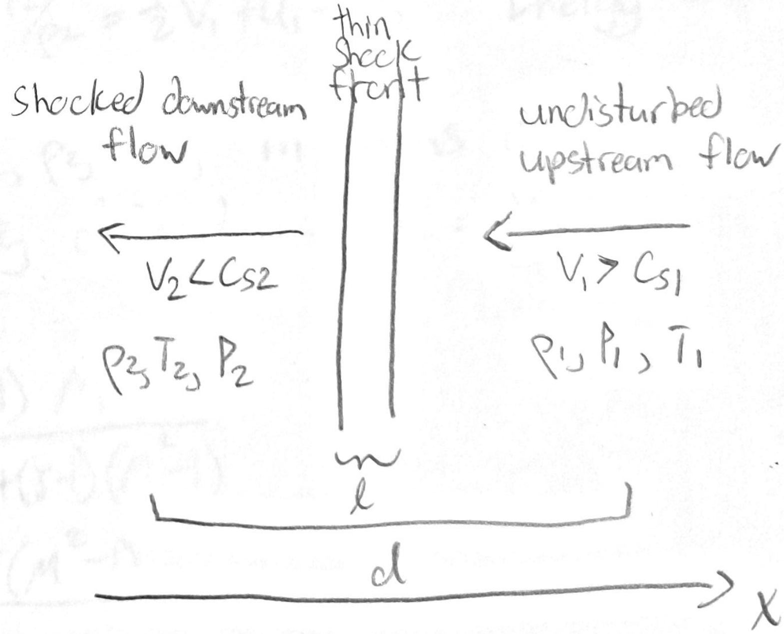
Lecture X: Shock Waves

Shock waves are sudden changes (on a viscous length scale) of the gas density/velocity, and shock fronts move supersonically relative to the unshocked gas. Hence, sound waves cannot "warn" the fluid in the upstream flow. Shocks are produced by steepening sound waves as discussed in the previous lecture, or by objects moving supersonically through a fluid (e.g., sonic booms) produced by jets, or by explosions.



To analyze shocks, we move to a frame co-moving with the shock front. Consider fluid on a scale d small compared to the pre/post-shock length scale, but large compared to the shock width, so the shock can be treated as a discontinuity. In this frame, the flow is steady, so $\frac{\partial}{\partial t}$ terms can be dropped. Away from the shock interface, the continuity, momentum, and energy equations become

$$\frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial}{\partial x}(\rho v^2 + P) = 0$$

$$\frac{\partial}{\partial x}\left[\rho v\left(\tfrac{1}{2}v^2 + h\right)\right] = 0$$

h = enthalpy $= \varepsilon + \frac{P}{\rho}$

ε = internal energy: $= \left(\frac{1}{\gamma-1}\right)\frac{P}{\rho}$

So we can equate the upstream quantities with the downstream quantities via the Rankine-Hugoniot jump conditions:

$$\rho_2 v_2 = \rho_1 v_1 \quad \text{Mass conservation}$$

$$\rho_2 v_2^2 + P_2 = \rho_1 v_1^2 + P_1 \quad \text{Momentum conservation}$$

$$\tfrac{1}{2}v_2^2 + u_2 + P_2/\rho_2 = \tfrac{1}{2}v_1^2 + u_1 + P_1/\rho_1 \quad \text{Energy conservation}$$

These can be solved for v_2, ρ_2, etc, in terms of the Mach number of the shock, defined as $M = \frac{v_1}{c_{s,1}}$.

For an ideal gas, $u = \frac{1}{\gamma-1}\frac{k_B T}{m}$

$$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M^2}{(\gamma+1)+(\gamma-1)(M^2-1)}$$

$$\frac{P_2}{P_1} = \frac{(\gamma+1)+2\gamma(M^2-1)}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{[(\gamma+1)+2\gamma(M^2-1)][(\gamma+1)+(\gamma-1)(M^2-1)]}{(\gamma+1)^2M^2}$$

Shocks must have $M>1$, which entails (for $\gamma>1$) that $P_2>P_1$, $\rho_2>\rho_1$, $T_2>T_1$, but $v_2<v_1$. The gas is compressed and heated, but decelerated by the shock (in the shock's frame).

