6.25. **(a)** Starting with the Lorentz force expression (6.114), show that in the dipole approximation the force acting on an atom can be expressed as

$$\frac{d\mathbf{P}_{\text{atom}}}{dt} = (\mathbf{d} \cdot \nabla)\,\mathbf{E} + \dot{\mathbf{d}} \times \mathbf{B}$$

where **d** is the atomic dipole moment and **E** and **B** are the electric and magnetic fields at the site of the atom.

(b) For a uniform plane wave of frequency ω in a nonmagnetic tenuous dielectric medium with index of refraction $n(\omega)$, show that the time rate of change of mechanical momentum per unit volume $\mathbf{g}_{\text{mech}}$ accompanying the electromagnetic momentum $\mathbf{g}_{\text{em}}$ (6.118) of the wave is

$$\frac{d\mathbf{g}_{\text{mech}}}{dt} = \frac{1}{2}(n^2 - 1)\frac{d\mathbf{g}_{\text{em}}}{dt}$$

[see Peierls (loc. cit.) for corrections for dense media and non-uniform waves]

APTER 7

ne Electromagnetic Waves d Wave Propagation

This chapter on plane waves in unbounded, or perhaps semi-infinite, media treats first the basic properties of plane electromagnetic waves in nonconducting media—their transverse nature, linear and circular polarization states. Then the important Fresnel formulas for reflection and refraction at a plane interface are derived and applied. This is followed by a survey of the high-frequency dispersion properties of dielectrics, conductors, and plasmas. The richness of nature is illustrated with a panoramic view (Fig. 7.9) of the index of refraction and absorption coefficient of liquid water over 20 decades of frequency. Then comes a simplified discussion of propagation in the ionosphere, and of magnetohydrodynamic waves in a conducting fluid. The ideas of phase and group velocities and the spreading of a pulse or wave packet as it propagates in a dispersive medium come next. The important subject of causality and its consequences for the dispersive properties of a medium are discussed in some detail, including the Kramers–Kronig dispersion relations and various sum rules derived from them. The chapter concludes with a treatment of the classic problem of the arrival of a signal in a dispersive medium, first discussed by Sommerfeld and Brillouin (1914) but only recently subjected to experimental test.

Plane Waves in a Nonconducting Medium

A basic feature of the Maxwell equations for the electromagnetic field is the existence of traveling wave solutions which represent the transport of energy from one point to another. The simplest and most fundamental electromagnetic waves are transverse, plane waves. We proceed to see how such solutions can be obtained in simple nonconducting media described by spatially constant permeability and susceptibility. In the absence of sources, the Maxwell equations in an infinite medium are

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\ \nabla \cdot \mathbf{D} &= 0, \qquad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = 0 \end{aligned} \tag{7.1}$$

Assuming solutions with harmonic time dependence $e^{-i\omega t}$, from which we can build an arbitrary solution by Fourier superposition, the equations for the amplitudes $\mathbf{E}(\omega, \mathbf{x})$, etc. read

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \qquad \nabla \times \mathbf{E} - i\omega\mathbf{B} = 0 \\ \nabla \cdot \mathbf{D} &= 0, \qquad \nabla \times \mathbf{H} + i\omega\mathbf{D} = 0 \end{aligned}$$

For uniform isotropic linear media we have $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, where ϵ and μ may in general be complex functions of ω. We assume for the present that they are real (no losses). Then the equations for **E** and **H** are

$$\nabla \times \mathbf{E} - i\omega\mathbf{B} = 0, \qquad \nabla \times \mathbf{B} + i\omega\mu\epsilon\mathbf{E} = 0 \tag{7.2}$$

The zero-divergence equations are not independent, but are obtained by taking divergences in (7.2). By combining the two equations we get the Helmholtz wave equation

$$(\nabla^2 + \mu\epsilon\omega^2)\begin{Bmatrix}\mathbf{E}\\ \mathbf{B}\end{Bmatrix} = 0 \tag{7.3}$$

Consider as a possible solution a plane wave traveling in the x direction, $e^{ikx-i\omega t}$. From (7.3) we find the requirement that the wave number k and the frequency ω are related by

$$k = \sqrt{\mu\epsilon}\,\omega \tag{7.4}$$

The *phase velocity* of the wave is

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}, \qquad n = \sqrt{\frac{\mu}{\mu_0}\frac{\epsilon}{\epsilon_0}} \tag{7.5}$$

The quantity n is called the *index of refraction* and is usually a function of frequency. The primordial solution in one dimension is

$$u(x, t) = ae^{ikx-i\omega t} + be^{-ikx-i\omega t} \tag{7.6}$$

Using $k = \omega v$ from (7.5), this can be written

$$u_k(x, t) = ae^{ik(x-vt)} + be^{-ik(x+vt)}$$

If the medium is nondispersive ($\mu\epsilon$ independent of frequency), the Fourier superposition theorem (2.44) and (2.45) can be used to construct a general solution of the form

$$u(x, t) = f(x - vt) + g(x + vt) \tag{7.7}$$

where $f(z)$ and $g(z)$ are arbitrary functions. Equation (7.7) represents waves traveling in the positive and negative x directions with speeds equal to the phase velocity v.

