# Dispersion measure: Confusion, Constants & Clarity

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**Abstract.** The dispersion measure (DM) is one of the key attributes of radio pulsars and Fast Radio Bursts (FRBs). There is a mistaken view that the DM is an accurate measure of the column density of electrons between the observer and the source. To start with, the DM, unlike a true column density, is not a Lorentz invariant. Next, the DM also includes contribution from ions and is sensitive to the temperature of the plasma in the intervening clouds. Separately, the primary observable is the dispersion slope, $\mathcal{D} \equiv \Delta(t)/\Delta(\nu^{-2})$, where $t(\nu)$ is the arrival time at frequency, $\nu$. A scaling factor composed of physical and astronomical constants is needed to convert $\mathcal{D}$ to DM. In the early days of pulsar astronomy the relevant constants were defined to parts per million (ppm). As a result, a convention arose in which this conversion factor was fixed. Over time, several such conventions came about – recipe for confusion. Meanwhile, over the past several years, the SI system has been restructured and the parsec is now exactly defined. As a result, the present accuracy of the conversion factor is below a part per billion – many orders of magnitude better than the best measurement errors of $\mathcal{D}$. We are now in an awkward situation wherein the primary "observable", the DM, has incorrect scaling factor(s). To address these two concerns I propose that astronomers report the primary measurement, $\mathcal{D}$ (with a suggested normalization of $10^{15}$ Hz), and not the DM. Interested users can convert $\mathcal{D}$ to DM without the need to know secret handshakes of the pulsar timing communities.

## 1 Motivation: confusion

UT 28 April 2020 was a memorable and fabulous day for the field of magnetars and fast radio bursts (FRBs). On this day, CHIME (400–800 MHz) and STARE2 (1.2–1.5 GHz) found a burst towards SGR 1935+2154. The CHIME project reported a fluence of a few kJy ms[1]. STARE2 reported a fluence of over $1.5 \times 10^6$ Jy ms. The STARE2 burst, if placed at the nearest FRB galaxy, could be reasonably argued to be at the faint end of the FRB

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[1]Revised three weeks later to $6 \times 10^5$ Jy ms

population and thus a case was developed in which at least some FRBs arise from active magnetars.

The motivation for this note came from my experience in relating the CHIME burst to the STARE2 burst. A scaling constant, $K$ or $a = K^{-1}$ (hereafter, "$K|a$", using the Unix programming convention where "|" stands for OR), composed of fundamental physical and astronomical constants, is need to extrapolate the arrival time at one frequency given the arrival time at another frequency. Being uninitiated I computed this number using the latest constants. I was unable to relate the two bursts. Eventually I learnt that, over five decades go, the value of $K$ was "fixed" with the slop taken up by the dispersion measure. The hoary arcana of pulsar timing and separately my frustrating experience provided me the impetus for inquiry which, in due course, led to a number of interesting forays into the revised SI system, plasma physics and special relativity. I thought the resulting study was of likely interest to astronomers who are interested in pulsars and FRBs, hence this report.

This simple report is organized as follows. I start off by reviewing the basic physics of propagation of radio waves in interstellar plasma and summarize the literature in regard to $a|K$ (§2). Then, in §3, I review the titanic changes that have taken place in the definition of constants in astronomy (2012) and in the SI system (2019). In §4 I explore phenomena other than electrons which could contribute to dispersive delay. The list includes ions, temperature of the plasma (motion of electrons), ambient magnetic fields and relative motion between the observer and interstellar plasma. In view of this situation there is little need for knowing the *absolute* value of the DM at the parts per million (ppm) level, let alone at the parts per thousand (ppt) level. With the demonstration that the DM is sensitive to a host of phenomena I suggest that we abandon the DM as the primary observable that gets reported (§5). Instead, I urge radio astronomers to report the experimental measure, $\mathcal{D}$, which carries no ideology or expectation with it. The DM can be straightforwardly deduced from $\mathcal{D}$, to a precision only limited by that of the physical constants (currently standing at under a part per billion).

Below, unless stated otherwise, I will be using the Gaussian framework with associated CGS units – the standard practice in astronomy.

