_{n+1} = y_n + k_2 \tag{16}$$

$$x_{n+1} = x_n + h \tag{17}$$

where h is the stepsize (presumed small, $\ll 1$). For our purposes f is a function of x only so $f(x_n, y_n) = f(x_n) = y'(x_n)$. Following the same procedure as above,

$$y_{n+1} = y_n + h \cdot y'\left(x_n + \frac{h}{2}\right) \tag{18}$$

We expand $y'(x_n + \frac{h}{2})$ in a Taylor series:

$$y'\left(x_n + \frac{h}{2}\right) = y'(x_n) + \frac{h}{2} \cdot y''(x_n) + \frac{1}{2!}\left(\frac{h}{2}\right)^2 \cdot y'''(x_n) + \cdots \tag{19}$$

So, we have,

$$y_{n+1} = y(x_n) + \frac{h^2}{2} \cdot y''(x_n) + \frac{h^3}{8} \cdot y'''(x_n) \tag{20}$$

Again taking the difference

$$y(x_n + h) - y_{n+1} = \frac{h^3}{8} \cdot y'''(x_n) + \cdots = O(h^3) \tag{21}$$

The error is of order h^3.

c) Integrate the equation

$$\frac{dy}{dx} = x^2 + sin(x) \tag{22}$$

where x runs from $0 \to 1$ radians.

$$y(0 < x < 1) = \int [x^2 \ dx + sin(x)] \ dx = \frac{x^3}{3} - cos(x) + C \tag{23}$$

I will assume here that $C = 0$, which means that $y(0) = -1$ (we will use this in part (d)). The plot follows together with part (d).

d) We will use the second order Runge-Kutta scheme outlined in part (b). Here the stepsize must be $h = 1/40$ if we are to have 40 steps in the interval $0 \to 1$. The function f is

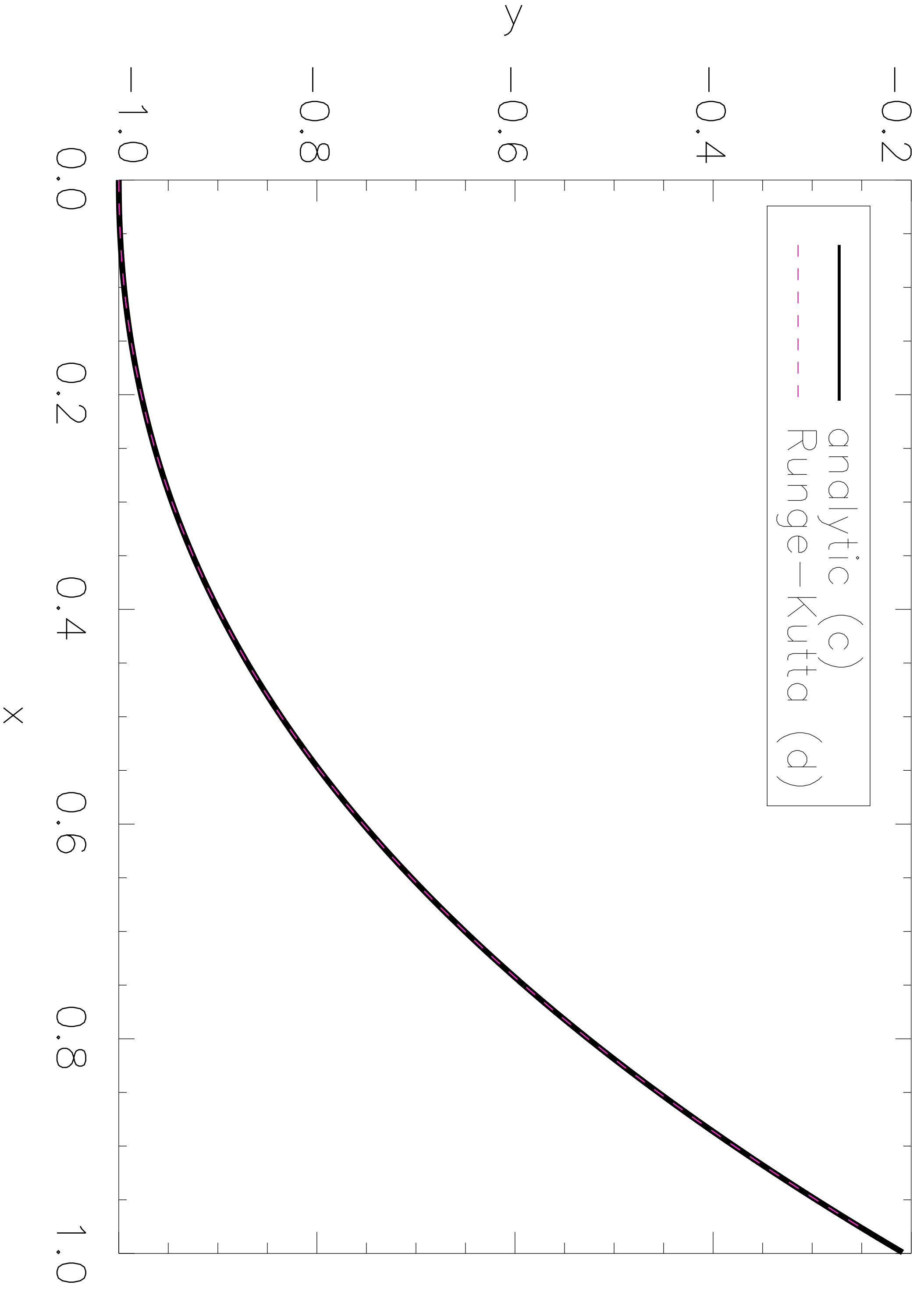
$$f(x_n) = y'(x_n) = \frac{dy}{dx} = x_n^2 + sin(x_n) \tag{24}$$

We begin with the initial condition $y(0) = -1$ from part (c), and step through x to get $y(x)$. The plot of the calculated y vs x follows. Comparing to the analytic solution, the largest absolute error in y is

$$\Delta y = 3.96 \times 10^{-5}, \tag{25}$$

which correspond to a fractional error

$$\frac{\Delta y}{y} = 0.016\%. \tag{26}$$


analytic (c)
Runge–Kutta (d)
x
y
0.0
0.2
0.4
0.6
0.8
1.0
−1.0
−0.8
−0.6
−0.4
−0.2