If the medium is dispersive, the basic solution (7.6) still holds, but when we build up a wave as an arbitrary function of x and t, the dispersion produces modifications. Equation (7.7) no longer holds. The wave changes shape as it propagates (see Sections 7.8, 7.9, and 7.11).

We now consider an electromagnetic plane wave of frequency ω and wave vector $\mathbf{k} = k\mathbf{n}$ and require that it satisfy not only the Helmholtz wave equation (7.3) but also all the Maxwell equations. The constraint imposed by (7.3) is essentially kinematic; those imposed by the Maxwell equations, dynamic. With the convention that the physical electric and magnetic fields are obtained by taking the real parts of complex quantities, we write the plane wave fields as

$$\begin{aligned}\mathbf{E}(\mathbf{x}, t) &= \boldsymbol{\mathscr{E}}e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t}\\ \mathbf{B}(\mathbf{x}, t) &= \boldsymbol{\mathscr{B}}e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t}\end{aligned} \tag{7.8}$$

where $\boldsymbol{\mathscr{E}}$, $\boldsymbol{\mathscr{B}}$, and **n** are constant vectors. Each component of **E** and **B** satisfies (7.3) provided

$$k^2\,\mathbf{n}\cdot\mathbf{n} = \mu\epsilon\omega^2 \tag{7.9}$$

To recover (7.4) it is necessary that **n** be a unit vector such that $\mathbf{n}\cdot\mathbf{n} = 1$. With the wave equation satisfied, there only remains the fixing of the vectorial properties so that the Maxwell equations (7.1) are valid. The divergence equations in (7.1) demand that

$$\mathbf{n}\cdot\boldsymbol{\mathscr{E}} = 0 \quad \text{and} \quad \mathbf{n}\cdot\boldsymbol{\mathscr{B}} = 0 \tag{7.10}$$

This means that **E** and **B** are both perpendicular to the direction of propagation **n**. Such a wave is called a *transverse wave*. The curl equations provide a further restriction, namely

$$\boldsymbol{\mathscr{B}} = \sqrt{\mu\epsilon}\,\mathbf{n}\times\boldsymbol{\mathscr{E}} \tag{7.11}$$

The factor $\sqrt{\mu\epsilon}$ can be written $\sqrt{\mu\epsilon} = n/c$, where n is the index of refraction defined in (7.5). We thus see that $c\mathbf{B}$ and **E**, which have the *same dimensions*, have the *same magnitude* for plane electromagnetic waves in free space and differ by the index of refraction in ponderable media. In engineering literature the magnetic field **H** is often displayed in parallel to **E** instead of **B**. The analog of (7.11) for **H** is

$$\boldsymbol{\mathscr{H}} = \mathbf{n}\times\boldsymbol{\mathscr{E}}/Z \tag{7.11'}$$

where $Z = \sqrt{\mu/\epsilon}$ is an impedance. In vacuum, $Z = Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 376.7$ ohms, the impedance of free space.

If **n** is real, (7.11) implies that $\boldsymbol{\mathscr{E}}$ and $\boldsymbol{\mathscr{B}}$ have the same phase. It is then useful to introduce a set of real mutually orthogonal unit vectors ($\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$, **n**), as shown in Fig. 7.1. In terms of these unit vectors the field strengths $\boldsymbol{\mathscr{E}}$ and $\boldsymbol{\mathscr{B}}$ are

$$\boldsymbol{\mathscr{E}} = \boldsymbol{\epsilon}_1 E_0, \qquad \boldsymbol{\mathscr{B}} = \boldsymbol{\epsilon}_2\sqrt{\mu\epsilon}\,E_0 \tag{7.12}$$

or

$$\boldsymbol{\mathscr{E}} = \boldsymbol{\epsilon}_2 E_0', \qquad \boldsymbol{\mathscr{B}} = -\boldsymbol{\epsilon}_1\sqrt{\mu\epsilon}\,E_0' \tag{7.12'}$$

where E_0 and E_0' are constants, possibly complex. The wave described by (7.8) and (7.12) or (7.12') is a transverse wave propagating in the direction **n**. It rep-

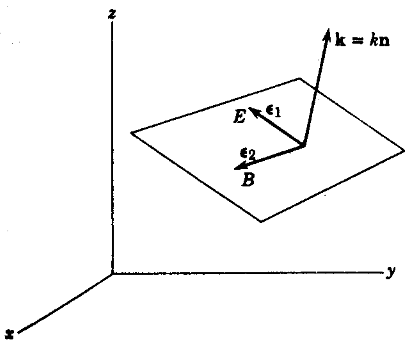


Figure 7.1 Propagation vector **k** and two orthogonal polarization vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$.

resents a time-averaged flux of energy given by the real part of the complex Poynting vector:

$$\mathbf{S} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^*$$

The energy flow (energy per unit area per unit time) is

$$\mathbf{S} = \sqrt{\frac{\epsilon}{\mu}}\,|E_0|^2\,\mathbf{n} \tag{7.13}$$

The time-averaged energy density u is correspondingly

$$u = \frac{1}{4}\left(\epsilon\mathbf{E}\cdot\mathbf{E}^* + \frac{1}{\mu}\mathbf{B}\cdot\mathbf{B}^*\right)$$

This gives

$$u = \frac{\epsilon}{2}|E_0|^2 \tag{7.14}$$

The ratio of the magnitude of (7.13) to (7.14) shows that the speed of energy flow is $v = 1/\sqrt{\mu\epsilon}$, as expected from (7.5).