## 2 Dispersion of Radio Signals

Electromagnetic pulses propagating through cold plasma obey the dispersion relation

$$\omega^2 = \omega_e^2 + c^2 k^2 \tag{1}$$

where

$$\omega_e^2 = \frac{4\pi n_e e^2}{m_e} \tag{2}$$

is the electron plasma angular frequency [B. Draine 2011, §11.2]. For typical astrophysical conditions, $n_e$, the density of electrons range from $10^{-4}\,\mathrm{cm}^{-3}$ to $10^4\,\mathrm{cm}^{-3}$ and the electron plasma frequency $\nu_e = \omega_e/(2\pi) \approx 9(n_e/\mathrm{cm}^{-3})^{1/2}\,\mathrm{kHz}$, well below any conceivable observing frequency. The group velocity, $\nu_g$, is

$$v_g(\nu) \equiv \frac{\partial \omega}{\partial k} = c\left(1 - \frac{\omega_e^2}{\omega^2}\right)^{1/2} \tag{3}$$

which leads to a frequency-dependent arrival time,

$$
\begin{aligned}
t(\nu) &= \int_0^L \frac{1}{v_g(\nu)} dl \approx \int_0^L \frac{dl}{c}\left(1 + \frac{1}{2}\frac{\omega_e^2}{\omega^2}\right), \\
\tau(\nu) &= t(\nu) - t(\infty) = \frac{e^2 \times \mathrm{parsec}}{2\pi m_e c}\frac{1}{\nu^2}\mathrm{DM}.
\end{aligned}
\tag{4}
$$

Here, $L$ is the distance to the source, $t(\infty) = L/c$ and $\mathrm{DM} = \int_0^L n_e dl$ with $n_e$ in $\mathrm{cm}^{-3}$ and $L$ in pc. The small value of $\epsilon = (\omega_e/\omega)^2 \approx 10^{-10}\nu_9^{-2}n_e$ justifies the Taylor approximation made above; here, $\nu = 10^9\nu_9\,\mathrm{Hz}$. There is a long history of astronomers searching for $\epsilon^2$ term [e.g., Goldstein & James 1969; Tuntsov 2014], but with no success.

From Equation 4 we see that the DM is a product of $\nu^2$ and the dispersive delay to that frequency:

$$\mathrm{DM} = K\nu_9^2\tau(\nu) \tag{5}$$

where

$$a = K^{-1} \equiv \frac{e^2}{2\pi m_e c} \times \mathrm{parsec}. \tag{6}$$

The tradition of fixing the value of $K$ can be traced to Manchester & Taylor (1972): "Because of the uncertainty in propagation constants, most accurate published dispersions are quoted as the dispersion constant, $DC = \Delta t/\Delta(1/f^2)$, which is a directly measured quantity. However, for consistency we have converted these values to the more commonly quoted dispersion measures, $DM$, using the relation $DM(\mathrm{cm}^{-3}\,\mathrm{pc}) = 2.41000 \times 10^{-16}\,DC\,(\mathrm{Hz})$."

This suggestion took traction within some precision pulsar timing groups. However, in the *Handbook of Pulsar Astronomy*, a textbook used by by students and researchers entering the field of pulsars, the authors quote $a = 4.148\,808(3)\,\mathrm{GHz}^2\,\mathrm{cm}^3\,\mathrm{pc}^{-1}\,\mathrm{ms}$ [D. R. Lorimer & M. Kramer (2004)]. Some pulsar programs even use $a \equiv 4.15\,\mathrm{GHz}^2\,\mathrm{cm}^3\,\mathrm{pc}^{-1}\,\mathrm{ms}$.

Within a given school of pulsar timing the specific choice of $K|a$ does not matter since $K|a$ is fully covariant with the DM. However, as explained in the previous section, difficulty will arise if one uses the published DM without knowing the associated $K|a$.

# 3    Fundamental Constants

More than five decades have passed since precision pulsar timing began in earnest. The situation in regard to physical constants has substantially changed over this period. To start with, the IAU in 2012, via Resolution B2, fixed[2] the the value of AU. The same resolution defined the parsec to be the small-angle-approximation[3] distance to a star which subtends a parallax of an arcsecond across an AU-wide baseline. Next, on 2019 May 20 (the "World Metrology Day"), the Système Internationale d'Unités (SI)[4] announced permanent values for four fundamental constants[5], which amongst other things, replaced the kilogram and the ampere; see D. Newell [2014] for an overview. These two developments provide additional impetus to investigate matters related to precision pulsar timing.

