Ay 20: Basic Astronomy and the Galaxy
Fall Term 2010

Solution Set 3

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(based on solutions by Swarnima Manohar, TA 2009)

Reporting an answer to unnecesary number of decimal places should be avoided. CGS units are popular among professional astronomers. SI system is considered universal, and so I will encourage you to stick to SI if you are presently using this system most frequently.

PROBLEM 1 (C&O Problem 8.6)

(a) $\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)/k T}$

Solve$\left[1 == \frac{2 \times 3^2}{2 \times 1^2} e^{-\left(\left(\frac{-13.6}{3^2} - \frac{-13.6}{1^2}\right) 1.602 \times 10^{-19}\right) / (1.381 \times 10^{-23}\,T)}, T\right]$

{{T → 63 823.5}}

(b) $\frac{N_3}{N_1} = \frac{g_3}{g_1} e^{-(E_1 - E_3)/k T}$**. Using** $T = 85,\ 400\ K$, $g_n = 2\, n^2$ **and** $N_1 = N$ **we get** $N_3 = 1.7\ N$.

(c) As $T \to \infty$, $e^{-E/k T} \to 1$. $\therefore N_b / N_a \to g_b / g_a = (n_b / n_a)^2$. Thus Boltzmann equation predicts the relative number of electrons in a given orbital to varyas the square of the orbital number. This prediction will not hold for $T \to \infty$ since the atom will be ionized and the electron will no longer be bound above $kT \sim 13.6$ eV. This will also be dictated by the recombination rate, depending on other factors like the density, radiation field, etc.

PROBLEM 2 (C&O Problem 8.11)

Use Saha equation,

$$\frac{\mathcal{N}_{III}}{\mathcal{N}_{II}} = \frac{2\, Z_{III}}{n_e\, Z_{II}} \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_{II}/k T} = \frac{2\, kT\, Z_{III}}{P_e\, Z_{II}} \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_{II}/k T}$$

where the subscript ' II ' refers to HeII and ' III ' refers to HeIII i.e. completely ionized Helium. From the data given in C & O Problem 8.10 we have,
$Z_{III} = 1$, $Z_{II} = 2$, $\chi_{II} = 54.4$ **eV. Also,**
$P_e = 1000\ N \cdot m^{-2}$. **Thus we get the temperature corresponding to** $\mathcal{N}_{III} = \mathcal{N}_{II}$ **as,**

Solve$\left[1 == 3.3 \times 10^{-5}\, T^{5/2}\, e^{-((54.4 \times 1.602 \times 10^{-19}) / (1.381 \times 10^{-23}\, T))}, T\right]$

{{T → 39 150.1}}

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x[T_] := 3.3 × 10^-5 T^5/2 e^-((54.4 × 1.602 × 10^-19)/(1.381 × 10^-23 T))
Plot[x[T], {T, 10000, 45000}, PlotStyle → Directive[Thick],
 FrameLabel → {Temperature[K], 𝒩III / 𝒩II}, Frame → True,
 GridLines → Automatic, GridLinesStyle → Directive[{Dashed}]]
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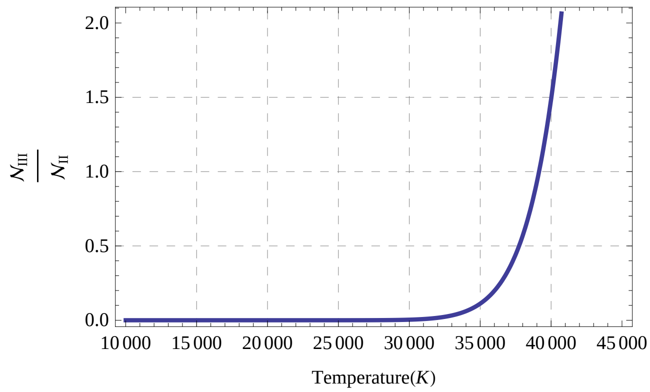

PROBLEM 3 (C&O Problem 8.14)

Giants are of spectral type G or later,
and therefore have effective temperatures of less than 6000 K. So we are essentially talking about hydrogen ionization in the atmosphere of these stars. The relevant Saha equation is,

$$\frac{\mathcal{N}_{II}}{\mathcal{N}_I} = \frac{2 Z_{II}}{n_e Z_I} \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_I / k T}$$

Since in this problem, the main sequence (MS) star and the giant (g) star are said to have the same spectral type (meaning that $\mathcal{N}_{II} / \mathcal{N}_I$ is the same for both stars), we can say that,

$$\left[\frac{T^{3/2}}{n_e} e^{-\chi_I / k T}\right]_g = \left[\frac{T^{3/2}}{n_e} e^{-\chi_I / k T}\right]_{MS}$$

$\chi_I / k = 13.6 \text{ eV} / k \approx 1.6 \times 10^5$ K, and ' n_e ' for the giant is less than that for the MS star due to its tenuous atmosphere. Therefore,

$$\left[T^{3/2} e^{-1.6\times10^5 \text{ K}/ T}\right]_g > \left[T^{3/2} e^{-1.6\times10^5 \text{ K}/ T}\right]_{MS}.$$