In the discussion that follows (7.11) we assumed that $\mathbf{n}$ was a real unit vector. This does not yield the most general possible solution for a plane wave. Suppose that $\mathbf{n}$ is complex, and written as $\mathbf{n} = \mathbf{n}_R + i\mathbf{n}_I$. Then the exponential in (7.8) becomes

$$e^{ik\mathbf{n}\cdot\mathbf{x}-i\omega t} = e^{-k\mathbf{n}_I\cdot\mathbf{x}}e^{ik\mathbf{n}_R\cdot\mathbf{x}-i\omega t}$$

The wave possesses exponential growth or decay in some directions. It is then called an *inhomogeneous plane wave*. The surfaces of constant amplitude and constant phase are still planes, but they are no longer parallel. The relations (7.10) and (7.11) still hold. The requirement $\mathbf{n}\cdot\mathbf{n} = 1$ has real and imaginary parts,*

$$\begin{aligned} n_R^2 - n_I^2 &= 1 \\ \mathbf{n}_R\cdot\mathbf{n}_I &= 0 \end{aligned} \tag{7.15}$$

The second of these conditions shows that $\mathbf{n}_R$ and $\mathbf{n}_I$ are orthogonal. The coordinate axes can be oriented so that $\mathbf{n}_R$ is in the x direction and $\mathbf{n}_I$ in the y direction. The first equation in (7.15) can be satisfied generally by writing

$$\mathbf{n} = \mathbf{e}_1\cosh\theta + i\mathbf{e}_2\sinh\theta \tag{7.16}$$

where θ is a real constant and $\mathbf{e}_1$ and $\mathbf{e}_2$ are real unit vectors in the x and y directions (not to be confused with $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$!). The most general vector $\mathscr{E}$ satisfying $\mathbf{n}\cdot\mathscr{E} = 0$ is then

$$\mathscr{E} = (i\mathbf{e}_1\sinh\theta - \mathbf{e}_2\cosh\theta)A + \mathbf{e}_3A' \tag{7.17}$$

where A and A' are complex constants. For $\theta \neq 0$, $\mathscr{E}$ in general has components in the direction(s) of $\mathbf{n}$. It is easily verified that for $\theta = 0$, the solutions (7.12) and (7.12′) are recovered.

We encounter simple examples of inhomogeneous plane waves in the discussion of total internal reflection and refraction in a conducting medium later in the chapter, although in the latter case the inhomogeneity arises from a com-

*Note that if $\mathbf{n}$ is complex it does not have unit magnitude, that is, $\mathbf{n}\cdot\mathbf{n} = 1$ does not imply $|\mathbf{n}|^2 = 1$

plex wave number, not a complex unit vector $\mathbf{n}$. Inhomogeneous plane waves form a general basis for the treatment of boundary-value problems for waves and are especially useful in the solution of diffraction in two dimensions. The interested reader can refer to the book by *Clemmow* for an extensive treatment with examples.

7.2 *Linear and Circular Polarization; Stokes Parameters*

The plane wave (7.8) and (7.12) is a wave with its electric field vector always in the direction $\boldsymbol{\epsilon}_1$. Such a wave is said to be linearly polarized with polarization vector $\boldsymbol{\epsilon}_1$. Evidently the wave described in (7.12′) is linearly polarized with polarization vector $\boldsymbol{\epsilon}_2$ and is linearly independent of the first. Thus the two waves,

$$\begin{aligned} \mathbf{E}_1 &= \boldsymbol{\epsilon}_1E_1e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \\ \mathbf{E}_2 &= \boldsymbol{\epsilon}_2E_2e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \end{aligned} \tag{7.18}$$

with

$$\mathbf{B}_j = \sqrt{\mu\epsilon}\,\frac{\mathbf{k}\times\mathbf{E}_j}{k}, \qquad j = 1, 2$$

can be combined to give the most general homogeneous plane wave propagating in the direction $\mathbf{k} = k\mathbf{n}$,

$$\mathbf{E}(\mathbf{x}, t) = (\boldsymbol{\epsilon}_1E_1 + \boldsymbol{\epsilon}_2E_2)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{7.19}$$

The amplitudes E_1 and E_2 are complex numbers, to allow the possibility of a phase difference between waves of different linear polarization.

If E_1 *and* E_2 have the *same phase*, (7.19) represents a *linearly polarized* wave, with its polarization vector making an angle $\theta = \tan^{-1}(E_2/E_1)$ with $\boldsymbol{\epsilon}_1$ and a magnitude $E = \sqrt{E_1^2 + E_2^2}$, as shown in Fig. 7.2.