Most astronomers routinely work within the Gaussian framework of equations (and related constants). However, with the latest revision to the SI system, some of the older SI to CGS conversions are no longer valid. Next, in the revised SI system, the charge of the electron, $e$, is exactly defined but in the Gaussian system, $e$ carries the uncertainty of the fine structure constant. I found that converting from SI units to CGS was prone to errors and so I will temporarily switch to the SI framework. The electron plasma frequency is then $\omega^2 = n_e e^2/(\epsilon_0 m_e)$ and so

$$K^{-1} = \frac{1}{8\pi^2} \frac{e^2}{\epsilon_0 m_e c} \times \text{parsec}. \tag{7}$$

The SI constants which constitute $K$ are

$$
\begin{aligned}
c &\equiv 299,792,458 \, \text{m s}^{-1}, \\
e &\equiv 1.602\ 176\ 634 \times 10^{-19} \, \text{Coulomb}, \\
\epsilon_0 &= 8.854\ 187\ 8128(13) \times 10^{-12} \, \text{F m}^{-1}, \\
m_e &= 9.109\ 383\ 7015(28) \times 10^{-31} \, \text{kg}, \\
\text{AU} &\equiv 149,597,870,700 \, \text{m}, \\
\text{parsec} &\equiv \frac{180 \times 3600}{\pi} \text{AU}.
\end{aligned}
$$

Substituting these constants into Equation 7 and using sensible normalizations (frequency

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[2]https://www.iau.org/public/themes/measuring/

[3]eschewing the proper trigonometric formula!

[4]https://www.nist.gov/si-redefinition

[5]With this announcement the seven defining constants of the SI system are: the hyperfine line frequency of Cesium-133 ($\Delta\nu_{\text{Cs}}$), $c$, $h$, $k_B$, $e$, Avogadro's number ($N_A$) and luminous efficacy ($K_{\text{cd}}$). See https://physics.nist.gov/cuu/Constants/index.html for values.

in GHz, $n_e$ in cm$^{-3}$, distance in parsec) I find

$$a = 4.148\,806\,4239(11) \text{ GHz}^2 \text{ cm}^3 \text{ pc}^{-1} \text{ ms}, \tag{8}$$

$$K = 241.033\,1786(66) \text{ GHz}^{-2} \text{ cm}^{-3} \text{ pc\,s}^{-1}. \tag{9}$$

Note that 8-byte floating point arithmetic should be used to preserve the high precision of the constants. The fractional uncertainty in $K|a$ of $2.8 \times 10^{-10}$ is a result of the experimental fractional uncertainties of $m_e$ ($3 \times 10^{-10}$) and $\epsilon_0$ ($1.5 \times 10^{-10}$). This uncertainty is so small, in relation to measurement errors of arrival times, that we should no longer be setting $K|a$ to nice round numbers.

# 4   What exactly is the DM measuring?

Astronomers are interested in DM because it appears to be a highly desirable physical quantity – the column of electrons from the source to the observer. However, a closer investigation shows that other phenomena also contribute to the DM. Some of these are intrinsic (ions, temperature, magnetic field) and others are extrinsic (relative motion). J. A. Phillips & A. Wolszczan [1992] discuss corrections arising from finite temperature of the interstellar plasma and the effect of ambient magnetic fields on the inferred DM. Below, I comment on this paper at appropriate points.

## 4.1   Ions

An external electromagnetic field induce motion of not only the electrons but also the ions. Including both excitations leads to the following dispersion relation:

$$\omega^2 = \omega_e^2 + \omega_i^2 + c^2 k^2 \tag{10}$$

where

$$\omega_i^2 = \frac{4\pi e^2}{m_p} \sum_{Z=1} \frac{n_Z q_Z^2}{A} \tag{11}$$

(see §A). The sum is over ions of atomic weight ($A$), atomic number ($Z$) and number density $n_Z$. We assume that each atomic species is represented by its dominant isotope and has only one dominant ionization state (charge $q_Z$). Equation 10 can be restated as

$$\omega^2 = \omega_p^2 + c^2 k^2 \tag{12}$$

where the "plasma frequency" is

$$\omega_p^2 = \frac{4\pi n_e e^2}{m_e} \left[ 1 + \frac{m_e}{m_p} \sum_{Z=1} \frac{n_Z}{n_e} \frac{q_Z^2}{A} \right]$$

| Z | A | Atom | X |
|---|---|---|---|
| 2 | 4 | He | $9.6 \times 10^{-2}$ |
| 8 | 16 | O | $5.4 \times 10^{-4}$ |
| 6 | 12 | C | $3.0 \times 10^{-4}$ |
| 10 | 20 | Ne | $9.3 \times 10^{-5}$ |
| 7 | 14 | N | $7.4 \times 10^{-5}$ |
| 12 | 24 | Mg | $4.4 \times 10^{-5}$ |
| 14 | 28 | Si | $3.6 \times 10^{-5}$ |
| 26 | 56 | Fe | $3.5 \times 10^{-5}$ |
| 16 | 32 | S | $1.5 \times 10^{-5}$ |

