Ay 20: Basic Astronomy and the Galaxy
Fall Term 2010

Solution Set 4

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(based on solutions by Swarnima Manohar, TA 2009)

Reporting an answer to unnecesary number of decimal places should be avoided. CGS units are popular among professional astronomers. SI system is considered universal, and so I will encourage you to stick to SI if you are presently using this system most frequently.

PROBLEM 1 (C&O Problem 7.6)

(a)

$$\frac{m_A}{m_B} = \frac{v_{Br}}{v_{Ar}} = \frac{22.4}{5.4} = 4.14$$

(b) Using Eq. 7.6 in Carroll and Ostlie,

$$m_A + m_B \to \frac{P}{2\pi G} \frac{(v_{Ar} + v_{Br})^3}{\mathrm{Sin}[i]^3} \mathrm{g} \;/.$$
$$\{P \to 6.31 \times 365.25 \times 24 \times 3600,\ G \to 6.67 \times 10^{-8},\ v_{Ar} \to 5.4 \times 10^5,\ v_{Br} \to 22.4 \times 10^5,\ i \to \pi / 2\}$$

$$m_A + m_B \to 1.02085 \times 10^{34}\ \mathrm{g}$$

which is 5.13 $M_\odot$.

(c)

$$\mathrm{Solve}\left[\left\{\frac{mA}{mB} == 4.14,\ mA + mB == 5.13\, M_{sun}\right\}, \{mA, mB\}\right]$$

$$\{\{mA \to 0. + 4.13195\, M_{sun},\ mB \to 0. + 0.998054\, M_{sun}\}\}$$

m_A = 4.13 $M_\odot$, m_B = 0.998 $M_\odot$.

(d) Assuming the orbital separation is much larger than the stellar radii, and that the orbits are circular, we can treat the velocity of the stars during eclipse as completely in the plane of the sky. For circular orbits, the maximum radial velocities given are the constant velocities throughout the orbit. The relative velocity is then in km/s

$$v \to 5.4 + 22.4$$

$$v \to 27.8$$

It takes a time $t_b - t_a$ for the smaller disk to enter the larger one, so the radius of the smaller star must be

$$r_s \to \frac{1}{2} v\, (\Delta t_{ab})\ \mathrm{cm} \;/.\ \{v \to 27.8 \times 10^5,\ \Delta t_{ab} \to 0.58 \times 24 \times 3600\}$$

$$r_s \to 6.96557 \times 10^{10}\ \mathrm{cm}$$

which is about 1 $R_\odot$. (Note that assuming i≈ 90° as we do here implies the transit is across a diameter of the large disk.) Now consider the point on the smaller disk that first eclipses the larger. From time t_a to t_c this point traces the diameter of the larger disk (see Fig. 7.9 in Carroll and Ostlie), so the radius of the larger disk is

$$r_l \to \frac{1}{2} v\,(\Delta t_{cb} + \Delta t_{ab})\ \text{cm}\ /.\ \{v \to 27.8 \times 10^5,\ \Delta t_{ab} \to 0.58 \times 24 \times 3600,\ \Delta t_{cb} \to 0.64 \times 24 \times 3600\}$$

$$r_l \to 1.46517 \times 10^{11}\ \text{cm}$$

which is 2.11 $R_\odot$.

(e) Following example 7.3.2 in Carroll and Ostlie, the ratio of fluxes between the primary minimum and maximum light is $B_p/B_0 = 100^{(5.40-9.20)/5} = 0.030$, and the ratio of fluxes between the secondary minimum and maximum light is $B_s/B_0 = 100^{(5.40-5.44)/5} = 0.964$. Then

$$\frac{T_s}{T_l} \to \left(\frac{1 - B_p / B_0}{1 - B_s / B_0}\right)^{1/4} \text{ /. } \{B_p / B_0 \to 0.030,\ B_s / B_0 \to 0.964\}$$

$$\frac{T_s}{T_l} \to 2.27833$$

The derivation in the text assumes that the smaller star is hotter, i.e. that the primary eclipse is when the smaller star passes behind the larger. Can we back this up with the data? Assuming this is true, then in the primary eclipse we see only the larger star, which gives $100^{(m_0 - m_p)/5} = 100^{(5.40 - 9.20)/5} = 3.02\%$ of the total brightness. In the secondary, we then expect to see the remaining 96.98% of the flux from the smaller star, plus the fraction $(r_s/r_l)^2 = 0.225$ (from part c) of the big star's 3.02% that is unobstructed, for a total of 97.7%. We observe $100^{(m_0 - m_s)/5} = 100^{(5.40 - 5.44)/5} = 96.38\%$, which is close considering we only two significant digits in the times.

If we instead imagine that the larger star is hotter, so that the primary eclipse occurs when the smaller star is in front, we find that the 96.38% of light visible in the secondary is due to the larger star, and that during the primary we expect 0.225 of this (plus the 3.62% contribution of the smaller star), which is far in excess of the 3.02% observed.