If E_1 *and* E_2 have *different phases*, the wave (7.19) is *elliptically polarized*. To understand what this means let us consider the simplest case, *circular polarization*. Then E_1 and E_2 have the same magnitude, but differ in phase by 90°. The wave (7.19) becomes:

$$\mathbf{E}(\mathbf{x}, t) = E_0(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2)e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} \tag{7.20}$$

with E_0 the common real amplitude. We imagine axes chosen so that the wave is propagating in the positive z direction, while $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are in the x and y directions, respectively. Then the components of the actual electric field, obtained by taking the real part of (7.20), are

$$\left.\begin{aligned} E_x(\mathbf{x}, t) &= E_0\cos(kz - \omega t) \\ E_y(\mathbf{x}, t) &= \mp E_0\sin(kz - \omega t) \end{aligned}\right\} \tag{7.21}$$

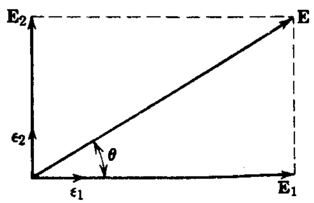


Figure 7.2 Electric field of a linearly polarized wave.

At a *fixed point in space*, the fields (7.21) are such that the electric vector is constant in magnitude, but sweeps around in a circle at a frequency ω, as shown in Fig. 7.3. For the upper sign ($\boldsymbol{\epsilon}_1 + i\boldsymbol{\epsilon}_2$), the rotation is counterclockwise when the observer is facing into the oncoming wave. This wave is called *left circularly polarized* in optics. In the terminology of modern physics, however, such a wave is said to have *positive helicity*. The latter description seems more appropriate because such a wave has a positive projection of angular momentum on the z axis (see Problem 7.29). For the lower sign ($\boldsymbol{\epsilon}_1 - i\boldsymbol{\epsilon}_2$), the rotation of $\mathbf{E}$ is clockwise when looking into the wave; the wave is *right circularly polarized* (optics); it has *negative helicity*.

The two circularly polarized waves (7.20) form an equally acceptable set of basic fields for description of a general state of polarization. We introduce the complex orthogonal unit vectors:

$$\boldsymbol{\epsilon}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2) \tag{7.22}$$

with properties

$$\begin{aligned} \boldsymbol{\epsilon}_\pm^* \cdot \boldsymbol{\epsilon}_\mp &= 0 \\ \boldsymbol{\epsilon}_\pm^* \cdot \boldsymbol{\epsilon}_3 &= 0 \\ \boldsymbol{\epsilon}_\pm^* \cdot \boldsymbol{\epsilon}_\pm &= 1 \end{aligned} \tag{7.23}$$

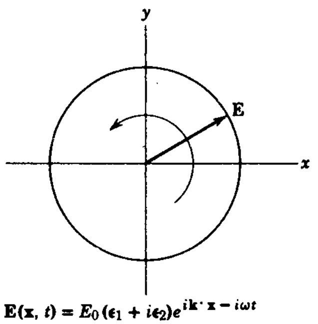
Then a general representation, equivalent to (7.19), is

$$\mathbf{E}(\mathbf{x}, t) = (E_+\boldsymbol{\epsilon}_+ + E_-\boldsymbol{\epsilon}_-)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t} \tag{7.24}$$

where E_+ and E_- are complex amplitudes. If E_+ and E_- have different magnitudes, but the same phase, (7.24) represents an elliptically polarized wave with principal axes of the ellipse in the directions of $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$. The ratio of semimajor to semiminor axis is $|(1 + r)/(1 - r)|$, where $E_-/E_+ = r$. If the amplitudes have a phase difference between them, $E_-/E_+ = re^{i\alpha}$, then it is easy to show that the ellipse traced out by the $\mathbf{E}$ vector has its axes rotated by an angle $(\alpha/2)$. Figure 7.4 shows the general case of elliptical polarization and the ellipses traced out by both $\mathbf{E}$ and $\mathbf{B}$ at a given point in space.

For $r = \pm 1$ we get back a linearly polarized wave.

The polarization content of a plane electromagnetic wave is known if it can be written in the form of either (7.19) or (7.24) with known coefficients (E_1, E_2) or (E_+, E_-). In practice, the converse problem arises. Given that the wave is of the form (7.8), how can we determine from observations on the beam the state of polarization in all its particulars? A useful vehicle for this are the four *Stokes*



$\mathbf{E}(\mathbf{x}, t) = E_0(\boldsymbol{\epsilon}_1 + i\boldsymbol{\epsilon}_2)e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$

Figure 7.3 Electric field of a circularly polarized wave.

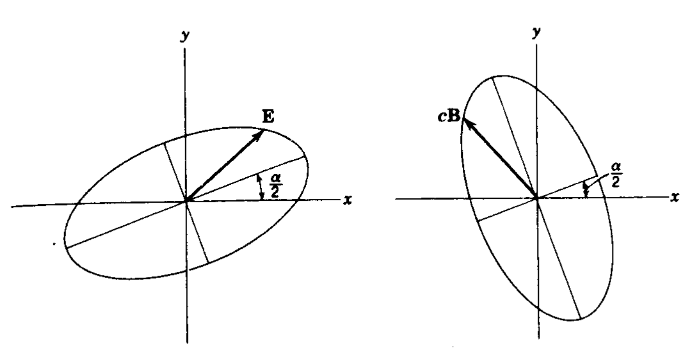


Figure 7.4 Electric field and magnetic induction for an elliptically polarized wave.

parameters, proposed by G. G. Stokes in 1852. These parameters are quadratic in the field strength and can be determined through intensity measurements only, in conjunction with a linear polarizer and a quarter-wave plate or equivalents. Their measurement determines completely the state of polarization of the wave.