Table 1: The abundance of elements with atomic number $A$ and atomic charge $Z$ by number, relative to Hydrogen. From Draine 2011, §1.2.

$$= \omega_e^2 \left[ 1 + \frac{m_e}{m_p} \sum_{Z=1} \frac{n_Z}{n_e} \frac{q_Z^2}{A} \right]. \tag{13}$$

In effect, Equation 13 informs us that the the square of the plasma frequency is equal to the charged particle number density but with each particle weighted by $Z^2/m$ where $Z$ and $m$ is the charge and the mass of the particle, respectively. Electrons dominate the sum because of their low mass. The contribution from protons is diminished by $m_e/m_p$ which amounts to 5 parts in $10^4$ or 500 part per million (ppm). In Table 1 I list the "Cosmic" or Solar abundance of significant elements.

In the Galactic Warm Ionized Medium (WIM) Helium is not ionized and so the dominant additional contribution is from protons. Moving on, the emerging view is that, at low redshift, a significant fraction of the baryons are distributed in an extended halo around galaxies (the "circumgalactic medium" or CGM; J. Tumlinson, M. Peeples & J. Werks 2017) as opposed to the traditional "intergalactic medium" (IGM; M. McQuinn 2016). The IGM is heated by light from active galactic nuclei whereas the CGM is heated by shocks generated by infall. For the low redshift Universe ($z \lesssim 3$) we can assume that Helium is fully ionized in the IGM in which case the additional contribution is $X(He)m_e/m_p$, about 55 pm. All metals, even if full ionized, will contribute, relative to electrons, no more than a few ppm.

## 4.2 Warm Plasma

The physics of warm plasma, $\epsilon_T = k_B T/(m_e c^2) \ll 1$, is not only complex but is rife with tedious algebra. A simple and qualitative understanding can be obtained by noting that the thermal motion of electrons, rms velocity $v_e$, results in an increased mass of the

electrons, $\gamma_e m_e$; here, $\gamma_e = (1 - \beta_e^2)^{-1/2}$ with $\beta_e = v_e/c$. As can be seen from Equation 2 a heavier electron leads to a smaller dispersive delay. The mean of the inverse mass of the electron is decreased by $\langle\gamma\rangle^{-1} \approx 1 - (1/2)\langle\beta_e^2\rangle$ where the averaging is done over the Maxwellian distribution. Since $\langle\beta_e^2\rangle = 3kT_e/(m_e c^2)$, the fractional contribution amounts to $O(-\epsilon_T)$.

Buneman (1980) derives dispersion relation for warm plasma, $\epsilon_T \ll 1$. Further simplifying this relation for low density plasma, $\epsilon \ll 1$, I find

$$\omega^2 = k^2 c^2 + \left[1 - \tfrac{3}{2}\epsilon_T\right]\omega_e^2 \tag{14}$$

(see §B). Equation 14 amazingly agrees with the simpler argument presented above. This correction amounts to *decreasing* the contribution by electrons by $f = -\tfrac{3}{2}\epsilon_T$. The result presented here differs from Phillips & Wolszczan [1992] who, with no justification, state $f = +\epsilon_T$.

Since $f \approx -253 T_6$ ppm the breakeven temperature is $2 \times 10^6$ K. At this temperature, the increased mass of the moving electrons compensates for the contribution to the dispersive delay by protons. For even hotter gas, say cluster gas with $T \approx 5 \times 10^7$ K, the hot electrons will decrease the dispersive delay by up to $-1\%$.

## 4.3 Magnetic Fields

The dispersion relation for electromagnetic waves through magnetized plasma is

$$\omega^2 = k^2 c^2 + \frac{\omega_e^2}{1 \pm (\omega_B/\omega)} \tag{15}$$

where $\omega_B = eB/(m_e c)$ is the electron gyro-frequency and the $\pm$ applies to the two senses of circular polarization [Draine 2011, §11.3]. For ISM parameters, $\eta = \omega_B/\omega \ll 1$, and so the above relation can be simplified to

$$\omega^2 = k^2 c^2 + \omega_e^2\left(1 \mp \frac{\omega_B}{\omega}\right) \tag{16}$$

leading to a dispersive delay that is both frequency and polarization dependent:

$$\tau(\nu) = a\frac{\mathrm{DM}}{\nu^2}\left[1 \pm 2\left(\frac{\nu_B}{\nu}\right)\right]. \tag{17}$$

The fractional change in the dispersive delay is $4\eta = 1.2 \times 10^{-8} B_\mu \nu_9^{-1}$ with $B_\mu$ being the magnetic field strength along the line-of-sight and in $\mu$G. Phillips & Wolszczan [1992] reach a similar conclusion.

