A & B coefficients, oscillator strength

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The basic material is given in Chapter 6 of Draine.

1 A & B Coefficients.

The usual derivation for the relation between A and B coefficients is as follows. Consider a two-level system immersed in a black body radiation field of temperature. The blackbody intensity is given by

$$B_\nu = 2k^2 \frac{h\nu}{\mathrm{e}^{h\nu/k_BT-1}} \tag{1}$$

where 2 accounts for polarizations, $k = \lambda^{-1}$ accounts for phase space per unit solid angle, $4\pi k^2/4\pi$ and $(\mathrm{e}^{h\nu/k_BT-1})^{-1}$ is the photon-occupation fraction. For CGS the units are $\mathrm{erg\,cm^{-2}\,s^{-1}\,Hz^{-1}\,ster^{-1}}$. The corresponding spectral energy density is $u_\nu = (4\pi/c)B_\nu$.

The upward transition rate, per unit volume, is given by $n_1 B_{12} u_{12}(\nu)$ where n_1 is the number density of a two-level atom. The radiation field is broad-band whilst the line width is narrow. So we can use $u(\nu_{12})$ in lieu of $\int u(\nu)\sigma(\nu)d\nu$ where $\sigma(\nu)$ is the absorption cross-section. Once thermal equilibrium is achieved between then the rate of upward transitions must equal rate of downward transitions. In 1905 Einstein considered the equilibrium between such a two-level atom and Planck's blackbody radiation field. The ratio of the population of particles in the excited (n_2) to those in the ground state (n_1) is

$$\frac{n_2/g_2}{n_1/g_1} = \mathrm{e}^{-E_{12}/k_BT} \tag{2}$$

where g_i is the degeneracy factor for level i and E_{12} is the energy of the excited level with respect to the ground state.

(Stimulated) absorption and radiative decay were already known by the turn of the century. Thus the equilibrium relation is

$$n_1 B_{12} \int u_\nu d\nu = n_2 A_{21} \tag{3}$$

where u_ν is the radiation energy density per unit frequency. Typically, the line-widths are very small and so we can approximate the integral by $u(\nu_{12})\delta\nu$ where $\delta\nu \ll \nu_{12}$. A logical conclusion of this equation is that B_{12} was temperature dependent (assuming that A_{21} was a constant of nature). In fact, B_{12} would go to infinity as the temperature dropped! There was no evidence for such a behavior. It took the genius of Einstein that adding a new process – stimulated emission – allowed for B_{12} to be temperature independent. Adding in stimulated emission results in the following equilibrium equation:

$$n_1 B_{12} u(\nu_{12})\delta\nu = n_2 B_{21} u(\nu_{12})\delta\nu + n_2 A_{21}\ . \tag{4}$$

Show that this equation demands the following relations to hold:

$$g_1 B_{12} = g_2 B_{21}, \tag{5}$$

$$B_{21} = \frac{c^3}{8\pi h\nu^3} A_{21}\ . \tag{6}$$

After you have proven the result, use the following more physically oriented approach: consider Equation 4 in the limiting case of $T \to 0$ and $T \to \infty$ to derive the same result.

Note: here, following Draine, we used $u(\nu)$. We could have used intensity, $I(\nu)$ in which case Equation 6 *would be different*; see R. C. Hilborn (Am. J. Phys. 1982) for wide-spread confusion on this topic.

Stimulated Emission. The photon occupation number, ν_γ, can be generally stated as

$$\bar{n}_\gamma \equiv \frac{c^2}{2h\nu^3}\bar{I}_\nu = \frac{c^3}{8\pi h\nu^3} u_\nu \tag{7}$$

where $\bar{I}_\nu$ is the intensity averaged of 4π steradians. We can see that the photon occupation number of radiation from a star decreases as the inverse square distance. For a radiation field that is isotropic the photon occupation number is fixed.

Incidentally, for black-body radiation field, $\bar{n}_\gamma = [\exp(h\nu/k_B T) - 1]^{-1}$. Detectors in astronomy fall into one of two categories: those which directly detect photons (e.g. CCDs, X-ray detectors, IR detectors) and those which detect the amplitude of the electric field (radio detectors). For the former $\bar{n}_\gamma \ll 1$ while for the latter $\bar{n}_\gamma \gg 1$.

The rate of excitation and de-excitation, per unit time, are

$$\frac{dn_1}{dt} = n_1 \frac{g_2}{g_1} A_{21} \bar{n}_\gamma \tag{8}$$

$$\frac{dn_2}{dt} = n_2 A_{21}(1 + \bar{n}_\gamma) \tag{9}$$

You can see that stimulated emission becomes important only when $\bar{n}_\gamma \gtrsim 1$. In astronomical settings stimulated emission is important at long-wavelengths, say $\gtrsim 100\,\mu$m.