The strong shock limit is $M_1 \gg 1$, in which we obtain

$$\frac{V_1}{V_2} = \frac{\rho_2}{\rho_1} \simeq \frac{\gamma+1}{\gamma-1} = 4 \quad \text{for } \gamma = 5/3$$

so shocks only compress gas by a factor ~ 4. In contrast,

$$\frac{P_2}{P_1} \simeq \frac{2\gamma M^2}{(\gamma+1)} \to \infty$$

so the pressure increase is arbitrarily large, as is the temperature increase

$$\frac{T_2}{T_1} \simeq \frac{2\gamma(\gamma-1)M^2}{(\gamma+1)^2} \to \infty$$

One can show for $\gamma = 5/3$

$$\frac{k_B T_2}{m} = \frac{3}{16} V_1^2$$

Hence, the flow's kinetic energy is converted to thermal energy by the shock. The downstream flow is sub-sonic,

$$\frac{V_2}{c_{s2}} = M_2 = \frac{(\gamma+1)+(\gamma-1)(M^2-1)}{(\gamma+1)+2\gamma(M^2-1)}$$

For $M \to 1$, $V_2 \to c_{s2} \simeq c_{s1} \simeq V_1$. For $M \gg 1$, $\frac{V_2}{c_{s2}} \to \frac{\gamma-1}{2\gamma} = \frac{1}{5}$ for ideal gas.

The entropy change $\frac{S_2}{S_1} = \frac{P_2}{P_1}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma} > 1$, so the shock is an irreversible process. The entropy jump is due to viscosity and thermal diffusion at the narrow shock interface.

Including viscosity/conduction modifies the conservation equations to

$$\frac{\partial}{\partial x}\left(\rho v^2 + P - \frac{4}{3}\rho\nu\frac{\partial v}{\partial x}\right) = 0$$

$$\frac{\partial}{\partial x}\left[\frac{1}{2}v^2 + h - \frac{4}{3}\nu\rho v\frac{\partial v}{\partial x} - K\frac{\partial T}{\partial x}\right] = 0$$

At the shock front, v and T are step functions and $\frac{dv}{dx}$ and $\frac{dT}{dx}$ are δ-functions, approximately. In reality, the shock front has width $\sim \ell$ the mean free path, so the fluid approximation starts to break down, and a kinetic treatment is needed for an accurate description.

More on shocks

We derived RK jump conditions

$$\frac{\rho_2}{\rho_1} = \frac{V_1}{V_2} = \frac{(\gamma+1)M^2}{(\gamma+1)+(\gamma-1)(M^2-1)} \simeq \frac{(\gamma+1)}{(\gamma-1)} = 4 \quad \text{for } M \gg 1,\ \gamma = 5/3$$

$$\frac{P_2}{P_1} = \frac{(\gamma+1)+2\gamma(M^2-1)}{\gamma+1} \simeq \frac{2\gamma M^2}{(\gamma+1)} \to \infty \quad \text{for } M \gg 1$$

$$\frac{T_2}{T_1} = \frac{[(\gamma+1)+2\gamma(M^2-1)][(\gamma+1)+(\gamma-1)(M^2-1)]}{(\gamma+1)^2 M_1^2} \simeq \frac{2\gamma(\gamma-1)M_1^2}{(\gamma+1)^2} \to \infty \quad \text{for } M \gg 1$$

So shocks can heat gas but can't actually compress it very well.

An exception to this rule occurs for radiative shocks, ie, shocks where the hot post-shock gas rapidly cools. See Shu Ch. 16 for details. In many cases radiative energy losses act such that the final (post-shock, post-cooling) temperature is $T_3 \simeq T_1$. But the conservation of mass/momentum still holds, so the RK conditions are

$$\rho_3 V_3 = \rho_1 u_1$$

$$\rho_3 V_3^2 + P_3 = \rho_1 V_1^2 + P_1$$

$$T_3 \simeq T_1 \Rightarrow \frac{P_3}{\rho_3} \simeq \frac{P_1}{\rho_1} \quad \text{for ideal gas}$$

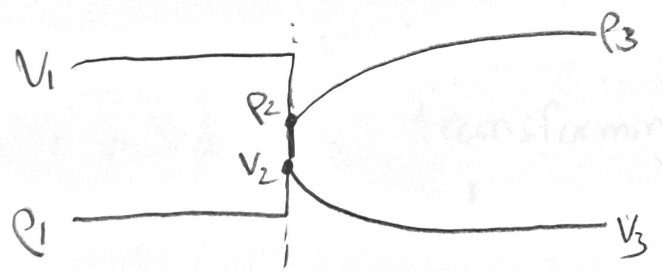
Solving these yields

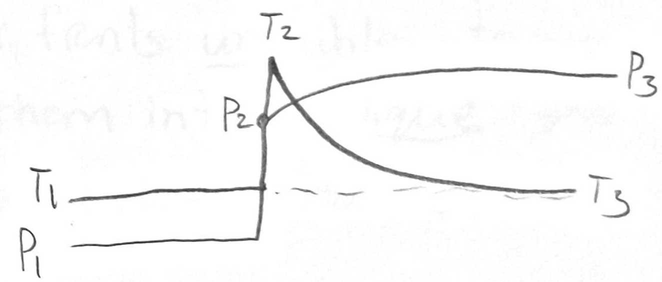
$$V_3 \simeq \frac{C_s^2}{V_1} \simeq \frac{C_s}{M} \quad \text{where} \quad C_s^2 = \frac{K_B T}{m} \quad \text{isothermal sound speed}$$