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This effect is only important if the intervening cloud is highly magnetized (perhaps a cloud local to FRB) and that too at low observing frequency. For instance, if the local magnetization is $1\,\mathrm{mG}$ then the effect is $10^{-4}(\nu/100\,\mathrm{MHz})$. This effect would be diluted if most of the contribution to the DM came from gas in ISM or IGM.

## 4.4   Relative Motion

The motion of the pulsar with respect to the intervening medium or the observer does not affect the inferred DM. However, the relative motion of the observer with respect to the intervening medium will lead the observer to infer a different value for the DM.

Consider an event which puts out a broadband pulse, radio through X-ray. We start by considering the simple case of an intervening plasma cloud that is stationary with respect to an observer located at the solar system barycenter (SSB). Owing to dispersive delay within the cloud, the radio pulse, frequency $\nu_0$, arrives $\tau_i$ after the X-ray pulse (which we assume traveled at the speed of light). Following Equation 5 the observer infers

$$\mathrm{DM}_i = K\tau_i\nu_0^2. \tag{18}$$

Next, consider the case of an observer stationed on Earth. The orbital velocity of Earth around the SSB can range up to $\pm 30\,\mathrm{km\,s^{-1}}$. The frequency of the radio pulse as perceived by the Earth-bound observer is given by the relativistic Doppler formula:

$$\nu = \frac{1}{\gamma}\left(\frac{1}{1+\beta_r}\right)\nu_0 \tag{19}$$

where $v_r = c\beta_\mathrm{r}$ is the radial velocity[6], $\beta = v_\mathrm{orb}/c$ and $\gamma = (1-\beta^2)^{-1/2}$.

Let us assume that we have arranged for a light pulse to be emitted when the radio pulse enters and exits the cloud. All quantities, unless stated otherwise, are in the frame of the observer. Let the distance from the observer to the cloud at the time of entry be $L$. The first light pulse arrives at time $t_1 = L/c$. From time dilation we know that the second pulse will be emitted $\Delta t = \gamma\tau_i$ (observer frame) later. Thus, the observer receives the second light pulse at time $t_2 = \Delta t + (L + v_r\Delta t)/c$. As a result, the dispersive delay measured by the observer, $t_2 - t_1$, is $\tau = (1+\beta_r)\gamma\tau_i$. As before (Equation 5) the inferred DM is given by the product of the dispersive delay and square of the frequency, both measured in the same frame, or

$$\mathrm{DM} = K\tau\nu^2 = \frac{1}{\gamma}\frac{1}{(1+\beta_r)}\mathrm{DM}_i. \tag{20}$$

Thus, the observer value of DM is the intrinsic value times the Doppler factor. It appears that the DM behaves like a spectral line. In cosmology, without large-scale structure, all

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[6]I follow the astronomical convention in which receding radial velocities are positive.

motions are radial ($\beta_r = \beta$) and so the observed DM is $(1 + z)$ smaller than the intrinsic value – a well known result [K. Ioka 2003, S. Inoue 2004].

Since $\beta_r$ is small we see that the fractional error in the inferred DM is $\pm\beta_r$. For routine observations such as searches the variation of DM with respect to SSB is not taken into account. However, precision timing programs such as `tempo` and `tempo2` are quite aware of Equation 20 (M. Bailes, pers. comm.).

Incidentally, FRB observers at two different Earth-based facilities observing the same burst will perceive slightly different frequencies, owing to differing (Earth) rotational velocities. The resulting fractional difference in the inferred DM is smaller than $v_{\rm rot}/c \approx 3 \times 10^{-6}$ or 3 parts per million. Parenthetically we note that the ionospheric vertical electron column density varies between 5 to $500\,{\rm TEC}$[7] depending on diurnal and sunspot phase. Likely, for almost all FRBs, the signal-to-noise ratio will not be high enough to make a material difference to this discussion.

Equation 20, thanks to the principle of relativity, equally applies for the case of the cloud moving with respect to the SSB. Notice that the fractional correction to DM has the sign of the radial velocity. So there will be a reduction of this effect, should the ray go through several clouds with opposing radial velocities. However, the second order corrections still remain at the level of $O(\beta^2)$.