Another way to estimate the temperature ratio is to note that the primary minimum is 0.030 of the maximum brightness, so that

$$0.030\left(T_s^4 R_s^2 + T_l^4 R_l^2\right) = T_l^4 R_l^2$$

$$\Rightarrow \frac{T_s}{T_l} = \left(\left(\frac{1}{0.030} - 1\right)\left(\frac{R_l}{R_s}\right)^2\right)^{1/4} = 3.5$$

using the radii from (d). This does not agree particularly well with the 2.3 we found above, indicating that the timing data in this problem are not consistent with the photometry under the assumption that the smaller star passes directly across a diameter of the larger star's disk. For this homework either answer is acceptable.

PROBLEM 2 (C&O Problem 7.10)

1) As given in Section 7.4 of C & O, The radial velocity variation in the parent staris induced by the gravitational tug of the orbiting planets.

$$v_{star} = \frac{M_{planet}}{M_{star}} v_{planet} = \frac{M_{planet}}{M_{star}} \cdot \frac{2\pi}{P}$$

2) Smaller period $\Longrightarrow$ Radial velocity variation can be measured at many points along the orbit in a smaller amount of time.

PROBLEM 3 (C&O Problem 7.11)

For simplicity, assume circular orbits. (In any case, eccentricity can be deduced from distortions of the radial velocity curve from sinusoidal.) Denoting the masses of the planet and star by M_p and M_s, Kepler's third law says

$$P^2 = \frac{4\pi^2}{G(M_p + M_s)} a^3 \approx \frac{4\pi^2}{G M_s} a^3$$

This is because $M_p \ll M_s$, so

$$a \approx \left(\frac{G M_s P^2}{4\pi^2}\right)^{1/3}$$

The orbital velocity of the planet is approximately

$$v_p \approx \frac{2\pi a}{P} \approx \left(\frac{2\pi G M_s}{P}\right)^{1/3}$$

and that of the star is reduced approximately by the mass ratio:

$$v_s \approx \frac{M_p}{M_s} v_p$$

We can measure the period P and estimate M_s by determining its spectral type with our spectrum. Thus, we know v_p. If we could measure v_s, then the above equations would allow us to solve for M_p. However, what we measure is $v_{rs} = v_s$ sin i, the radial component of the stars̊ velocity, and the orbital inclination i is in general unknown. Since v_{rs} is a lower limit on v_s, we get a lower limit on $M_p = M_s v_s / v_p$, but cannot say more.

PROBLEM 4 (C&O Problem 8.12)

$$\frac{\mathcal{N}_{II}}{\mathcal{N}_I} \to \frac{2 Z_{II}}{n_e Z_I} \left(\frac{2 \pi m_e k T}{h^2}\right)^{3/2} e^{-\chi_i / k T} \; /. \; \{Z_I \to 2,\ Z_{II} \to 1,\ n_e \to 6.1 \times 10^{31},$$
$$h \to 6.63 \times 10^{-34},\ m_e \to 9.11 \times 10^{-31},\ k \to 1.38 \times 10^{-23},\ \chi_i \to 13.6 \times 1.6 \times 10^{-19},\ T \to 15.7 \times 10^6\}$$

$$\frac{\mathcal{N}_{II}}{\mathcal{N}_I} \to 2.44$$

Therefore,

$$\frac{\mathcal{N}_{II}}{\mathcal{N}_t} = \frac{2.44}{2.44 + 1} = 0.71$$

The Saha equation describes the degree of ionization of this plasma as a function of the temperature, density, and ionization energies of the atoms.The Saha equation only holds for weakly ionized plasmas for which the Debye length is large (meaning that the screening of the coulomb charge of ions and electrons by other ions and electrons is negligible). Since in this case, the degree of ionization is large, Saha equation is not expected to give the correct answer (fully ionized plasma).

PROBLEM 5 (C&O Problem 9.17)