and

$$\frac{\rho_3}{\rho_1} = \left(\frac{V_1}{C_s}\right)^2 = M^2$$

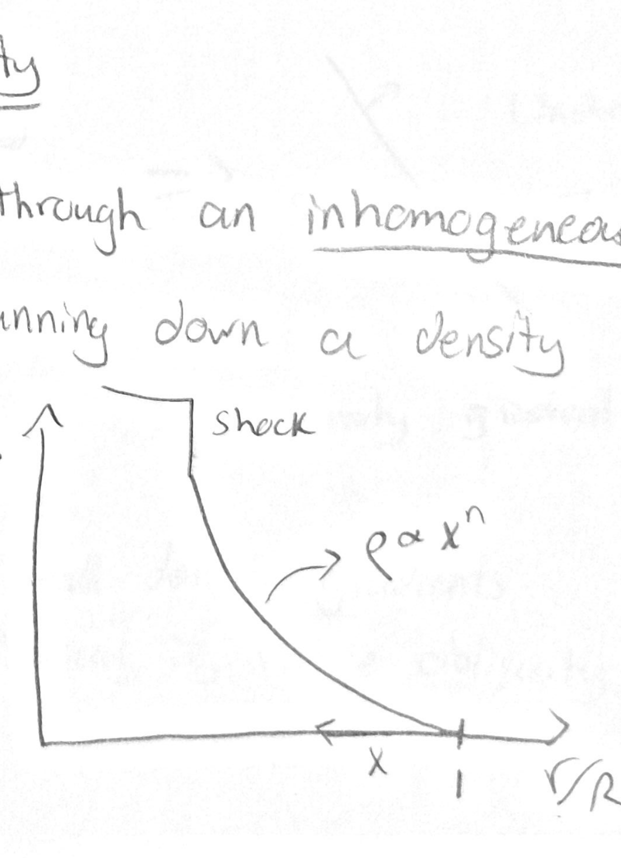
so radiative shocks can produce arbitrarily large compression, after radiative cooling has acted. This allows for dense shell formation, dust formation, instabilities, MHD effects, etc. The overall structure will be





Shock Acceleration and stability

Consider a shock propagating through an <u>inhomogeneous</u> medium. For instance, a shock running down a density gradient. At edge of star with adiabatic index $\gamma = 1 + 1/n$, we have seen $\rho \propto x^n$ where $x = (1 - r/R)$ and $n = \frac{1}{\gamma - 1}$



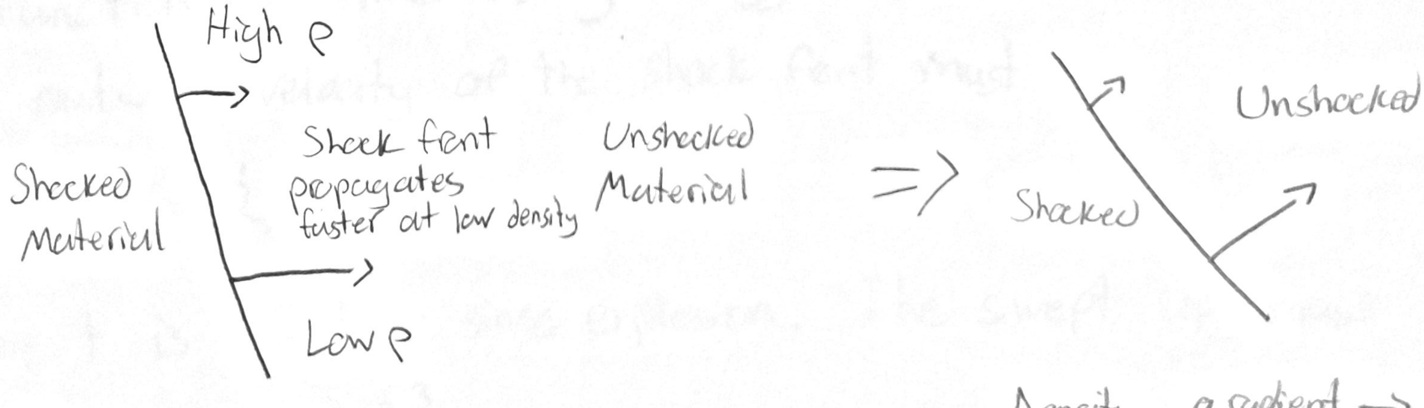
Sakurai 1960 showed that the shock velocity scales as (assuming a very strong shock with $M \gg 1$)

