- Synchrotron emission: A brief history
- Examples
  - Cyclotron radiation
  - Synchrotron radiation
  - Synchrotron power from a single electron
  - Relativistic beaming
  - Relativistic Doppler effect
  - Spectrum of synchrotron emission from a single electron
1933: Karl Jansky (Bell Labs) builds an antenna designed to receive radio waves at 20.5 MHz

- Discovers a steady source of radio emission from the sky that varied on a cycle of 23 hours 56 minutes (sidereal day)

- Comparing his observations to optical astronomical maps, he realized that this radiation was coming from the Milky Way and was strongest towards Sagittarius, the center of our Galaxy

- What is this mysterious radiation?
Synchrotron Radiation

- Synchrotrons are large particle accelerators that accelerate particles in a magnetic field

- 1947: General Electric discover “an arc in the tube” of their synchrotron accelerator

- coined the term ‘synchrotron radiation’

- 1950: Hannes Alfven and Bernt Herlofsen suggest that the mysterious ‘radio stars’ generate their radio emission by synchrotron radiation

- 1950: Kiepenheuer suggests that the galactic radio emission was due to cosmic-ray electrons gyrating in the galactic magnetic field producing synchrotron radiation

- Synchrotron radiation now known to be one of dominant sources of radiation in our universe

---

Cosmic Rays as the Source of General Galactic Radio Emission

K. G. Kiepenheuer
Yerkes Observatory, University of Chicago, Wisconsin, Wisconsin
June 30, 1930

The galactic radio emission is not a thermal free-free radiation of interstellar gas, as was first believed. The electron temperature would have to be of the order of 100,000° in contradiction to all spectroscopic evidence which gives values around 10,000°. Stelae could be considered as sources only under very artificial assumptions. The observed intensities, which must come from the outermost layers of stellar atmospheres, could not be blackbody radiation and might be understood only in terms of coherent plasma oscillations of extended regions. The formation and maintenance of these oscillations is hardly possible in stellar atmospheres.

It will now be shown that the general cosmic radiation of our star system is a high-frequency source of sufficient power. In interstellar space, at least outside the interstellar clouds which occupy about 1% of space, the mean density of kinetic energy ought to be of the same order as the magnetic field energy; therefore, fields of around 10^-4 gauss are to be expected. An energetic electron with energy W = 10^9 eV, which is decelerating in this field, is radiating electromagnetic energy into a very narrow cone whose angular aperture is 10^-15 W in the direction of motion. Therefore, an observer at rest receives very short pulses corresponding to a frequency which is very much higher than the classical Larmor frequency, σo. The mean spectral intensity distribution of this radiation will then be

\[ P(\nu) = \frac{1}{2} \sigma_o E \frac{W}{\nu^2} \]

for \( W > \sigma_o E \), where \( W \) is the radius of the electron's circular orbit and \( \sigma_o = 4\pi^2 \mu_0 m_e c^3 \). If \( n_e \) is the number of electrons per cm^3 with energy \( W \), the emissivity of high-frequency radiation will be

\[ \varepsilon = n_e \sigma_o E \frac{W}{\nu^2} \]

This increases steadily with frequency until \( \nu = \sigma_o \), and then decreases rapidly. The observed distribution within the frequency range of 10 to 3000 Mc/second turns to be \( \nu^{-2} \). We therefore expect to be already in a region with \( \nu \gg \sigma_o \). Also the
Synchrotron Radiation in Astrophysics (Examples)

- Radio emission (<30 GHz) from ‘normal’ galaxies

- Non-thermal optical emission from the Crab Nebula (and pulsar)

- Optical and X-ray continuum emission from AGN

Synchrotron radiation of relativistic and ultra-relativistic electrons is a dominant process in high energy astrophysics
Why? Magnetic fields permeate our universe!