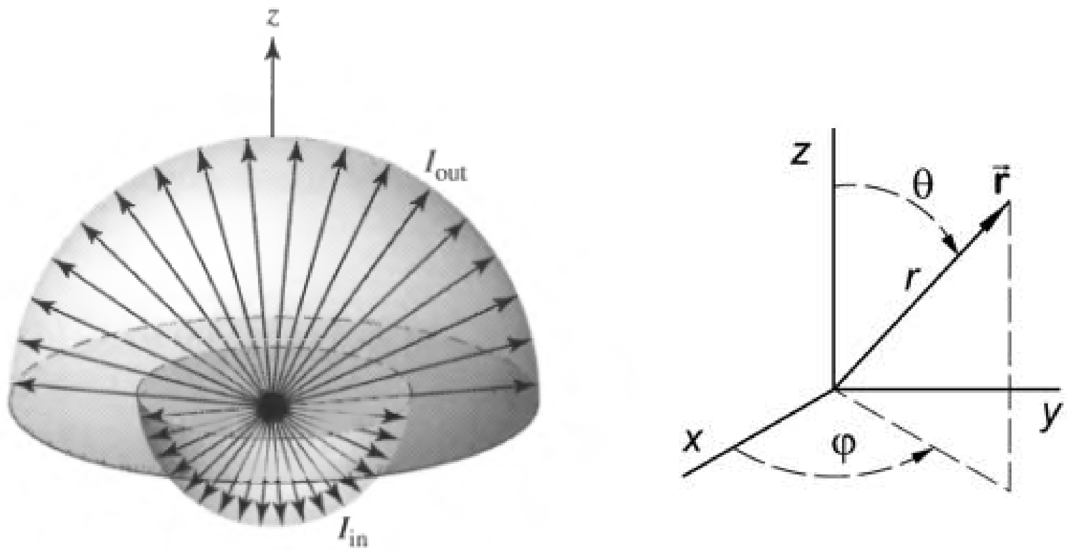

FIGURE 9.15 The Eddington approximation.

For I_{out} : $\theta = 0 \longrightarrow \frac{\pi}{2}$, $\phi = 0 \longrightarrow 2\pi$

For I_{in} : $\theta = \frac{\pi}{2} \longrightarrow \pi$, $\phi = 0 \longrightarrow 2\pi$

$\vec{I} = \vec{I}_{in} + \vec{I}_{out}$

From C & O eqn 9.3 :

$$< I > = \frac{1}{4\pi} \int \vec{I} \, d\Omega = \int_{\pi/2}^{\pi} \int_0^{2\pi} I_{in} \sin\theta \, d\theta \, d\phi + \int_0^{\pi/2} \int_0^{2\pi} I_{out} \sin\theta \, d\theta \, d\phi = \frac{1}{2} (I_{out} + I_{in})$$

From C & O eqn 9.8 :

$$F_{rad} = \int \vec{I} \cos\theta \, d\Omega = \pi (I_{out} - I_{in})$$

From C & O eqn 9.9 :

$$P_{rad} = \frac{1}{c} \int \vec{I} \cos^2\theta \, d\Omega = \frac{2\pi}{3c} (I_{out} + I_{in})$$

PROBLEM 6 (C&O Problem 10.11)

The goal is to approximate the expression (eqn 10.62)

$$\epsilon_{3\alpha} = 50.9 \rho^2 Y^3 T_8^{-3} f_{3\alpha} e^{-44.027 T_8^{-1}}$$

as a power law of the form:

$$\epsilon_{3\alpha} \propto T_8^{\alpha}$$

near $T = 10^8$ ($T_8 = 1$). We equate the two equations, take log of both sides, require their log-derivatives to match, and plug in $T_8 = 1$ to solve for α:

$$\frac{d}{d\ln T_8}\left[\ln\left(T_8^{\alpha}\right) = \text{const} + \ln\left(T_8^{-3}\, e^{-44.027\, T_8^{-1}}\right)\right]$$

$$\alpha = -3 + \frac{44.027}{T_8} = 41$$

5. C&O Problem 10.23

Eddington luminosity is given by (10.114):

$$L_{Edd} = \frac{4\pi G c}{\kappa} M$$

Where the units of κ are such that $1/(\kappa\,\rho) = \lambda$ (mean free path).

(a) Plugging in numbers for the low-mass star gives (in cgs units)

$$L_{Edd} \to \frac{4\pi G c}{\kappa} M\, \text{erg} \;/.\; \{G \to 6.673\times 10^{-8},\ c \to 2.9979\times 10^{10},\ M \to 0.072\times 1.989\times 10^{33},\ \kappa \to 0.01\}$$

$$L_{Edd} \to 3.60011\times 10^{38}\ \text{erg}$$

which is almost $10^5 L_\odot$, which is much much higher than the MS luminosity for this star given in 10.21.

(b) For the high mass star, we have to use the fact that the opacity is due to electron scattering, for which the fundamental parameter is the Thomson cross-section σ_T ($6.65\ \times 10^{-25}$ cm^{-2}). Recall that $1/(n\sigma)$ also = λ, so that $\kappa = n_e\,\sigma_T/\rho$ for electron scattering. Because $\rho = m_p\, n_p$ and we assume $n_e = n_p$, this gives $\kappa = \sigma_T/\, m_p$=0.40 cm^2 g^{-1}= 0.040 m^2 kg^{-1}, giving

$$L_{Edd} \to \frac{4\pi G c}{\kappa} M\, \text{erg} \;/.\; \{G \to 6.673\times 10^{-8},\ c \to 2.9979\times 10^{10},\ M \to 120\times 1.989\times 10^{33},\ \kappa \to 0.4\}$$

$$L_{Edd} \to 1.50004\times 10^{40}\ \text{erg}$$

or $3.8 \times 10^6\ L_\odot$. The given luminosity of this star is about $1.8 \times 10^6\ L_\odot$, which is about half its Eddington luminosity, which is significant. In fact, it is the Eddington luminosity that sets the upper limit on how massive stars can be.