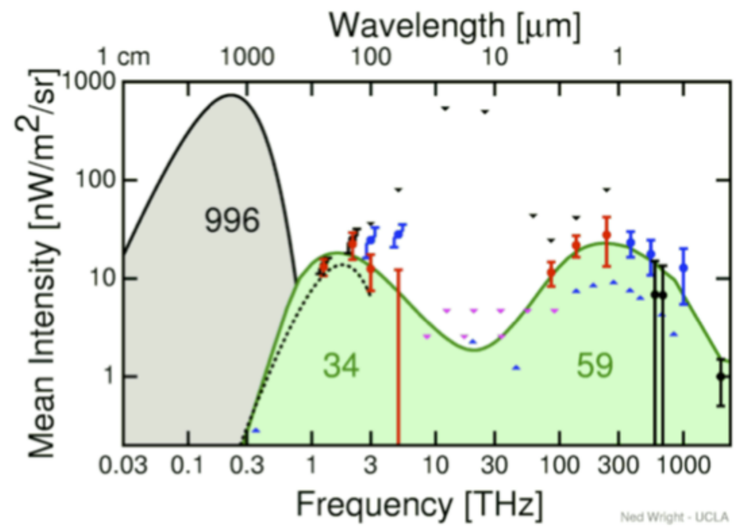

Figure 1: The sky background from 1 μm through 1 cm. The peaks, left to right, are Cosmic Microwave Background Radiation, far infra-red background (due to heating of dust by star-light) and near infra-red background (due to stars). Note that ordinate refers to νI_ν. The integrals, $\int I_\nu d\nu$ are noted in the text (units: nW/m^2/sr). The integral From Ned Wright, UCLA. `https://astro.ucla.edu/~wright/CIBR/`.

Problem: Stimulated Emission. The sky brightness (at high latitude) is displayed in Figure 1. In what spectral regions does $\bar{n}_\gamma$ exceed unity?

Problem: Energy density of the CMB. The temperature of CMB is 2.7260 ± 0.0013 K. Verify that $\int I_\nu d\nu = 996\,\mathrm{nW\,m^{-2}\,sr^{-1}}$ (as stated in Figure 1). What is the energy density (erg/cm^3) of the CMB?

Problem: Radio Background. At radio (low frequency) bands, the intensity is quoted in terms of "brightness" temperature. The brightness temperature can be converted to intensity via the Rayleigh-Jeans formula:

$$I_\nu = 2\frac{k_B T}{\lambda^2}\ . \tag{10}$$

The radio background at high latitude is shown in Figure 2. This is primarily due to Galactic diffuse synchrotron emission. Compute the frequency range over which stimulated emission dominates.

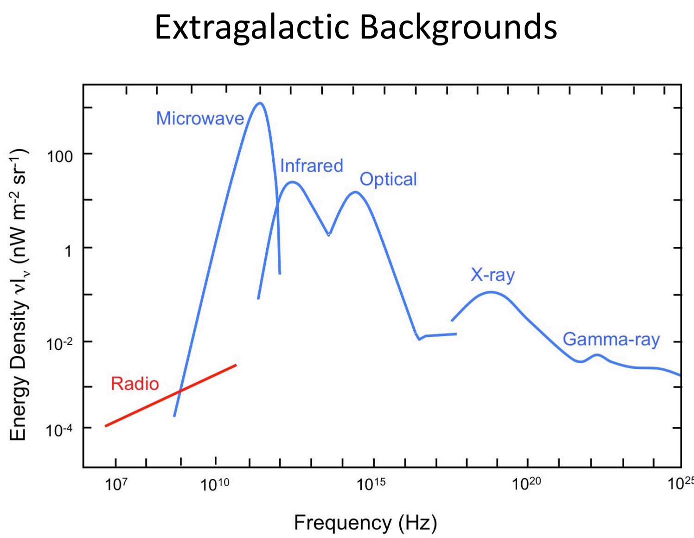

Figure 2: Radio background.

2 Absorption Cross-section.

It can be readily demonstrated that the absorption cross-section, σ, is related to the A-coefficient (Draine, 6.18):

$$\int d\nu\sigma(\nu) = \frac{g_u}{g_l}\frac{c^2}{8\pi\nu_{12}^2}A_{21} \tag{11}$$

One way to took at this equation is that at $\nu = \nu_{12}$ the absorption cross-section is $\approx \lambda_{12}^2$ (leaving aside, g_1, g_2 and 8π) but is limited to a bandwidth of A_{21}. Thus, "permitted" transitions with large A_{21} have stronger absorption relative to "forbidden" transitions with smaller A_{21} coefficients.

$$\sigma(\nu) = \frac{g_u}{g_l}\frac{c^2}{8\pi\nu_{12}^2}A_{21}\phi(\nu) \tag{12}$$

with $\int \phi_\nu d\nu = 1$. Consider the atoms to be at rest. In this case, $\phi(\nu)$ is frequency profile of the radiative decay process. Connecting to classical mechanics the oscillator strength, f_{12}, is defined as

$$\sigma(\nu) = \frac{\pi e^2}{m_e c}f_{12}\phi_\nu \tag{13}$$

The relation between oscillator strength and A-coefficient is

$$A_{21} = \frac{8\pi^2 e^2 \nu_{21}^2}{m_e c^3}\frac{g_1}{g_2}f_{12}\ . \tag{14}$$

Note that the oscillator strength for emission ($2 \rightarrow 1$) is negative:

$$g_1 f_{12} = -g_2 f_{21} \ . \tag{15}$$

Note that in some papers $gf = |g_1 f_{12}| = |g_2 f_{21}|$ is quoted.

The oscillator strength is useful because of the Thomas-Reich-Khun sum rule. Specifically, for a single valence electron

$$\sum_j f_{ij} = 1 \tag{16}$$

where the initial level is i and the sum is over all transitions from this state to all levels j (above and below i and including to the continuum). Consider, the ground state of hydrogen, $1\,^2\mathrm{S}_{1/2}$. The sum of the the oscillator strengths (Draine, Table 9.1) is ≈ 0.6. This means that the remaining 0.4 is due to transitions to the continuum.