Given cosmographical parameters and assuming that most of the baryons are in the IGM a formal formula can be written down as a a function of $(1 + z)$ [K. Ioka 2003, S. Inoue 2004]. Convenient fitting formula are readily available [e.g. Z. Zheng et al. 2014]. However, for any given FRB, owing to large- and small-scale structure in (baryonic and dark) matter, significant deviations in DM, with respect to such formulae, are expected. The deviations will depend on the number of times the line-of-sight crosses clusters of galaxies and intersects CGM halos of galaxies; see M. McQuinn [2017].

From Equation 20 we see that peculiar velocities (velocities deviating from pure Hubble radial flow) will result in additional but minor perturbations. Examples of peculiar velocities include the rotation curve of the Milky Way ($250\,{\rm km\,s^{-1}}$), the infall of our Galaxy towards M31 ($100\,{\rm km\,s^{-1}}$) and the peculiar velocity due to structure on the local Baryon Acoustic Oscillation (BAO) scale (about $400\,{\rm km\,s^{-1}}$). The latter is most elegantly measured by observations of the Cosmic Background Radiation (CMB) "dipole" [e.g. Planck Collaboration 2014]. For gas at higher redshift all such kinematic effects are suppressed by $(1 + z)$. I wonder, whether in a decade from now, when FRBs are routinely localized to arcsecond accuracy every hour (and redshifts of host galaxies already determined from massively-multiplexed spectroscopic surveys), we will be able to sense the CMB dipole via this effect.

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[7]The TEC or "total electron column" is a unit used by aeronomers and is equal to a column density of $10^{12}\,{\rm cm^{-2}}$; see `http://solar-center.stanford.edu/SID`.

## 4.5  Deviations from $\nu^{-2}$

Above we have assumed that the pulsed signal arrives strictly as $\nu^{-2}$. However, there are indications of increased timing noise at low frequencies and one of the possibilities is differences in path taken by rays at low and high frequencies [R. M. Shannon & J. M. Cordes 2017]. Low frequency timing has its own additional opportunities and complications. This topic is beyond the scope of the paper.

# 5  Conclusion & Way Forward

## 5.1  Accurate Dispersion Measure: (F)utility

PSR 1909-3744 (DM=$10.4\,\mathrm{cm}^{-3}\,\mathrm{pc}$) probably has the most precise DM measurement, about $10^{-5}\,\mathrm{cm}^{-3}\,\mathrm{pc}$ per epoch [M. L. Jones et al. 2017]. Most pulsars, when carefully monitored, show changes in the DM at the level of $10^{-4}\,\mathrm{cm}^{-3}\,\mathrm{pc}\,\mathrm{yr}^{-1}$ [e.g. V. M. Kaspi, J. H. Taylor & M. F. Ryba 1994; Lam et al. 2016]. Such annual changes at the level of 1 to 100 ppm primarily arise from secular evolution of the pulsar-observer line of sight and/or gradual changes in the ISM itself. Furthermore, as discussed earlier in the paper, the DM, in addition to being the measure of the electron column density, is also sensitive to protons (500 ppm), motions of intervening clouds (10 to $10^3$ ppm) and the temperature (up to $-1\%$) of the intervening gas.

Thus, for most purposes, an accurate value of DM is not all that useful. On the other hand, astronomers undertaking precision pulsar timing have to account for changes in DM and for this purpose the measured quantity

$$\mathcal{D}(\nu_1, \nu_2) \equiv \frac{t(\nu_1) - t(\nu_2)}{\nu_1^{-2} - \nu_2^{-2}} \tag{21}$$

is a sensible and apt quantity. $\mathcal{D}$ carries the unit of Hz. A convenient normalization is $\mathcal{D}$ is $10^{15}\,\mathrm{Hz}$ since it corresponds to about $0.4\,\mathrm{cm}^{-3}\,\mathrm{pc}$. Thus, I suggest that $\mathcal{D}_{15} \equiv \mathcal{D} \times 10^{-15}\,\mathrm{Hz}$ be recorded and reported.

Separately, colloquially it is often stated that "the dispersion measure is the column density of electrons". One also hears of frequent press announcements proclaiming that "FRBs allow astronomers to count every electron along the line of sight". However, as extensively discussed in §4.1 and §4.2, the DM is sensitive to ions as well and has temperature dependence. Furthermore, unlike true column densities, the DM is not a Lorentz invariant (§4.4). We conclude that the DM is a good proxy for the electron column density for routine astronomical purposes, say to one ppt, but highly accurate measurements of the DM, say to one ppm, do not carry proportionally valuable information.

