

Ay 20: Basic Astronomy and the Galaxy Fall Term 2010

Solution Set 2

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(based on solutions by Swarnima Manohar, TA 2009)

Reporting an answer to unnecessary number of decimal places should be avoided. Notice here that I have, while evaluating most of the answers, used cgs units, which is popular among professional astronomers. SI system is considered universal, and so I will encourage you to stick to SI if you are presently using this system most frequently.

PROBLEM 1 (C&O Problem 3.9)

(a) The blackbody luminosity is given by eqn 3.17 (everything is in cgs units):

$$L \rightarrow 4 \pi R^2 \sigma T_e^4 \text{ erg} \cdot \text{s}^{-1} / . \{R \rightarrow 5.16 \times 10^{11}, \sigma \rightarrow 5.67 \times 10^{-5}, T_e \rightarrow 28000\}$$

$$L \rightarrow 1.16607 \times 10^{38} \text{ erg} \cdot \text{s}^{-1} = 1.2 \times 10^{38} \text{ erg} \cdot \text{s}^{-1}$$

-OR-

$$L = 1.2 \times 10^{31} \text{ W}$$

In solar units:

$$L \rightarrow \frac{1.16607 \times 10^{38} \text{ erg} \cdot \text{s}^{-1}}{L_{\odot}} / . \{L_{\odot} \rightarrow 3.84 \times 10^{38} \text{ erg} \cdot \text{s}^{-1}\}$$

$$\therefore L \rightarrow 3.0 \times 10^4 L_{\odot}$$

(b) Absolute magnitude is defined as:

$$M = -2.5 \text{ Log}_{10} \left[\frac{L}{4 \pi (10 \text{ pc})^2} \right] + \text{constant}$$

Using equation 3.8 and the absolute magnitude of Sun, $M_{\text{sun}} = 4.76$:

$$M = M_{\text{sun}} - 2.5 \text{ Log}_{10} \left[\frac{L}{L_{\odot}} \right]$$

$$\therefore M \rightarrow -6.4$$

(c) Apparent magnitude is given by eqn 3.6:

$$m - M = 5 \text{ Log}_{10} \left[\frac{d}{10 \text{ pc}} \right]$$

$$m \rightarrow M + 5 \text{ Log} \left[10, \frac{d}{10} \right] / . \{M \rightarrow -6.45, d \rightarrow 123\}$$

$$m \rightarrow -1.0$$

(d) Distance modulus:

$$m - M = 5 \text{ Log}_{10} \left[\frac{d}{10 \text{ pc}} \right] = 5 \text{ Log}_{10} \left[\frac{123 \text{ pc}}{10 \text{ pc}} \right] = 5.45$$

(e) Because of the Stefan Boltzmann law, we know that $F = \sigma T_e^4$ at the surface of the star, therefore the flux is:

$$F \rightarrow \sigma T_e^4 \text{ erg s}^{-1} \text{ cm}^{-2} / \{ \sigma \rightarrow 5.67 \times 10^{-5}, T_e \rightarrow 28\,000 \}$$

$$F \rightarrow 3.5 \times 10^{13} \text{ erg s}^{-1} \text{ cm}^{-2}$$

Thus, the flux is $3.5 \times 10^{13} \text{ erg s}^{-1} \text{ cm}^{-2}$.

(f) Radiant flux at the Earth's Surface in $\text{erg s}^{-1} \text{ cm}^{-2}$:

$$F \rightarrow \frac{L}{4\pi d^2} / \{ L \rightarrow 1.16607 \times 10^{38}, d \rightarrow 123 \times 3.084 \times 10^{18} \}$$

$$F \rightarrow 0.0000644875$$

This is $6.45 \times 10^{-5} \text{ erg s}^{-1} \text{ cm}^{-2}$. Compare this to the solar constant, $1.36 \times 10^6 \text{ erg s}^{-1} \text{ cm}^{-2}$. The star's flux is only 2.22×10^{-11} of the solar constant.

(g) Wein's law (eqn 3.15) gives:

$$\lambda_{\max} \rightarrow \frac{2.9 \text{ mm}}{T} / \{ T \rightarrow 28\,000 \text{ K} \}$$

$$\lambda_{\max} \rightarrow 1.03571 \times 10^{-7} \text{ m}$$

or about 1036 \AA .