<table>
<thead>
<tr>
<th></th>
<th>Magnetic Field (Gauss)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interstellar medium</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>Crab Nebula:</td>
<td>$10^{-3}$</td>
</tr>
<tr>
<td>This room:</td>
<td>0.3</td>
</tr>
<tr>
<td>Earth’s magnetic pole:</td>
<td>0.5</td>
</tr>
<tr>
<td>Jupiter’s magnetic pole:</td>
<td>14</td>
</tr>
<tr>
<td>Refrigerator magnet:</td>
<td>50</td>
</tr>
<tr>
<td>Sunspot:</td>
<td>1500</td>
</tr>
<tr>
<td>Brown Dwarf:</td>
<td>5000</td>
</tr>
<tr>
<td>Strongest lab magnetic field (without destroying the lab):</td>
<td>$10^6$</td>
</tr>
<tr>
<td>White Dwarf:</td>
<td>up to $10^9$</td>
</tr>
<tr>
<td>Neutron Star:</td>
<td>up to $10^{15}$</td>
</tr>
</tbody>
</table>

1 Gauss = $10^{-4}$ Tesla
Consider an electron moving with velocity \( \ll c \).

The magnetic force is given by the Lorentz force:

\[
\mathbf{F} = e\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)
\]

The magnetic force is perpendicular to the particle velocity:

\[\mathbf{F} \cdot \mathbf{v} = 0\]

No power is transferred to the electron and its kinetic energy \((m_e \mathbf{v}^2/2)\) remains constant.

Therefore, \( \mathbf{v} \) remains constant. However, \( \mathbf{v}_\perp \) is also constant, therefore \( \mathbf{v}_\parallel \) is constant.

In a constant magnetic field, the electron moves along the magnetic field line on a uniform helical path with constant linear and angular speeds.

In the inertial frame of the electron, the electron orbits in a circle perpendicular to the magnetic field with angular velocity \( \omega \) needed to balance centripetal and magnetic forces.

Electron cyclotron frequency

\[
\omega_{cyc} = \frac{eB}{m_e c}
\]

Let \( \nu_{cyc} = \omega_{cyc}/2\pi \):

\[
\nu_{cyc} \text{ MHz} = 2.8 \left(\frac{B}{\text{Gauss}}\right)
\]
Non-relativistic case: Cyclotron radiation

- Example 1: Hercules X-1 (X-ray binary)
- Absorption feature at 34 KeV in X-ray spectrum
- Cyclotron absorption by hot gas near poles of magnetized neutron star
- Can be used to measure magnetic field strength of the neutron star

\[ \nu = \frac{E}{\hbar} \approx \frac{34 \times 10^3 \text{ eV} \times 1.60 \times 10^{-12} \text{ erg eV}^{-1}}{6.63 \times 10^{-27} \text{ erg s}} \approx 8.2 \times 10^{18} \text{ Hz} \]

\[ \nu_{\text{cyc}} \left( \text{MHz} \right) = 2.8 \left( \frac{B}{\text{Gauss}} \right) \]

\[ B \approx 2.9 \times 10^{12} \text{ Gauss} \]

Recall Lorentz transformations:

\[
\begin{align*}
    x &= \gamma (x' + vt') \\
    y &= y' \\
    z &= z' \\
    t &= \gamma (t' + \beta x' / c)
\end{align*}
\]

\[
\begin{align*}
    x' &= \gamma (x - vt) \\
    y' &= y \\
    z' &= z \\
    t' &= \gamma (t - \beta x / c)
\end{align*}
\]

where \( \beta \equiv v / c \) Lorentz factor \( \gamma \equiv (1 - \beta^2)^{-1/2} \)

- Electron rest energy \( E_0 = m_e c^2 = 0.51 \text{ MeV} \)
- Relativistic energy \( E = \gamma m_e c^2 \)

\[
\omega_{cy} = \frac{eB}{m_e c} \quad \Rightarrow \quad \omega_g = \frac{eB}{\gamma m_e c}
\]

- A cosmic ray with energy \( 10^{10} \text{ eV} \) has \( \gamma = 20,000 \)

- In the Galactic magnetic field \((5 \mu\text{G})\) has gyrofrequency of \( 7 \times 10^{-4} \text{ Hz} \)!
- In the lab frame $K$, an electron is moving in a magnetic field of $B$, with a velocity $\beta \equiv \frac{u}{c}$.

- We know how to calculate the power from an accelerating electron when non-relativistic.

