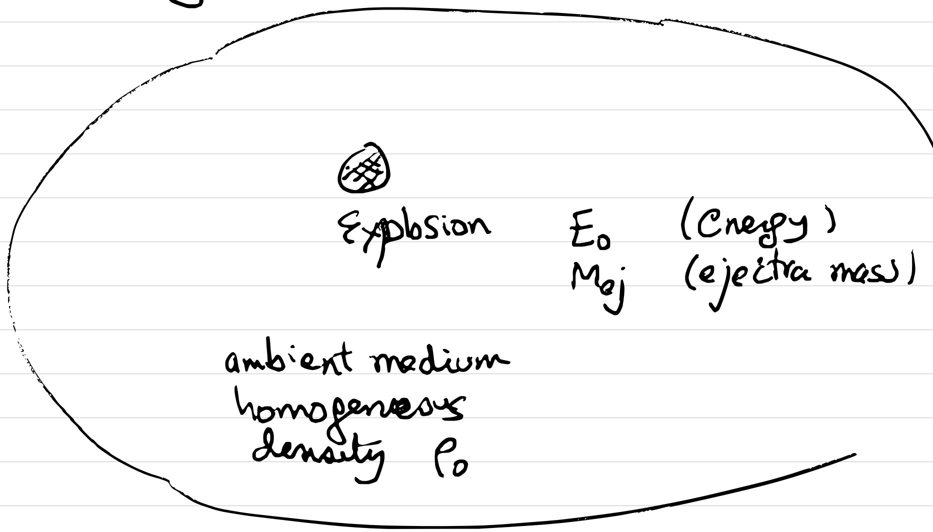
Sedov–Taylor solution

DRAINE, ch 39



Stage 1: Free expansion

Ejecta expands at velocity, V_{rms}

$$\frac{1}{2} M_{ej} V_{rms}^2 = E_0$$

$$V_{rms} \simeq \left(\frac{2E_0}{M_{ej}}\right)^{1/2}$$

For Supernovae

$E_0 = 10^{51}$ erg ("foe", Bethe)

M_{ej} … normalized to $M_\odot$

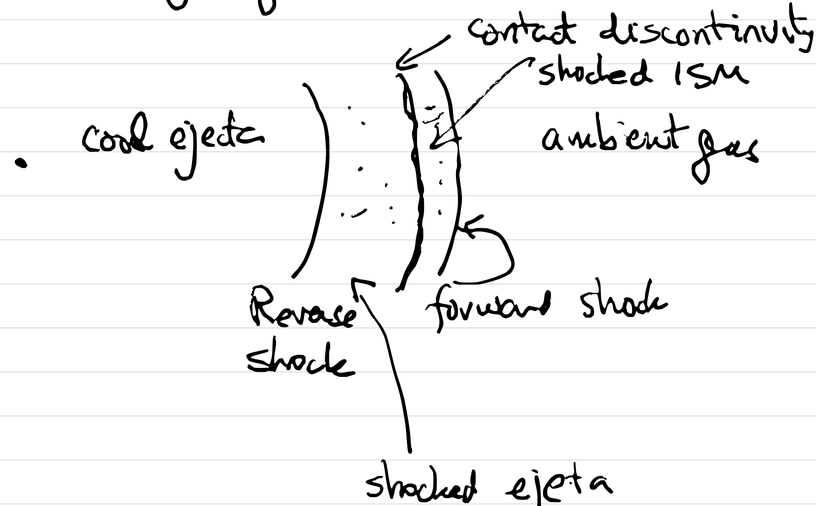
$$V_{rms} = 10^4 \text{ km/s } E_{51}^{1/2} \left(\frac{M_{ej}}{M_\odot}\right)^{-1/2}$$

As ejecta ballistically expands the ~~dens~~ mean density of ejecta falls as t^{-3}

The ejecta cools rapidly $(PV^{\gamma} = \text{constant})$

Once the ram pressure, $\rho(t) v_{rms}^2$ approaches mean ambient pressure, P_{amb}, a "reverse" shock sets in.

[The leading edge of ejecta slows down. The trailing edge starts to catch up]



Radius ~~of~~ at which ~~f~~ reverse shock reaches the center

$$R_1 = (3M_{ej} / 4\pi\rho_0)^{1/3}$$

$$R_1 \approx 2\,\mathrm{pc} \left(\frac{M_{ej}}{M_\odot}\right)^{1/3} n_0^{-1/3}$$

atoms/cm^3

Corresponding to epoch

$$t_1 \approx \frac{R_1}{V_{rms}} = 186\,\mathrm{yr} \left(\frac{M_{ej}}{M_\odot}\right)^{5/6} E_{51}^{-1/5} n_0^{-1/3}$$

Stage 2: Sedov-Taylor.