## 5.2   A Way Forward

In view of the arguments presented in the previous section I suggest that we abandon DM as one of the key precision parameters of FRBs and pulsars. Instead, I urge my colleagues to report $\mathcal{D}(\nu_1, \nu_2)$ for both FRBs and pulsars. Since $\mathcal{D}$ is not a Lorentz invariant it is essential to report the topocentric and barycentric values. I note that Table 1 of R. N. Manchester [1971] is a fine example for reporting $\mathcal{D}$ (followed by the DM).

Given $\mathcal{D}$, astronomers can readily obtain the DM from

$$\text{DM} = K\mathcal{D} \tag{22}$$

with the full assurance that $K$ (Equation 9) is known to better than one part per billion. For almost all purposes that I can think of there is little need for *accuracy* of DM, say, beyond even a part per thousand. Neophytes can compute the dispersive delay to any frequency via the equation $\tau(\nu) = a\mathcal{D}\nu^{-2}$. Fastidious users can apply appropriate corrections, both Special relativistic and General relativistic, to $\mathcal{D}$.

The old name for $\mathcal{D}$ was "dispersion constant" [see R. N. Manchester & J. H. Taylor 1972]. However, $\mathcal{D}$ is not a time invariant for a given pulsar, being affected by the secular evolution of the line-of-sight from the observer to the pulsar and gradual changes in the ISM (§5.1). Obviously different pulsars have different values of $\mathcal{D}$. In view of this I suggest the term "Dispersion Slope" for $\mathcal{D}$.

The proposal made here has the distinct advantage of preserving the precision of the measured quantity, $\mathcal{D}$ (or $\mathcal{D}_{15}$), in *published* literature. Perhaps, equally importantly, this proposal will result in sparing tyros, attempting to link the burst at different frequencies, from the need to learn secret handshakes of pulsar timing clubs.

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# A    Dispersion relation including the contribution from ions

F. F. Chen [1974; §4.12] provides the starting point for this section. As an external electromagnetic field propagates through the plasma it will excite small currents $\mathbf{j}_1$ which in turn generate electromagnetic fields. We will assume that there is no external magnetic field threading the plasma ($B_0 = 0$) and also that the plasma is "cold" (no gas pressure).

The relevant Maxwell's equations for the electromagnetic fields are

$$\nabla \times \mathbf{E_1} = -\frac{1}{c}\dot{\mathbf{B}}_1, \tag{23}$$

$$\nabla \times \mathbf{B_1} = \frac{1}{c}\dot{\mathbf{E}}_1 + \frac{4\pi}{c}\mathbf{j}_1. \tag{24}$$

The curl of Equation 23 is

$$\nabla \times (\nabla \times \mathbf{E}_1) = \nabla(\nabla \cdot \mathbf{E}_1) - \nabla^2 \mathbf{E}_1 = -\frac{1}{c}\nabla \times \dot{\mathbf{B}}_1$$

whereas the time derivative of Equation 24 is

$$\nabla \times \dot{\mathbf{B}}_1 = \frac{1}{c}\ddot{\mathbf{E}}_1 + \frac{4\pi}{c}\dot{\mathbf{j}}_1.$$

Eliminating $\nabla \times \dot{\mathbf{B}}_1$ between the last two equations and letting $\mathbf{E}_1 \propto \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$ yields

$$-\mathbf{k} \cdot (\mathbf{k} \cdot \mathbf{E}_1) + k^2\mathbf{E}_1 = \frac{4\pi i\omega}{c^2}\mathbf{j}_1 + \frac{\omega^2}{c^2}\mathbf{E}_1.$$

Electromagnetic waves are transverse waves and so $\mathbf{k} \cdot \mathbf{E}_1 = 0$. Thus,

$$(\omega^2 - c^2k^2)\mathbf{E}_1 = -4\pi i\omega\mathbf{j}_1. \tag{25}$$

In the absence of plasma, the RHS is zero and we would then find $\omega^2 = k^2c^2$, as expected.