PROBLEM 2 (C&O Problem 5.1):

(a) The $H\alpha$ absorption line is observed at a wavelength of $\lambda_0 = 656.281 \text{ nm}$ (rest wavelength). The radiation from the star is seen at a wavelength of $\lambda = 656.034 \text{ nm}$ in air. The radial velocity of the star is

$$v_r \rightarrow c \frac{\lambda - \lambda_0}{\lambda_0} / \{ c \rightarrow 2.9979 \times 10^{10} \text{ cm/s}, \lambda_0 \rightarrow 656.281 \times 10^{-7} \text{ cm}, \lambda \rightarrow 656.034 \times 10^{-7} \text{ cm} \}$$

$$v_r \rightarrow - \frac{1.1283 \times 10^7 \text{ cm}}{\text{s}}$$

Thus, the radial velocity of Barnard's star is about -113 km/s .

Why the negative sign? As per convention, a negative value of V_r implies that Barnard's star is approaching earth. Note that the velocity of earth in its orbit around the sun is $30 \text{ km} \cdot \text{s}^{-1}$, and cannot be neglected. This is why the velocities of stars are usually reported in the barycentric frame. The refractive index of air, $n = 1.000297$ only affects the speed of light by a slight value, and can be neglected in this case as all measurements are made in air.

(b) The parallax of Barnard's star is $p = 0.549''$ giving a distance $d = 1.82 \text{ pc}$. Then, we use the proper motion μ to calculate linear velocity:

$$v_t \rightarrow d \mu / \{ d \rightarrow 1.82 \times 3.086 \times 10^{18} \text{ cm}, \mu \rightarrow \frac{10.3577}{206\,265 \times 24 \times 3600 \times 365.25} \text{ rad} \cdot \text{s}^{-1} \}$$

$$v_t \rightarrow \frac{8.93719 \times 10^6 \text{ cm}}{\text{s}}$$

which is 89.4 km/s .

(c) The total speed is simply obtained by vector addition of the two perpendicular components of velocity. Thus,

$$v \rightarrow \sqrt{v_x^2 + v_t^2} / . \{v_t \rightarrow 89.4, v_x \rightarrow -113\}$$

$$v \rightarrow 144.088$$

Space velocity is 144 km/s.

PROBLEM 3 (C&O Problem 8.5):

This is a straightforward application of the Boltzmann Equation. For having only 1% atoms in the first excited state,

$$\frac{P[E_b]}{P[E_a]} = \frac{g_b}{g_a} e^{-(E_a - E_b)/kT}$$

$$\text{solve} \left[10^{-2} == \frac{2 \times 2^2}{2 \times 1^2} e^{-\left(\frac{-13.6}{2^2} - \frac{-13.6}{1^2}\right) 1.602 \times 10^{-12}} / (1.381 \times 10^{-16} T)}, T \right]$$

$$\{\{T \rightarrow 19748.59\}\}$$

Hence we get 2.0×10^4 K. For having 10% atoms in the excited state, we get

$$\text{solve} \left[10^{-1} == \frac{2 \times 2^2}{2 \times 1^2} e^{-\left(\frac{-13.6}{2^2} - \frac{-13.6}{1^2}\right) 1.602 \times 10^{-12}} / (1.381 \times 10^{-16} T)}, T \right]$$

$$\{\{T \rightarrow 32075.58\}\}$$

Thus, a temperature of 3.2×10^4 K is needed.

PROBLEM 4 (C&O Problem 8.9):

We are given: $n_e V = N_{II}$

$$N_I = \rho V (m_p + m_e) \approx \rho V / m_p$$

$$\rho = 10^{-6} \text{ kg/m}^{-3} = 10^{-9} \text{ g/cm}^{-3}$$

(a) Starting from Eqn 8.8:

$$\frac{N_{i+1}}{N_i} = \frac{2 Z_{i+1}}{n_e Z_i} \left(\frac{2 \pi m_e k T}{h^2} \right)^{3/2} e^{-\chi_i/kT}$$

$$\frac{N_{II}}{N_t - N_{II}} = \frac{2 \times 1}{n_e 2} \left(\frac{2 \pi m_e k T}{h^2} \right)^{3/2} e^{-\chi_I/kT}$$

But,

$$n_e = \frac{N_{II}}{V} = \frac{N_{II} \rho}{m_p N_t}$$

So we have:

$$\left(\frac{1}{N_t / N_{II} - 1}\right) = \frac{m_p N_t}{N_{II} \rho} \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T}$$

$$1 = \left(\frac{N_t}{N_{II}} - 1\right) \frac{N_t}{N_{II}} \left(\frac{m_p}{\rho}\right) \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T}$$

$$\left(\frac{N_t}{N_{II}}\right)^2 \left(\frac{m_p}{\rho}\right) \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T} - \frac{N_t}{N_{II}} \left(\frac{m_p}{\rho}\right) \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T} = 1$$

$$\therefore \left[\left(\frac{N_{II}}{N_t}\right)^2 + \frac{N_{II}}{N_t} \left(\frac{m_p}{\rho}\right) \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T} - \left(\frac{m_p}{\rho}\right) \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T}\right] = 0$$