That is, for $x = T_g / T_{MS}$,

$$x^{3/2} > \exp\left(-(1 - x)\, 1.6 \times 10^5 \text{ K} / T_{MS}\right)$$

As stated above, $T_{MS} \lesssim 6000$ K. Therefore,

$$x^{3/2} \exp(300 (1 - x)) \gtrsim 1.$$

This requires xto be slightly less than 1,
and thus the inconsistency in spectral type and temperature.

```
a[x_] := x^(3/2) e^(300 (1-x))
Plot[a[x], {x, 0.995, 1.01}, PlotStyle → Directive[Thick], FrameLabel → {x, a},
 Frame → True, GridLines → Automatic, GridLinesStyle → Directive[{Dashed}]]
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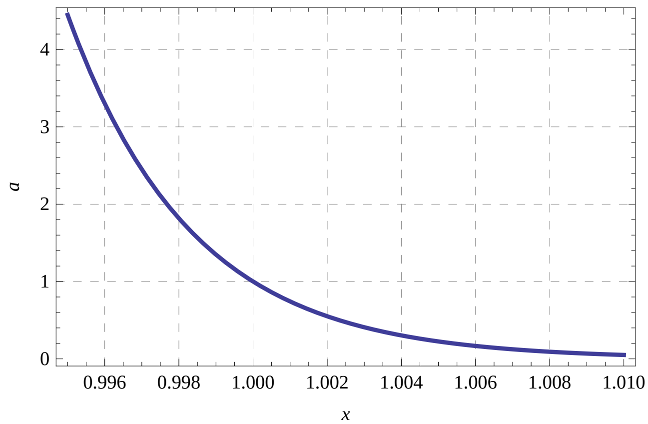

PROBLEM 4 (C&O Problem 9.11)

$\rho_c = 1.53 \times 10^5\ \text{kg}/m^3 = 153\ g/\text{cm}^3$

$\bar{\kappa} = 0.217\ m^2\ \text{kg}^{-1} = 2.17\ \text{cm}^2\ g^{-1}$

(a) Mean Free path is given by:

$$l \to \frac{1}{\kappa \rho} \ /.\ \{\kappa \to 2.17\ \text{cm}^2\ g^{-1},\ \rho \to 153\ g/\text{cm}^3\}$$

$l \to 0.00301196\ \text{cm}$

(b) Displacement, d after size of step l (eqn 9.29)

$$d \to l\sqrt{N}$$

Radius of Sun is the distance that the photon would have to traverse, $d = R_\odot = 6.96 \times 10^{10}$ cm and l is the mean free path. Number of steps taken in total:

$$N \to \left(\frac{d}{l}\right)^2 \ /.\ \{d \to 6.96 \times 10^{10}\ \text{cm},\ l \to 0.00301196\ \text{cm}\}$$

$N \to 5.33974 \times 10^{26}$

Each step takes l/c time. Average total time taken for photon to escape from the Sun:

$$t_{total} \to \mathcal{N} \frac{l}{c} \frac{yr}{24 \times 3600 \times 365.25\, s} \;/.\; \{l \to 0.00301196\, cm,\; c \to 2.9979 \times 10^{10}\, cm/s,\; \mathcal{N} \to 5.33974 \times 10^{26}\}$$

$$t_{total} \to 1.7 \times 10^{6}\, yr$$

Not much, just about 2 million years.

PROBLEM 5 (C&O Problem 9.16)

(a) The radiative transfer equation, integrated over wavelength, says

$$-\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda$$

where τ_λ is measured along the light path. It is easy to show that ds is related to dr by ds = dr/ cos θ'. (Note that is relation is true differentially, but a statement like s = r/ cos θ' does not make sense. As one moves along the light path, the local radial vector and hence θ' are changing.) Finally, the optical depth is defined by $d\tau_\lambda = \kappa_\lambda \rho$ ds. Putting the pieces together yields

$$-\frac{dI_\lambda}{\kappa_\lambda \rho\, ds} = I_\lambda - S_\lambda$$

$$-\frac{\text{Cos}[\theta']}{\kappa_\lambda \rho} \frac{dI_\lambda}{dr} = I_\lambda - S_\lambda$$

(b) To obtain the pressure gradient we would like to take the first moment. We might as well pick a point to study, place the origin there, and orient the polar axis along the outward radial vector so that there is no distinction between θ and θ'. Multiplying the result from (a) by cos θ' and integrating over the sphere gives

$$\int -\frac{\text{Cos}^2[\theta']}{\bar{\kappa}\rho} \frac{dI}{dr} d\Omega = \int I\, \text{Cos}[\theta']\, d\Omega - \int S\, \text{Cos}[\theta']\, d\Omega$$

where the λ dependence has been removed by integration over wavelength. (We can take κ to be the Rosseland mean opacity $\bar{\kappa}$.) Note that the solid angle element is d = sin θ'dθ' dϕ .The first term on the RHS is just the flux F_{rad}. Since S is assumed isotropic, it has no angular dependence and can be taken outside the integral, which then vanishes since

$$\int \text{Cos}[\theta']\, d\Omega = \int_0^{2\pi} \int_{\theta'=-\pi/2}^{\theta'=\pi/2} \text{Cos}[\theta']\, \text{Sin}[\theta']\, d\theta'\, d\phi = 0$$

Finally, recalling that

$$\int I\, \text{Cos}^2[\theta']\, d\Omega = c\, P_{rad}$$

we get that

$$\frac{d}{dr} \int I\, \text{Cos}^2[\theta']\, d\Omega = \frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c} F_{rad}$$