$$\frac{V_1}{V_0} = \left(\frac{\rho_1}{\rho_0}\right)^{-\beta}$$

with $\beta = 0.19$ for both $n=3$ ($\gamma = 4/3$) and $n = 3/2$ ($\gamma = 5/3$), where V_0 is the shock velocity at a reference density ρ_0.

So the shock <u>accelerates</u> as it propagates into lower density material.

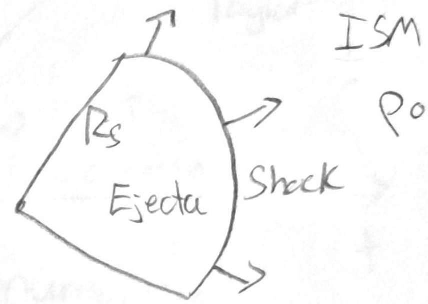
This also tends to make shock fronts <u>unstable</u> to tilt instabilities, transforming them into <u>oblique</u> shocks that propagate at an angle relative to the density gradient:



Deep in star, spherical geometry and small density gradients can act to make shocks more spherical, ie, reduce obliquity.

Blast Waves

Explosions such as bombs or supernovae drive supersonic ejecta into an ambient medium, with a shock front propagating into the surroundings. Initially, this shock front increases in radius at the initial explosion speed. However, as the ambient medium is swept up by the shock, it is accelerated outwards, so by momentum conservation the shock begins to slow down when the swept up mass becomes comparable to the ejecta mass.



After this phase, we can estimate the shock radius as a function of time using energy conservation.

The outward velocity of the shock front must be

$$V_s = \dot{R}_s \sim \frac{R_s}{t}$$

where t is the time since explosion. The swept up mass is

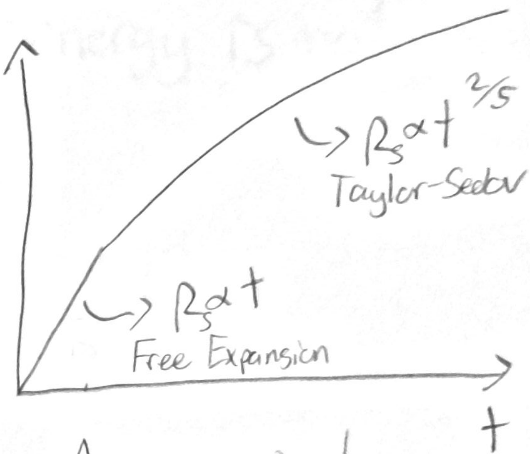
$$M_s \sim \rho_0 R_s^3$$

The energy of the swept up gas is

$$E_s \sim M_s V_s^2 \sim \rho_0 \frac{R_s^5}{t^2}$$

because we saw from the jump conditions that both the kinetic and internal energy of post-shock gas is $\sim \rho V_s^2$. Since the energy of the swept up gas is simply the explosion energy E, we obtain

$$R_s \sim \left(E/\rho_0\right)^{1/5} t^{2/5}$$

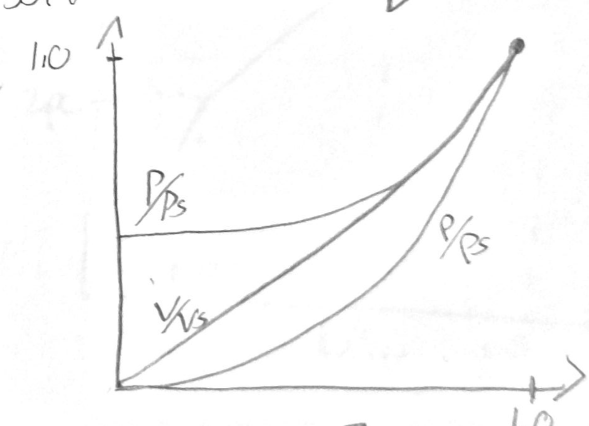


and the shock velocity

$$V_s \sim \left(E/\rho_0\right)^{1/5} t^{-3/5}$$

This is known as the Sedov-Taylor phase. A numerical solution shows $R_s = \left(\frac{E}{0.49\rho_0}\right)^{1/5} t^{2/5}$ for $\gamma = 5/3$.