The Stokes parameters can be motivated by observing that for a wave propagating in the z direction, the scalar products,

$$\boldsymbol{\epsilon}_1 \cdot \mathbf{E}, \qquad \boldsymbol{\epsilon}_2 \cdot \mathbf{E}, \qquad \boldsymbol{\epsilon}_+^* \cdot \mathbf{E}, \qquad \boldsymbol{\epsilon}_-^* \cdot \mathbf{E} \tag{7.25}$$

are the amplitudes of radiation, respectively, with linear polarization in the x direction, linear polarization in the y direction, positive helicity, and negative helicity. Note that for circular polarization the *complex conjugate* of the appropriate polarization vector must be used, in accord with (7.23). The squares of these amplitudes give a measure of the intensity of each type of polarization. Phase information is also needed; this is obtained from cross products. We give definitions of the Stokes parameters with respect to both the linear polarization and the circular polarization bases, in terms of the projected amplitudes (7.25) and also explicitly in terms of the magnitudes and relative phases of the components. For the latter purpose we define each of the scalar coefficients in (7.19) and (7.24) as a magnitude times a phase factor:

$$\begin{aligned} E_1 &= a_1 e^{i\delta_1}, \qquad & E_2 &= a_2 e^{i\delta_2} \\ E_+ &= a_+ e^{i\delta_+}, \qquad & E_- &= a_- e^{i\delta_-} \end{aligned} \tag{7.26}$$

In terms of the linear polarization basis ($\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$), the Stokes parameters are*

$$\begin{aligned} s_0 &= |\boldsymbol{\epsilon}_1 \cdot \mathbf{E}|^2 + |\boldsymbol{\epsilon}_2 \cdot \mathbf{E}|^2 = a_1^2 + a_2^2 \\ s_1 &= |\boldsymbol{\epsilon}_1 \cdot \mathbf{E}|^2 - |\boldsymbol{\epsilon}_2 \cdot \mathbf{E}|^2 = a_1^2 - a_2^2 \\ s_2 &= 2\,\mathrm{Re}[(\boldsymbol{\epsilon}_1 \cdot \mathbf{E})^*(\boldsymbol{\epsilon}_2 \cdot \mathbf{E})] = 2a_1a_2\cos(\delta_2 - \delta_1) \\ s_3 &= 2\,\mathrm{Im}[(\boldsymbol{\epsilon}_1 \cdot \mathbf{E})^*(\boldsymbol{\epsilon}_2 \cdot \mathbf{E})] = 2a_1a_2\sin(\delta_2 - \delta_1) \end{aligned} \tag{7.27}$$

If the circular polarization basis ($\boldsymbol{\epsilon}_+, \boldsymbol{\epsilon}_-$) is used instead, the definitions read

$$\begin{aligned} s_0 &= |\boldsymbol{\epsilon}_+^* \cdot \mathbf{E}|^2 + |\boldsymbol{\epsilon}_-^* \cdot \mathbf{E}|^2 = a_+^2 + a_-^2 \\ s_1 &= 2\,\mathrm{Re}[(\boldsymbol{\epsilon}_+^* \cdot \mathbf{E})^*(\boldsymbol{\epsilon}_-^* \cdot \mathbf{E})] = 2a_+a_-\cos(\delta_- - \delta_+) \\ s_2 &= 2\,\mathrm{Im}[(\boldsymbol{\epsilon}_+^* \cdot \mathbf{E})^*(\boldsymbol{\epsilon}_-^* \cdot \mathbf{E})] = 2a_+a_-\sin(\delta_- - \delta_+) \\ s_3 &= |\boldsymbol{\epsilon}_+^* \cdot \mathbf{E}|^2 - |\boldsymbol{\epsilon}_-^* \cdot \mathbf{E}|^2 = a_+^2 - a_-^2 \end{aligned} \tag{7.28}$$

*The notation for the Stokes parameters is unfortunately not uniform. Stokes himself used (*A, B, C, D*); other labelings are (*I, Q, U, V*) and (*I, M, C, S*). Our notation is that of *Born and Wolf*.

The expressions (7.27) and (7.28) show an interesting rearrangement of roles of the Stokes parameters with respect to the two bases. The parameter s_0 measures the relative intensity of the wave in either case. The parameter s_1 gives the preponderance of x-linear polarization over y-linear polarization, while s_2 and s_3 in the linear basis give phase information. We see from (7.28) that s_3 has the interpretation of the difference in relative intensity of positive and negative helicity, while in this basis s_1 and s_2 concern the phases. The four Stokes parameters are not independent, since they depend on only three quantities, a_1, a_2, and $\delta_2 - \delta_1$. They satisfy the relation

$$s_0^2 = s_1^2 + s_2^2 + s_3^2 \tag{7.29}$$

Discussion of the operational steps needed to measure the Stokes parameters and so determine the state of polarization of a plane wave would take us too far afield. We refer the reader to Section 13.13 of *Stone* for details. Also neglected, except for the barest mention, is the important problem of quasi-monochromatic radiation. Beams of radiation, even if monochromatic enough for the purposes at hand, actually consist of a superposition of finite wave trains. By Fourier's theorem they thus contain a range of frequencies and are not completely monochromatic. One way of viewing this is to say that the magnitudes and phases (a_i, δ_i) in (7.26) vary slowly in time, slowly, that is, when compared to the frequency ω. The observable Stokes parameters then become averages over a relatively long time interval, and are written as