(b) Let $x = N_{II} / N_t$.

$$x^2 + \left(\frac{m_p}{\rho}\right) \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T} x - \left(\frac{m_p}{\rho}\right) \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T} = 0$$

$$a = 1$$

$$b = \left(\frac{m_p}{\rho}\right) \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T}$$

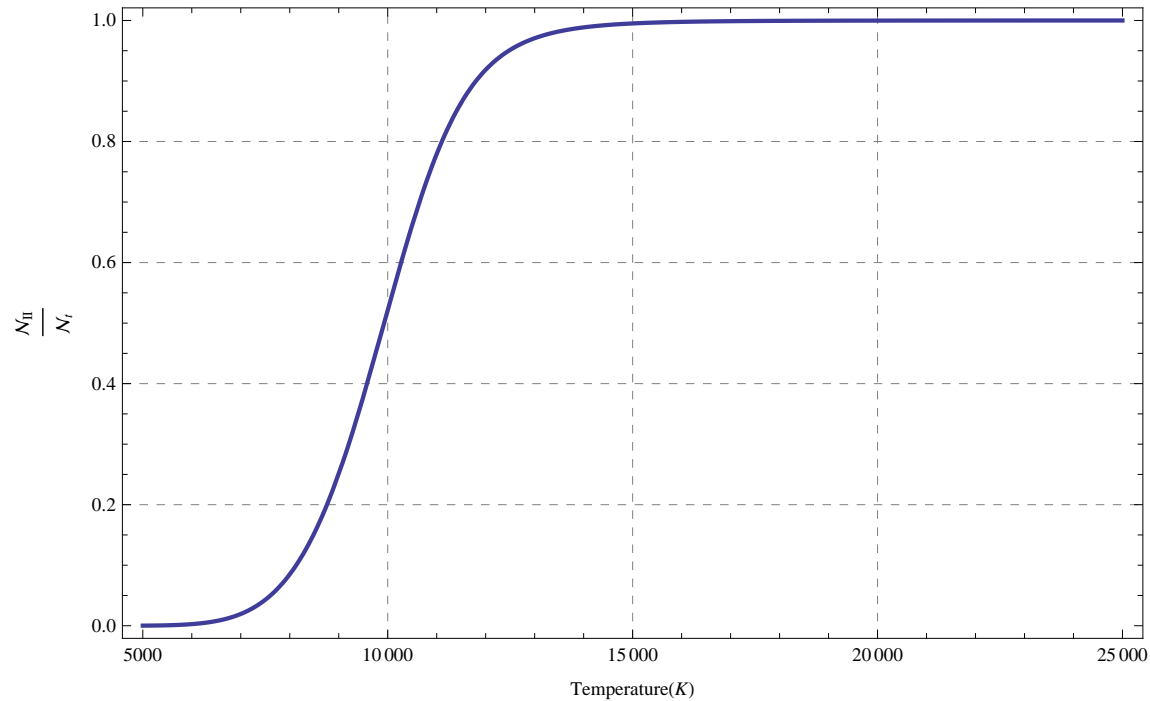
$$c = -b$$

Since $x > 0$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{b^2 + 4b}}{2}$$

$$x[T_] := \frac{-b + \sqrt{b^2 + 4b}}{2} /. \left\{ b \rightarrow \left(\frac{m_p}{\rho} \right) \left(\frac{2 \pi m_e k T}{h^2} \right)^{3/2} e^{-\frac{\chi_1}{kT}} \right\}$$

Plot[x[T] /. {mp → 1.673 × 10⁻²⁴, ρ → 10⁻⁹, me → 9.109 × 10⁻²⁸,
 k → 1.381 × 10⁻¹⁶, h → 6.626 × 10⁻²⁷, χ₁ → 13.6 × 1.602 × 10⁻¹²}, {T, 5000, 25000},
 PlotStyle → Directive[Thick], FrameLabel → {Temperature[K], N_{II} / N_I},
 Frame → True, GridLines → Automatic, GridLinesStyle → Directive[{Dashed}]]



It matches well with Fig 8.8.