The ejecta is now heated, again

Similarity analysis:

This is a more powerful approach compared to dimensional analysis

The free parameters are

$$E_0, \rho_0$$

(M_{ej} has disappeared since $M_{swept\text{-}up} \gg M_{ej}$)

Step 1: Dimensional analysis

$$R_s = A E^\alpha \rho^\beta t^\eta$$

radius of shock front

With modest effort

$$R_s = \xi E^{1/5} \rho_0^{-1/5} t^{2/5}$$

Surprisingly with a lot of handwork

$$\xi \simeq 1.15167 \quad (!)$$

$$R_s \simeq 5\,\mathrm{pc}\,(E_{51}/n_0)^{1/5}\, t_3^{2/5}$$

t_3: 10^3 yr

$$V_s \simeq 1950\ \mathrm{km/s} \left(\frac{E_{51}}{n_0}\right)^{1/5} t_3^{-3/5}$$

$$T_s = 5.2 \times 10^7\ \mathrm{K} \left(\frac{E}{n_0}\right)^{2/5} t_3^{-6/5}$$

obtained using shock jump equations

Step 2:

Similarity analysis

$$\rho(r) = \rho_0 f(x) \qquad x = \frac{r}{R_s}$$

$$v(r) = \frac{R_s}{t} g(x)$$

$$p(r) = \frac{\rho_0 R_s^2}{t^2} h(x)$$

sensible choices

Sedov-Taylor formulation: Shu, Volume II, Ch 17

Strong-shock jump conditions:

$$\rho_2 = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_1$$

$$u_2 = \left(\frac{2}{\gamma+1}\right)V_s$$

$$P_2 = \frac{2}{(\gamma+1)}\rho_1 V_s^2$$

The equations

(1) $$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u) = 0$$

(2) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{dP}{dr}$$

(3) $$\frac{\partial}{\partial t}\left[\rho\left(\varepsilon + \frac{1}{2}u^2\right)\right] + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \rho u\left(\varepsilon + \frac{P}{\rho} + \frac{1}{2}u^2\right)\right] = 0$$

$$\varepsilon = \left(\frac{P}{\gamma-1}\right)\frac{1}{\rho}$$

We now apply self-similarity expectations

$$\rho(r,t) = \left(\frac{\gamma+1}{\gamma-1}\right)\rho_1 \alpha(\xi) = \rho_2 \alpha(\xi)$$

$$u(r,t) = \frac{4}{5(\gamma+1)} \frac{r}{t} v(\xi) = u_2 \frac{r}{R_s} v(\xi)$$

$$P(r,t) = \frac{8}{25(\gamma+1)} \rho_1 \left(\frac{r}{t}\right)^2 p(\xi) = P_2 \left(\frac{r}{R_s}\right)^2 p(\xi)$$

Substitute these into (1)–(3).

Solve numerically (Taylor) or analytically (Sedov.)

In the Sedov-Taylor phase the interior is hot. $\Rightarrow$ very little radiative loss.

As shock speed falls, cooling increases

S-T ends, say when, $\Delta E/E_0 \simeq -1/3$

$$t_{end} = 5\times10^4\,\text{yr}\; E_{51}^{0.22}\, n_0^{-0.55}$$

$$R_{end} = 24\,\text{pc}\; E_{51}^{0.29}\, n_0^{-0.42}$$

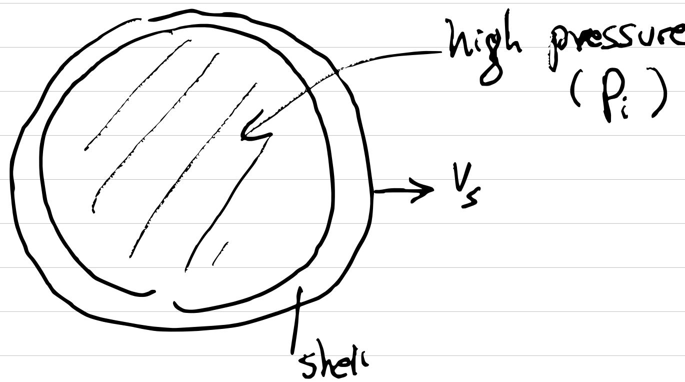
$$V_s = 188\,\text{km/s}\,(E_{51} n_0)^{0.07}$$

$$kT_s \simeq 41\,\text{eV}\,(E_{51} n_0^2)^{0.13}$$

Stage 3: Snow-plow phase

The post-shocked gas cools strongly (recall $V_{shock} \simeq 188$ km/s at the start of this phase)

The post-shock gas collapses to a high density shell.



The hot interior drives the shell outward

$$P V^{\gamma} = \text{constant} \qquad \gamma = 5/3$$

$$V \propto R^3$$

$$\underset{\nwarrow \text{interior}}{P_i} = P_0(t = t_{rad}) \left(\frac{R_{rad}}{R_s}\right)^5$$

$$t_{rad} = 49 \times 10^3 \, E_{51}^{0.22} \, n_0^{-0.55} \, \text{yr}$$

$$R_{rad} = 7.3 \times 10^{19} \, \text{cm} \, E_{51}^{0.29} \, n_0^{-0.42}$$

The force equation is

$$\frac{d}{dt}(M_s V_s) = P_i 4\pi R_s^2 .$$

$$= 4\pi\, p_0(t = t_{rad})\, R_{rad}^5\, R_s^{-3}$$

Note $V_s = \frac{dR_s}{dt}$

$$\dot{M}_s = R_s^2 V_s = R_s^2 \dot{R}_s$$

Assume

$$R_s \propto t^\eta$$

$$\Rightarrow \dot{M}_s \propto t^{3\eta-1}$$

$$\Rightarrow M_s \propto t^{3\eta}$$

$$\therefore \frac{d}{dt}(M_s \dot{R}_s) \propto t^{4\eta-2} \propto R_s^{-3}$$

$$\therefore \eta = 2/7$$

In "snow plow" phase $R_s \propto \left(\frac{t}{t_{rad}}\right)^{2/7}$

$$V_s = \frac{2}{7}\frac{R_s(t_{rad})}{t_{rad}}\left(\frac{t}{t_{rad}}\right)^{-5/7}$$

Stage 4: Fade-out.

(Dumps remaining kinetic energy into the ISM)