$$s_2 = 2\langle a_1 a_2 \cos(\delta_2 - \delta_1)\rangle$$

for example, where the angle brackets indicate the macroscopic time average. One consequence of the averaging process is that the Stokes parameters for a quasi-monochromatic beam satisfy an inequality,

$$s_0^2 \geq s_1^2 + s_2^2 + s_3^2$$

rather than the equality, (7.29). "Natural light," even if monochromatic to a high degree, has $s_1 = s_2 = s_3 = 0$. Further discussion of quasi-monochromatic light and partial coherence can be found in *Born and Wolf*, Chapter 10.

An astrophysical example of the use of Stokes parameters to describe the state of polarization is afforded by the study of optical and radiofrequency radiation from the pulsar in the Crab nebula. The optical light shows some small amount of linear polarization, while the radio emission at $\omega = 2.5 \times 10^9\ \text{s}^{-1}$ has a high degree of linear polarization.* At neither frequency is there evidence for circular polarization. Information of this type obviously helps to elucidate the mechanism of radiation from these fascinating objects.

7.3 *Reflection and Refraction of Electromagnetic Waves at a Plane Interface Between Dielectrics*

The reflection and refraction of light at a plane surface between two media of different dielectric properties are familiar phenomena. The various aspects of the phenomena divide themselves into two classes.

*See *The Crab Nebula and Related Supernova Remnants*, eds. M. C. Kafatos and R. B. C. Henry, Cambridge University Press, New York (1985).

1. Kinematic properties:
   (a) Angle of reflection equals angle of incidence.
   (b) Snell's law: $(\sin i)/(\sin r) = n'/n$, where i, r are the angles of incidence and refraction, while n, n' are the corresponding indices of refraction.
2. Dynamic properties:
   (a) Intensities of reflected and refracted radiation.
   (b) Phase changes and polarization.

The kinematic properties follow immediately from the wave nature of the phenomena and from the fact that there are boundary conditions to be satisfied. But they do not depend on the detailed nature of the waves or the boundary conditions. On the other hand, the dynamic properties depend entirely on the specific nature of electromagnetic fields and their boundary conditions.

The coordinate system and symbols appropriate to the problem are shown in Fig. 7.5. The media below and above the plane $z = 0$ have permeabilities and permittivities μ, ϵ and μ', ϵ', respectively. The indices of refraction, defined through (7.5), are $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$ and $n' = \sqrt{\mu'\epsilon'/\mu_0\epsilon_0}$. A plane wave with wave vector $\mathbf{k}$ and frequency ω is incident from medium μ, ϵ. The refracted and reflected waves have wave vectors $\mathbf{k}'$ and $\mathbf{k}''$, respectively, and $\mathbf{n}$ is a unit normal directed from medium μ, ϵ into medium μ', ϵ'.

According to (7.18), the three waves are:

INCIDENT

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$
$$\mathbf{B} = \sqrt{\mu\epsilon}\,\frac{\mathbf{k} \times \mathbf{E}}{k} \tag{7.30}$$

REFRACTED

$$\mathbf{E}' = \mathbf{E}_0' e^{i\mathbf{k}'\cdot\mathbf{x} - i\omega t}$$
$$\mathbf{B}' = \sqrt{\mu'\epsilon'}\,\frac{\mathbf{k}' \times \mathbf{E}'}{k'} \tag{7.31}$$

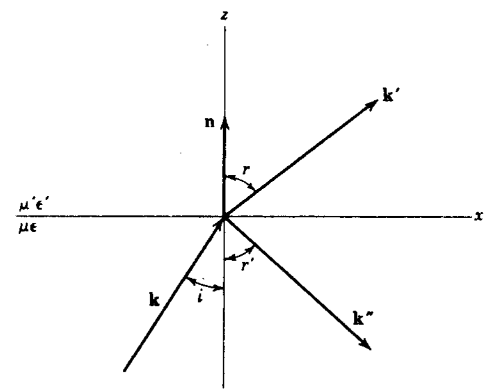


Figure 7.5 Incident wave **k** strikes plane interface between different media, giving rise to a reflected wave **k**″ and a refracted wave **k**′.

REFLECTED

$$\mathbf{E}'' = \mathbf{E}_0'' e^{i\mathbf{k}''\cdot\mathbf{x} - i\omega t}$$

$$\mathbf{B}'' = \sqrt{\mu\epsilon}\,\frac{\mathbf{k}'' \times \mathbf{E}''}{k} \tag{7.32}$$

The wave numbers have the magnitudes

$$\begin{aligned} |\mathbf{k}| &= |\mathbf{k}''| = k = \omega\sqrt{\mu\epsilon} \\ |\mathbf{k}'| &= k' = \omega\sqrt{\mu'\epsilon'} \end{aligned} \tag{7.33}$$