- We can:
  1) transform to an inertial frame $K'$, in which the particle is instantaneously at rest
  2) calculate the radiation field,
  3) calculate the power (Lorentz-invariant)
  4) transform back

$$P = \frac{2q^2u^2}{3c^3}$$

Larmor’s formula
Synchrotron power from a single electron

\[ P = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 U_B \]

- For \( \gamma \gg 1 \rightarrow \beta \approx 1 \rightarrow P \propto \gamma^2 \)

- A cosmic ray with energy \( 10^{10} \) eV has \( \gamma = 20,000 \)

- \( 4 \times 10^8 \) more power than the limit \( \ll c \)!
Relativistic Beaming

- How can synchrotron radiation be emitted at frequencies >> \( \omega_g = \omega_{cyc}/\gamma \)

- Relativistic aberration causes the Larmor dipole pattern in the electron frame to become beamed sharply in the direction of motion as \( v \) approaches \( c \)

- For an electron moving in the x direction (from Lorentz transformation for differentials)

\[
\begin{align*}
\nu_x &\equiv \frac{dx}{dt} = \frac{dx'}{dt'} \frac{dt'}{dt} \\
\nu_x &= \gamma \left( \frac{dx'}{dt'} + \nu \frac{dt'}{dt} \right) \left( \frac{dt}{dt'} \right)^{-1} \\
\nu_x &= \nu_x' + \nu \left( 1 + \beta \frac{\nu_x'}{c} \right)^{-1} \\
\nu_x &= (\nu_x' + \nu) \left( 1 + \beta \nu_x' \frac{\nu}{c} \right)^{-1}.
\end{align*}
\]

- Consider the synchrotron photons emitted with speed \( c \) at an angle \( \Theta' \) from the x’ axis. Let \( \nu_x' \) and \( \nu_y' \) be the projections of the photon speed onto the x’ and y’ axes. Then

\[
\begin{align*}
\cos \theta' &= \frac{\nu_x'}{c} \\
\sin \theta' &= \frac{\nu_y'}{c} \\
\cos \theta &= \frac{\nu_x}{c} \\
\sin \theta &= \frac{\nu_y}{c}
\end{align*}
\]

in the electron's frame in observer's frame
Using our equations for velocities, we get

\[ \cos \theta = \left( \frac{\gamma^2 + \gamma'}{1 + \beta \gamma'} \right) \frac{1}{c} = \left( \frac{c \cos \theta' + \nu}{1 + \beta c \cos \theta' / c} \right) \frac{1}{c} \]

\[ \cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'} \]

\[ \sin \theta = \frac{\nu}{c \gamma (1 + \beta \gamma') / c} \]

\[ \sin \theta = \frac{\sin \theta'}{\gamma (1 + \beta \cos \theta')} \]

Image credit: Giampaolo Pisano

Dipole radiation pattern

The observer sees a short pulse of radiation emitted during only the tiny fraction

\[ \frac{2}{2\pi \gamma} = \frac{1}{\pi \gamma} \]
- The beamed radiation from a relativistic electron is visible only while the electron’s velocity points within $\pm \frac{1}{\gamma}$ of the line of sight. During that time $\Delta t$, the electron moves a distance $\Delta x = v \Delta t$ towards the observer, almost keeping up with the radiation that travels a distance $c \Delta t$. As a result, the observed pulse duration is shortened by a factor $(1-v/c)$.

$$\Delta t_p = t(\text{end of observed pulse}) - t(\text{start of observed pulse})$$

$$\Delta t_p = \frac{\Delta x}{v} + \frac{(x - \Delta x)}{c} - \frac{x}{c}$$

Note that

$$\Delta t_p = \frac{\Delta x}{v} - \frac{\Delta x}{c} = \frac{\Delta x}{v} \left(1 - \frac{v}{c}\right) \ll \frac{\Delta x}{v}$$

In the limit $v \to c$,

$$\left(1 - \frac{v}{c}\right) = \left(1 - \frac{v}{c}\right) \frac{1 + v/c}{1 + v/c} = \frac{1 - v^2/c^2}{1 + v/c} \approx \frac{\gamma^2}{2} = \frac{1}{2\gamma^2}$$

$$\Delta t_p = \frac{\Delta x}{v} \frac{1}{2\gamma^2} = \frac{\Delta \theta}{1} \frac{1}{\omega_g} \frac{1}{2\gamma^2}.$$  

Recall that $\Delta \theta \approx 2/\gamma$ so

$$\Delta t_p = \frac{2}{\gamma \omega_g 2\gamma^2} = \frac{1}{\gamma^3 \omega_g} = \frac{1}{\gamma \omega_{cyc}}$$