PROBLEM 5: Stellar Photosphere

(a) For pure hydrogen, $g_n = 2n^2$. Using Boltzmann Eqn:

$$\frac{P[E_b]}{P[E_a]} = \frac{g_b}{g_a} e^{-(E_a - E_b)/kT}$$

$$\text{Solve}\left[1 = \frac{2 \times 2^2}{2 \times 1^2} e^{-\left(\frac{-13.6}{2^2} - \frac{-13.6}{1^2}\right) 1.602 \times 10^{-12}} / (1.381 \times 10^{-16} T)}, T\right]$$

$$\{\{T \rightarrow 85352.\}\}$$

This happens at an extremely high temperature of 8.5×10^4 K.

(b) $P_e = 200$ dyne/cm². Saha's equation (eqn 8.9):

$$\frac{N_{i+1}}{N_i} = \frac{[H_{II}]}{[H_I]} = \frac{2 k T Z_{i+1}}{P_e Z_i} \left(\frac{2 \pi m_e k T}{h^2} \right)^{3/2} e^{-\chi_i/kT}$$

$$\text{NSolve}\left[1 == \frac{2 k T}{P_e} \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\frac{\chi_1}{k T}} / .\right. \\ \left.\{m_e \rightarrow 9.109 \times 10^{-28}, k \rightarrow 1.381 \times 10^{-16}, h \rightarrow 6.626 \times 10^{-27}, \chi_1 \rightarrow 13.6 \times 1.602 \times 10^{-12}, P_e \rightarrow 200\}, T\right] \\ \{T \rightarrow 9552.62\}$$

Ionized and neutral hydrogen fractions are equal at about 10^4K .

PROBLEM 6: P60

(a) The central wavelengths for U , B , V filters are approximately $(\lambda_U, \lambda_B, \lambda_V) = (365, 440, 550)$ nm. The theoretical diffraction limit is $\theta_{\text{lim}} = 1.22 \lambda/D$ radians. This is equal to $0.168''(\lambda/1 \mu\text{m})$ for P60 with $D=1.5\text{m}$. Thus, the required answer is $\theta_{\text{lim}}(U, B, V) = (0.061'', 0.074'', 0.092'')$.

(b) Focal ratio is $f/8.75$, which means that $f = 8.75 \times D = 13.1 \text{ m}$

(c) Plate scale = $(1 \text{ rad})/f = (0.0763 \text{ rad/m}) = 0.262 \text{ arcmin/mm}$

(d) Part (c) $\Rightarrow 1 \text{ mm}$ on the detector $\equiv 0.262'$ on the sky.

$\therefore 24 \text{ mm}$ on the detector $\equiv 24 \times 0.262' = 6.30'$ on the sky.

(e) From Appendix G of C & O, an A0 star has $R = 2.2 R_{\odot}$ and $T_e = 9800 \text{ K}$. Therefore,

$$L \rightarrow \frac{4 \pi R^2 \sigma T_e^4}{L_{\odot}} / . \\ \{R \rightarrow 2.2 (6.96 \times 10^8 \text{ m}), \sigma \rightarrow 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ Kelvin}^{-4}, T_e \rightarrow 9800 \text{ Kelvin}, L_{\odot} \rightarrow 3.84 \times 10^{31} \text{ W}\}$$

$\therefore L \rightarrow 40.1 L_{\odot}$

The expected value from Appendix A is $39.4 L_{\odot}$, and the small difference between these two values is due to the fact that stars are not blackbodies.

(f) The flux at the Earth's surface is then,

$$F \rightarrow \frac{L}{4 \pi d^2} / . \{L \rightarrow 40.1 \times (3.84 \times 10^{33} \text{ erg s}^{-1}), d \rightarrow 500 \text{ pc} \times (3.084 \times 10^{18} \text{ cm / pc})\} \\ \therefore F \rightarrow 5.15 \times 10^{-9} \text{ erg s}^{-1} \text{ cm}^{-2}$$

(g) Assuming (falsely) that there is no atmospheric attenuation and that photons from all parts of the electromagnetic spectrum fall on the primary mirror of P60, the energy collected by the mirror per unit time will be,

$$E / t = F \times \pi D^2 / 4 = (5.15 \times 10^{-9} \text{ erg s}^{-1} \text{ cm}^{-2}) \times \pi (75 \text{ cm})^2 = 9.2 \times 10^{-5} \text{ erg s}^{-1} = 9.2 \times 10^{-12} \text{ W}$$

Compare this with the output of a dim light - bulb,

which is about 10 W , or a common light emitting diode (LED) of 0.1 W .

$$(h) M_V = m_V - 5 \text{ Log} \left[10, \frac{d}{10} \right] / . \{m_V \rightarrow 9, d \rightarrow 500\}$$

$\therefore M_V = 0.51 \text{ mag}$