The existence of boundary conditions at $z = 0$, which must be satisfied at all points on the plane at all times, implies that the spatial (and time) variation of all fields must be the same at $z = 0$. Consequently, we must have the phase factors all equal at $z = 0$,

$$(\mathbf{k}\cdot\mathbf{x})_{z=0} = (\mathbf{k}'\cdot\mathbf{x})_{z=0} = (\mathbf{k}''\cdot\mathbf{x})_{z=0} \tag{7.34}$$

independent of the nature of the boundary conditions. Equation (7.34) contains the kinematic aspects of reflection and refraction. We see immediately that all three wave vectors must lie in a plane. Furthermore, in the notation of Fig. 7.5,

$$k \sin i = k' \sin r = k'' \sin r' \tag{7.35}$$

Since $k'' = k$, we find $i = r'$; the angle of incidence equals the angle of reflection. Snell's law is

$$\frac{\sin i}{\sin r} = \frac{k'}{k} = \sqrt{\frac{\mu'\epsilon'}{\mu\epsilon}} = \frac{n'}{n} \tag{7.36}$$

The dynamic properties are contained in the boundary conditions—normal components of **D** and **B** are continuous; tangential components of **E** and **H** are continuous. In terms of fields (7.30)–(7.32) these boundary conditions at $z = 0$ are:

$$\begin{aligned} &[\epsilon(\mathbf{E}_0 + \mathbf{E}_0'') - \epsilon'\mathbf{E}_0']\cdot\mathbf{n} = 0 \\ &[\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'' - \mathbf{k}'\times\mathbf{E}_0']\cdot\mathbf{n} = 0 \\ &(\mathbf{E}_0 + \mathbf{E}_0'' - \mathbf{E}_0')\times\mathbf{n} = 0 \\ &\left[\frac{1}{\mu}(\mathbf{k}\times\mathbf{E}_0 + \mathbf{k}''\times\mathbf{E}_0'') - \frac{1}{\mu'}(\mathbf{k}'\times\mathbf{E}_0')\right]\times\mathbf{n} = 0 \end{aligned} \tag{7.37}$$

In applying these boundary conditions it is convenient to consider two separate situations, one in which the incident plane wave is linearly polarized with its polarization vector perpendicular to the plane of incidence (the plane defined by **k** and **n**), and the other in which the polarization vector is parallel to the plane of incidence. The general case of arbitrary elliptic polarization can be obtained by appropriate linear combinations of the two results, following the methods of Section 7.2.

We first consider the electric field perpendicular to the plane of incidence, as shown in Fig. 7.6*a*. All the electric fields are shown directed away from the viewer. The orientations of the **B** vectors are chosen to give a positive flow of energy in the direction of the wave vectors. Since the electric fields are all parallel

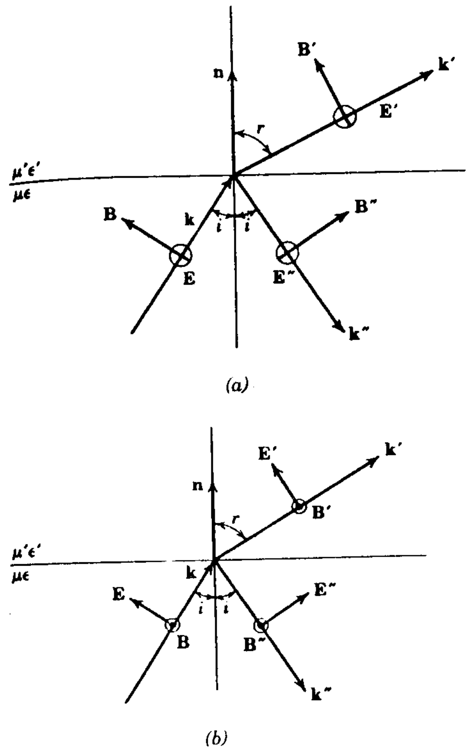


Figure 7.6 Reflection and refraction with polarization (*a*) perpendicular and (*b*) parallel to the plane of incidence.

to the surface, the first boundary condition in (7.37) yields nothing. The third and fourth equations in (7.37) give

$$\begin{aligned} E_0 + E_0'' - E_0' &= 0 \\ \sqrt{\frac{\epsilon}{\mu}}(E_0 - E_0'')\cos i - \sqrt{\frac{\epsilon'}{\mu'}}E_0'\cos r &= 0 \end{aligned} \tag{7.38}$$

while the second, using Snell's law, duplicates the third. The relative amplitudes of the refracted and reflected waves can be found from (7.38). These are:

E PERPENDICULAR TO PLANE OF INCIDENCE

$$\begin{aligned} \frac{E_0'}{E_0} &= \frac{2n\cos i}{n\cos i + \dfrac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \\ \frac{E_0''}{E_0} &= \frac{n\cos i - \dfrac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}}{n\cos i + \dfrac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2 i}} \end{aligned} \tag{7.39}$$

The square root in these expressions is $n'\cos r$, but Snell's law has been used to express it in terms of the angle of incidence. For optical frequencies it is usually permitted to put $\mu/\mu' = 1$. Equations (7.39), and (7.41) and (7.42) below, are most often employed in optical contexts with real n and n', but they are also valid for complex dielectric constants.

