Module: Maxwell-Boltzmann Distribution

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Background & Motivation. *The interstellar (and intergalactic) medium is the most rarified gas in the Universe. Even so, collisions ensure that the plasma reaches kinetic equilibrium, which means that the velocity distribution of each species of particles (protons, electrons, atoms, ions) are given by a Maxwell-Boltzmann (MB) distribution with the same temperature, T, for all species. Only on very short timescales and nascent phenomenon (e.g. gas parcel behind a shock front, a bow shock etc) is the gas not in kinetic equilibrium. The purpose of this exercise is to make the student become intimately familiar with the properties of the MB distribution. The basic source for this material is the entry for "Maxwell-Boltzmann distribution" in Wikipedia.*

[1] **Measures of the MB distribution**. The MB probability distribution is

$$f(\vec{v})d^3v = \frac{1}{\pi^{3/2}} v_p^{-3}\, \mathrm{e}^{-v^2/v_p^2} d^3v \tag{1}$$

where $v_p = \sqrt{2k_BT/m}$. Since the velocity distribution is isotropic, we switch to spherical coordinates in which case $d^3v = dv_x dv_y dv_z = 4\pi v^2 dv$ where $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$ is the speed. The probability distribution for speed is given by

$$f_v(v)dv = \frac{4}{\sqrt{\pi}}\Big(\frac{v}{v_p}\Big)^2 \mathrm{e}^{-v^2/v_p^2} d\Big(\frac{v}{v_p}\Big) dv \tag{2}$$

It is important to always keep an eye on the dimensionality of equations (if only to catch typos). Clearly, $f_v(v)$ has zero dimension, as expected of any probability distribution.

Show that the mode, mean and mean-square speed $f_v(v)$ are v_p, $\langle v\rangle = 2v_p/\sqrt{\pi}$ and $\langle v^2\rangle = 3v_p/2$. Sometimes the modal speed is called as the "most probable" velocity (reflecting that the mode represent the highest point in a probability distribution).

[2] **Characteristic Values.** Almost all physical measures have a range of velocities, e.g., the velocity of particles in plasma. So when you want to compute a quantity in which kinetic is involved you need to integrate over the velocity distribution. However, before

doing any integration you should review the distribution and determine the range of velocity for a good fraction of particles. If the speed range is within a factor of two then you can, as a first estimate, use a characteristic value for the speed and make the necessary first estimate of the quantity you are interested in.
Compute the median and 25% and 75% velocities.

The kinetic energy is $E = {}^1\!/\!_2 mv^2$ where m is the mass of the particles. Show that probability distribution for the energy is

$$f_E(E)dE = \frac{2}{\pi^{1/2}}\left(\frac{E}{k_BT}\right)^{1/2} \mathrm{e}^{-E/k_BT} d\left(\frac{E}{k_BT}\right) \tag{3}$$

Demonstrate that $\langle E\rangle = {}^3\!/\!_2 k_BT$.

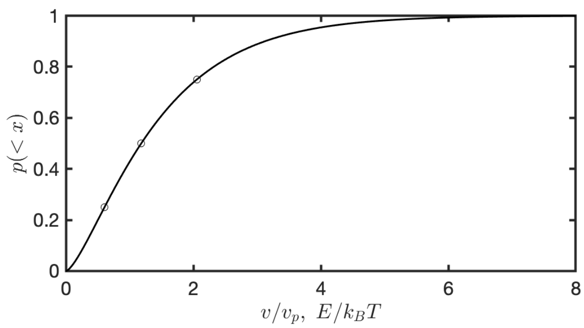

Figure 1: The cumulative probability distribution for v/v_p and E/k_BT The 25%- 50%- and 75%-percentile points are marked by open circle.

As can be seen from Figures 1 50% of the particles have energy (velocity) within less than a factor of two of the median value. So it is reasonable to use a characteristic value (e.g., mean or median) as representative of the entire population.

[3] **Line Profile.** Astronomical observations via spectral lines capture only one axis of velocity. The thermal velocities are isotropic. So we simply integrate over two axis, say y and z. It is easy to show that

$$f_{v_x}(v_x)dv_x = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-(v_x/v_p)^2} d\left(\frac{v_x}{v_p}\right) \tag{4}$$

where $-\infty < v_x < \infty$. Astronomers usually (but not always!) quote "full width at half maximum" (FWHM) which we find is $\sqrt{\ln(256)k_BT/m} \approx 2.35\sqrt{k_BT/m}$. Show that the same result is obtained by switching to cylindrical coordinates and integrating over, say, $\rho^2 = x^2 + y^2$.