Within the shocked region, the solution must scale with $\xi = r/R_s$, and it is thus self-similar, ie it has the same shape at all times. The solution looks like ⤵



The Sedov-Taylor solution is "adiabatic" in the sense that energy is conserved. At late times, the post-shock gas cools via radiative emission, e.g., Bremsstrahlung radiation for supernova shocks. The cooling time is

$$t_{cool} = \frac{E}{L_{rad}}$$

For Bremsstrahlung radiation,

$$L \propto \rho^2 R^3 T^{-1/2} \propto t^{9/5} \quad \text{for Sedov-Taylor}$$

So the cooling time $t_{cool} \propto t^{-9/5}$, ie, it gets smaller until $t > t_{cool}$, at which point the ejecta radiates away its energy. After this phase, the momentum of the ejecta is conserved even though the energy is not.

Hence

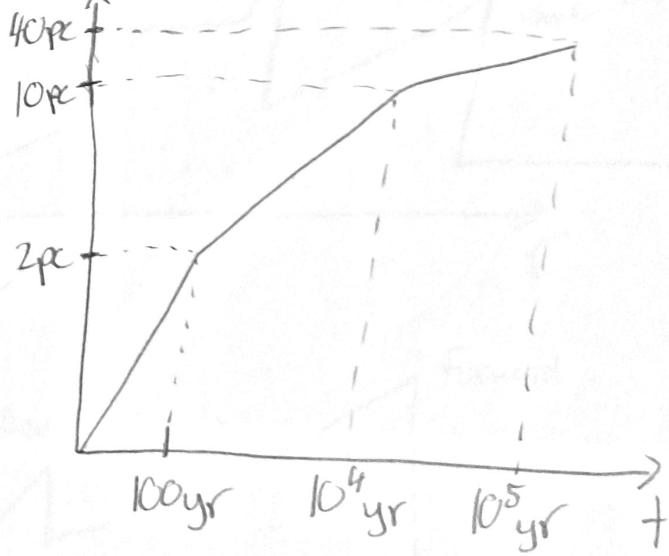
$$p = M_s V_s = \text{constant}$$

$$\Rightarrow \rho_0 R_s^3 \frac{R_s}{t} \sim \text{const}$$

$$\Rightarrow R_s \propto t^{1/4}, \quad V_s \propto t^{-3/4}$$

Once the shock velocity decreases below c_{s0}, the ambient sound speed, it is no longer a shock.

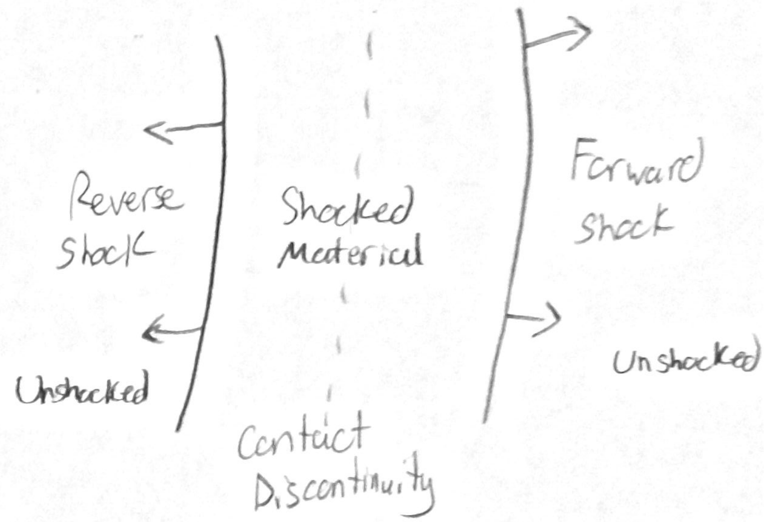
For a supernova with $E \sim 10^{51}$ erg exploding into the ISM with $n \sim 1\,\text{cm}^{-3}$, the expansion looks like



Within the ejecta, there are a few important features

The forward shock separates shocked ISM from unshocked ISM.

The contact discontinuity is the boundary between original ejecta and ISM,



The reverse shock propagates back into the ejecta (in a Lagrangian sense) when the swept up mass becomes comparable to the ejecta mass It causes the ejecta to slow down.

All this happens before the Sedov-Taylor phase. When the reverse shock reaches the center of the ejecta, the Sedov-Taylor phase begins.