If the electric field is parallel to the plane of incidence, as shown in Fig. 7.6b, the boundary conditions involved are normal D, tangential E, and tangential H [the first, third, and fourth equations in (7.37)]. The tangential E and H continuous demand that

$$\begin{aligned} \cos i(E_0 - E_0'') - \cos r\, E_0' &= 0 \\ \sqrt{\frac{\epsilon}{\mu}}(E_0 + E_0'') - \sqrt{\frac{\epsilon'}{\mu'}}\, E_0' &= 0 \end{aligned} \tag{7.40}$$

Normal D continuous, plus Snell's law, merely duplicates the second of these equations. The relative amplitudes of refracted and reflected fields are therefore

E PARALLEL TO PLANE OF INCIDENCE

$$\begin{aligned} \frac{E_0'}{E_0} &= \frac{2nn'\cos i}{\dfrac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \\ \frac{E_0''}{E_0} &= \frac{\dfrac{\mu}{\mu'}n'^2\cos i - n\sqrt{n'^2 - n^2\sin^2 i}}{\dfrac{\mu}{\mu'}n'^2\cos i + n\sqrt{n'^2 - n^2\sin^2 i}} \end{aligned} \tag{7.41}$$

For normal incidence ($i = 0$), both (7.39) and (7.41) reduce to

$$\left.\begin{aligned} \frac{E_0'}{E_0} &= \frac{2}{\sqrt{\dfrac{\mu\epsilon'}{\mu'\epsilon}} + 1} \rightarrow \frac{2n}{n' + n} \\ \frac{E_0''}{E_0} &= \frac{\sqrt{\dfrac{\mu\epsilon'}{\mu'\epsilon}} - 1}{\sqrt{\dfrac{\mu\epsilon'}{\mu'\epsilon}} + 1} \rightarrow \frac{n' - n}{n' + n} \end{aligned}\right\} \tag{7.42}$$

where the results on the right hold for $\mu' = \mu$. For the reflected wave the sign convention is that for polarization parallel to the plane of incidence. This means that if $n' > n$ there is a phase reversal for the reflected wave.

7.4 Polarization by Reflection and Total Internal Reflection; Goos–Hänchen Effect

Two aspects of the dynamical relations on reflection and refraction are worthy of mention. The first is that for polarization parallel to the plane of incidence there is an angle of incidence, called *Brewster's angle*, for which there is no reflected wave. With $\mu' = \mu$ for simplicity, we find that the amplitude of the reflected wave in (7.41) vanishes when the angle of incidence is equal to Brewster's angle,

$$i_B = \tan^{-1}\left(\frac{n'}{n}\right) \tag{7.43}$$

For a typical ratio $n'/n = 1.5$, $i_B \simeq 56°$. If a plane wave of mixed polarization is incident on a plane interface at the Brewster angle, the reflected radiation is *completely* plane-*polarized* with polarization vector *perpendicular* to the plane of incidence. This behavior can be utilized to produce beams of plane-polarized light but is not as efficient as other means employing anisotropic properties of some dielectric media. Even if the unpolarized wave is reflected at angles other than the Brewster angle, there is a tendency for the reflected wave to be predominantly polarized perpendicular to the plane of incidence. The success of dark glasses that selectively transmit only one direction of polarization depends on this fact. In the domain of radiofrequencies, receiving antennas can be oriented to discriminate against surface-reflected waves (and also waves reflected from the ionosphere) in favor of the directly transmitted wave.

The second phenomenon is called *total internal reflection*. The word "internal" implies that the incident and reflected waves are in a medium of larger index of refraction than the refracted wave ($n > n'$). Snell's law (7.36) shows that, if $n > n'$, then $r > i$. Consequently, $r = \pi/2$ when $i = i_0$, where

$$i_0 = \sin^{-1}\left(\frac{n'}{n}\right) \tag{7.44}$$

For waves incident at $i = i_0$, the refracted wave is propagated parallel to the surface. There can be no energy flow across the surface. Hence at that angle of incidence there must be total reflection. What happens if $i > i_0$? To answer this we first note that, for $i > i_0$, $\sin r > 1$. This means that r is a complex angle with a purely imaginary cosine.

$$\cos r = i\sqrt{\left(\frac{\sin i}{\sin i_0}\right)^2 - 1} \tag{7.45}$$

The meaning of these complex quantities becomes clear when we consider the propagation factor for the refracted wave:

$$e^{i\mathbf{k}'\cdot\mathbf{x}} = e^{ik'(x\sin r + z\cos r)} = e^{-k'[(\sin i/\sin i_0)^2 - 1]^{1/2}z}\, e^{ik'(\sin i/\sin i_0)x} \tag{7.46}$$

This shows that, for $i > i_0$, the refracted wave is propagated only parallel to the surface and is attenuated exponentially beyond the interface. The attenuation occurs within a very few wavelengths of the boundary, except for $i \simeq i_0$.

Even though fields exist on the other side of the surface there is no energy flow through the surface. Hence total internal reflection occurs for $i \geq i_0$. The lack of energy flow can be verified by calculating the time-averaged *normal* component of the Poynting vector just inside the surface:

$$\mathbf{S}\cdot\mathbf{n} = \frac{1}{2}\,\mathrm{Re}[\mathbf{n}\cdot(\mathbf{E}'\times\mathbf{H}'^*)] \tag{7.47}$$