Solution: HW 6
AY 123, Spring 2003

Problem 1.

Solution (or, rather, rough guide to the solution):

a) COLD WHITE DWARF

This is a polytrope problem, much like that of HW 2, Problem 3 (2.3). The equation of state for a relativistic, completely degenerate electron is:

$$P_e = 1.24 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3} \Rightarrow P \propto \rho^{4/3} \tag{1}$$

which means that $(n+1)/n = 4/3 \Rightarrow n = 3$, just as in our previous problem. As an added bonus we now know the constant of proportionality from Eq. 17 of Problem 2.3: $K = 1.24 \times 10^{15} \mu_e^{-4/3}$ (cgs). The plot of ρ/ρ_c vs r/R are in the solution to Problem 2.3. To get $\rho(r)$ one must calculate ρ_c:

$$\rho_c = \frac{1}{3} \frac{M}{(4/3\pi R^3)} \left(\frac{\xi_1}{-\theta'(\xi_1)}\right), \tag{2}$$

which is where the solar mass part comes in. From $\rho(r)$, $P(r)$ follows trivially. The mass $M(r)$ can be calculated similarly to Problem 2.2, Eq. 24, where we do not plug in ξ_1, but leave the variable ξ. We are looking at a "cold" white dwarf, and have assumed a completely degenerate gas, which means that the temperature is $T = 0$ independent of radius. For more details see section 7.2 of HK.

b) FULLY CONVECTIVE $0.2M_\odot$ ZAMS STAR

For this problem we have the following four differential equations we must simultaneously solve using a runge-kutta method, or something similar:

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \tag{3}$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \tag{4}$$

$$\frac{dL}{dM} = \epsilon = \epsilon_{pp} \tag{5}$$

$$\frac{dT}{dM} = -\frac{GMT}{4\pi r^4 P}\nabla \tag{6}$$

Here, $M = M_r$, $L = L_r$, etc, and I will use M_* to denote the total mass $M_* = 0.2M_\odot$. We can assume an ideal gas, which lets us eliminate ρ in the first equation:

$$\rho = \frac{\mu m_u P}{kT} \Rightarrow \frac{dr}{dM} = \frac{kT}{4\pi \mu m_u r^2 P} \tag{7}$$

We are also told to use PP energy generation, so that

$$\epsilon_{pp} = \frac{2.4 \times 10^4 \rho X^2}{T_9^{2/3}} e^{-3.38T_9^{-1/3}} = 2.4 \times 10^{13} \frac{kT_9^{1/3} X^2}{\mu m_u P} e^{-3.38T_9^{-1/3}} \quad \frac{\text{erg}}{\text{g} \cdot \text{s}} \tag{8}$$

We know that the star is fully convective; if we assume that the convection is effecient, so that essentially all of the heat is carried by convection, as well as adiabatic, then we can use the adiabatic temperature gradient of an ideal gas:

$$\nabla = \nabla_{ad} = 0.4 \ \ \text{(ideal gas)} \tag{9}$$

So, we now have four differential equations for r, P, T and L in terms of M. In order to integrate these using a numerical integration routine, we will need boundary conditions. The problem is that we don't have boundary conditions for all the variables at one point. We know $r = 0, L = 0$ at $M = 0$, but we do not know $P_c = P(M = 0)$ or $T_c = T(M = 0)$. On the other hand, we do know P and T at the surface, from photospheric boundary conditions at $\tau = 2/3$:

$$T(\tau = 2/3) = T_{eff}, \quad P(\tau = 2/3) \approx \frac{2}{3}\frac{g_s}{\kappa_p} \approx \frac{2}{3}\frac{GM_*}{R^2 \kappa_p} \tag{10}$$

where κ_p is the opacity at the photosphere ($\tau = 2/3$). See HK section 7.3.2 for details.

The solution to our problem is to do two integrations: one from the center out (guessing at the boundary conditions we don't know) and another from the surface in (again guessing the unknown values). We try to meet at some intermediate radius (or mass), and refine our guessing until the two integrations meet. A more rigorous method is to use a difference matrix to refine our initial guesses, as described in HK 7.2.4.

Note that we can't start at $M = 0$ because this will cause singularities in the differential equations (at $r(M = 0) = 0$), therefore we pick some small M and calculate the corresponding r, L for small M:

$$r(M \ll M_*) \approx \left(\frac{3}{4\pi} \frac{M}{\rho_c} \right)^{1/3} \tag{11}$$

$$L(M \ll M_*) \approx \frac{4\pi}{3} \rho_c \epsilon_c r^3 \tag{12}$$

However, these clearly require knowing ρ_c and ϵ_c, which we don't. A simple way around this dilema is to pick starting values from a similar model, e.g. from Allen. Some good initial boundary conditions to start with are:

CENTER:

- $M_c = 1.2 \times 10^{29}$ g
- $r_c = 4.87 \times 10^8$ cm
- $L_c = 1.1 \times 10^{28}$ erg/s

$P_c = 4.18 \times 10^{17}$ dyne/cm^2

$T_c = 1.67 \times 10^7$K

SURFACE:

- $M_s = 0.2M_\odot$

$R = r_s = 0.17R_\odot$

$L_s = 6.83 \times 10^{33}$ erg/s

- $P_s = 4.5 \times 10^{10}$ dyne/cm^2
- $T_{eff} = T_s = 2826K$

Bulleted boundary conditions indicatee fixed, predetermined values. The other boundary conditions are to be varied in the following way:

1) Choose initial boundary conditions and an appropriate mass step size.
2) Using Runge-Kutta or some other method, numerically integrate Eqs 3-6 from $M \approx 0$ to $M = M_*$, with the chosen boundary conditions.
3) Repeat Step 2, but integrate from $M = M_*$ to $M \approx 0$.
4) Compare the results of Steps 2 & 3 at some intermediate mass $M(r \approx R/2)$.
5) Vary the boundary conditions by some small amount, repeat, and calculate the corrections to the boundary conditions.
6) Iterate until the corrections are small.

Finally, we need to check that the star is indeed fully convective. To do this we will calculate ∇_{rad} and compare it to $\nabla_{ad} = 0.4$. For convection, we should find $\nabla_{rad} > \nabla_{ad}$ everywhere.

$$\nabla_{rad} = \frac{3}{16\pi acG} \frac{P\kappa}{T^4} \frac{L}{M} \tag{13}$$

The only value here that we do not now know as a function of radius is κ. Technically we should check which opacity (bound-free, free-free, electron scattering, etc) is dominant at each density and temperature. As an approximation, it should be ok to assume that bound-free dominates throughout most of the star. This has a Kramer's form:

$$\kappa_{bf} \approx 4 \times 10^{25} Z(1+X)\rho T^{-3.5} \text{ cm}^2/\text{g} \tag{14}$$

On the ZAMS typical values are $X \approx 0.7$, $Z \approx 0.03$. At temperatures above 10^7K we could also add in the constant electron scattering term:

$$\kappa \approx 4 \times 10^{25} Z(1+X)\rho T^{-3.5} + 0.2(1+X) \text{ cm}^2/\text{g} \tag{15}$$

Plugging all this in, one should find $\nabla_{rad} > \nabla_{ad}$ everywhere (although there may be problems near the surface where these opacities don't really hold).