The electric current is due to electrons and ions

$$\mathbf{j}_1 = -n_e e\mathbf{v}_{e1} + n_i e q_i \mathbf{v}_{i1} \tag{26}$$

where $n_e$ and $\mathbf{v}_{e1}$ is the number density and velocity of the electrons and $n_i$, $\mathbf{v}_{i1}$ and $q_i e$ is the number density, velocity and charge of the ions. We use the fluid mechanics equation of force to compute the velocities:

$$m_e\left[\frac{\partial\mathbf{v}_{e1}}{\partial t} + (\mathbf{v}_{e1} \cdot \nabla)\mathbf{v}_{e1}\right] = -e\left[\mathbf{E}_1 + \frac{\mathbf{v}_{1e}}{c} \times \mathbf{B}_1\right]. \tag{27}$$

In the small amplitude approximation we only retain terms which are linear in perturbed quantities. The second term in the LHS, being $O(v_{e1}^2)$, can be ignored. We now turn to the RHS. To start with note that there is no pressure term on the RHS, consistent with the assumption of cold plasma. Next, as can be deduced from Equation 23, the strength of $E_1$ is similar to that of $B_1$. However, the force due to magnetic field is reduced by $v_{e1}/c$. In our small amplitude approximation, the velocities of the fluid are small relative to the speed of light, $v_{e1}/c \ll 1$. Thus, in this approximation the force on the electron due to (propagating) magnetic fields can be neglected. The result is

$$m_e\frac{\partial\mathbf{v}_{e1}}{\partial t} = -e\mathbf{E}_1, \qquad m_i\frac{\partial\mathbf{v}_{i1}}{\partial t} = q_i e\mathbf{E}_1.$$

The electron and ion velocities are excited by the incident electromagnetic field and thus they should also have the same functional form or $\mathbf{v}_{e1} \propto \exp[i\mathbf{k} \cdot \mathbf{x} - i\omega t]$ and so

$$\mathbf{v}_{e1} = \frac{e\mathbf{E}_1}{im_e\omega}, \qquad \mathbf{v}_{q1} = -\frac{qe\mathbf{E}_1}{im_i\omega}.$$

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Substituting these velocities into Equation 26 we find that the total current is

$$\mathbf{j}_1 = -\frac{1}{i\omega}\left[\frac{n_e e^2}{m_e} + \frac{n_i q_i^2 e^2}{m_i}\right]\mathbf{E}_1 \tag{28}$$

which, when substituted into Equation 25, leads to the dispersion law:

$$\omega^2 = k^2 c^2 + \frac{4\pi n_e e^2}{m_e} + \frac{4\pi n_i q_i^2 e^2}{m_i}. \tag{29}$$

As can be seen from Equation 28 it is easy to extend the dispersion relation to include contributions from several species of ions.

# B    Dispersion Relation for Warm Plasma

O. Buneman [1980] provides the following dispersion relation of warm plasma which is accurate to $O(\xi)$ where $\xi = \langle v^2 \rangle / c^2$ and $\langle v^2 \rangle$ is thermal velocity dispersion of the electrons in the plasma:

$$\frac{\omega^2}{\omega_e^2} = \frac{1}{1-n^2}\left(1 - \frac{\xi}{2}\right) - \frac{\xi}{3}. \tag{30}$$

Here, $\omega$ is the angular frequency; $k$, the wavenumber; $\omega_e$ the electron plasma angular frequency; $n = c/v_p$ is the refractive index where the phase velocity is $v_p = \omega/k$.

Let, as in the main text, $\epsilon = \omega_e^2/\omega^2$. For our purpose, we seek to further simplify Equation 30 for the case of $\epsilon \ll 1$. Noting that $n = kc/\omega$, Equation 30 can be recast as

$$\frac{\omega^2}{\omega_e^2}\left(1 - \frac{k^2 c^2}{\omega^2}\right) = \left(1 - \frac{\xi}{2}\right) - \frac{\xi}{3}\left(1 - \frac{k^2 c^2}{\omega^2}\right)$$

which can be re-arranged to yield

$$\omega^2 - k^2 c^2 - \omega_e^2\left(1 - \frac{5}{6}\xi\right) = \frac{1}{3}\xi\epsilon k^2 c^2.$$

We see that Equation 31 simplifies to the usual plasma dispersion law when $\xi = 0$. Further rearranging whilst dropping $O(\epsilon^2)$ results in

$$\begin{aligned}
k^2 c^2 &= \omega^2\left[1 - \epsilon(1 - \tfrac{5}{6}\xi)\right]\Big/\left[1 + \tfrac{1}{3}\xi\epsilon\right] \\
&\approx \omega^2\left[1 - \epsilon\left(1 - \tfrac{5}{6}\xi\right)\right]\left[1 - \tfrac{1}{3}\xi\epsilon\right] \\
&= \omega^2\left[1 - \epsilon + \tfrac{1}{2}\epsilon\xi + \tfrac{1}{3}\xi\epsilon^2 - \tfrac{5}{18}\xi^2\epsilon^2\right] \\
&\approx \omega^2 - \omega_e^2\left[1 - \tfrac{1}{2}\xi\right].
\end{aligned} \tag{31}$$

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