Allowing for the motion of the electron parallel to the magnetic field, we replace the total magnetic field by its perpendicular component $B \sin \alpha$, yielding

$$\Delta t_p = \frac{1}{\gamma^2 \omega_{cyc}} \sin \alpha$$

- Remember our cosmic ray with energy $10^{10}$ eV and $\gamma = 20,000$

- In the Galactic magnetic field (5 $\mu$G) has gyrofrequency of $7 \times 10^{-4}$ Hz

$$\Delta t_p = \frac{1}{\gamma^3 \omega_g} = 1.8 \times 10^{-10} \text{ s}$$
- Observed radiation distribution can be decomposed by Fourier analysis in sum of dipoles radiating at harmonics of the gyrofrequency.

**Critical frequency**: The maximum Fourier component of the pulse is expected to be:

\[ \nu_c \approx \Delta t_p^{-1} \approx \gamma^3 \nu_g \]

The exact calculation generalized for any pitch angle gives:

\[ \nu_c = \frac{3}{2} \gamma^2 \nu_{cy} \sin \alpha \]

The synchrotron power spectrum of a single electron is:

\[ P(\nu) = \frac{\sqrt{3} e^3 B \sin \alpha}{mc^2} \left( \frac{\nu}{\nu_c} \right) \int_{\nu/\nu_c}^{\infty} K_{5/3}(\eta) \eta d\eta \]

where \( K_{5/3} \) is a modified Bessel function.

Note \( \nu_c \propto \gamma^2 \propto E^2 \)

High energy electrons radiate at higher frequencies!
Let’s consider a real source...

The observed spectrum is the convolution of the electron energy distribution with the spectrum from a single electron.

What kind of spectrum is typical for an astrophysical source?
Spectral energy of cosmic rays electrons approximated by power-law distribution

\[ N(E)dE = \kappa E^{-p}dE \]

- See Readhead (Ch. 3) for discussion of why cosmic rays have a power law energy distribution.

- Because \( N(E) \) is nearly a power law over more than two decades of energy and the critical frequency \( \nu_c \) is proportional to \( E^2 \), we expect the synchrotron spectrum to reflect this power law over a frequency range of at least \( (10^2)^2 = 10^4 \)

- Remo \[ P = \frac{-dE}{dt} = \frac{4}{3} \sigma \nu_c \beta^2 \gamma^2 U_B \]

- Let’s make a crude approximation: Each electron radiates all its power at a frequency close to the critical frequency

\[ \nu \approx \gamma^2 \nu_{\text{cy}c} \]

The energy spectrum of cosmic-ray electrons in the local interstellar medium (Casadei, D., & Bindi, V. 2004, ApJ, 612, 262). In the energy range above a few GeV, \( N(E) \) is a power law with slope \( p = 2.4 \).
Power Law electron energy distributions

The emission coefficient for an ensemble of electrons is

\[ j_\nu d\nu = \frac{-dE}{dt} N(E) dE \]

where \( E = \gamma m_e c^2 \approx \left( \frac{\nu}{\nu_{cyc}} \right)^{-1/2} m_e c^2 \)

Differentiating \( E \) gives

\[ \frac{dE}{d\nu} \approx \frac{m_e c^2 \nu^{-1/2}}{2 \nu_{cyc}^{1/2}} \]

so \( j_\nu \approx \left( \frac{4}{3} \sigma_T c^2 \gamma^2 U_B \right) (KE^{-p}) \frac{m_e c^2 \nu^{-1/2}}{2 \nu_{cyc}^{1/2}} \)

Eliminating \( E \) in favor of \( \nu/\nu_{cyc} \) and ignoring the physical constants in this equation for results in the proportionality

\[ j_\nu \propto \left( \frac{\nu}{\nu_{cyc}} \right)^{p/2} (\nu_{cyc})^{-1/2} \]

since \( \nu_{cyc} \propto B \) we get:

\[ j_\nu \propto \left( \frac{\nu}{B} \right)^{p/2} (\nu B)^{-1/2} \]

\[ j_\nu \propto B^{(p+1)/2} \nu^{(1-p)/2} \]

Remember spectral index defined such that \( S \propto \nu^{-\alpha} \propto \nu^{(1-p)/2} \)

\[ \alpha = \left( \frac{p-1}{2} \right) \]